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ELEMENTS

OF

ARITHMETIC AND ALGEBRA.

LONDON:
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ELEMENTS

OF

ARITHMETIC AND ALGEBRA.

For the Use of the Royal Military College.

BY

WILLIAM SCOTT, M.A. F.R.A.S.

PROFESSOR OF MATHEMATICS IN THE INSTITUTION.

LONDON:
PRINTED FOR
LONGMAN, BROWN, GREEN, AND LONGMANS,
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ADVERTISEMENT.

THE following Treatise on the Elements of Arithmetic and Algebra is the first of a series which is to constitute a General Course of Mathematics for the use of the gentlemen cadets and the officers in the senior department of this Institution. The course, when completed, will comprehend the subjects whose titles are subjoined: I. Arithmetic and Algebra. II. Geometry.* III. Plane and Spherical Trigonometry, with Mensuration. IV. Analytical Geometry and the Differential and Integral Calculus, with the Properties of the Conic Sections. V. Practical Astronomy and Geodesy. VI. The Principles of Mechanics; and VII. Physical Astronomy.

ROYAL MILITARY COLLEGE,
1844.

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* Published.

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PART I.

ELEMENTS OF ARITHMETIC.

SECTION I.

NUMERATION.

ARTICLE 1. The terms magnitude, quantity, are employed to describe that property of objects by means of which they are susceptible of increase or diminution.

Quantity is either discontinuous, that is, composed of separate parts; or continuous, that is, whole, or without any distinction of parts.

In discontinuous quantity, a separate part may be taken, and an exact notion of the quantity formed, by observing how often this part is contained in the whole.

In continuous quantity, an arbitrary part is assumed and employed in the same manner as the separate part in the case of discontinuous quantity.

The name given to this separate or arbitrary part, is one or unit, or unity.

Unity, therefore, is a magnitude of any kind whatever, either given by the constitution of the quantity, or arbitrarily assumed, and which serves as the means of comparison between all quantities of the same kind.

2. Number is the relation of a quantity to its unit.

If the magnitude, compared with its unit, is described as being of some particular sort, the number resulting from such comparison is said to be concrete.

A number, considered without reference to any determinate species of unit, is termed abstract.

When different quantities of the same kind are compared with the same unit, the resulting numbers must be different. Whence it becomes necessary to assign to these results different names, to the end that the numbers may be distinguished each from another.

The formation and arrangement of a system of names of numbers is the subject matter of numeration.

Names of numbers, like all other words of any language, are purely conventional.

3. One is the name employed to designate a single individual.

All whole numbers are formed by the combination of one with itself, then with the result of this combination, and so on.

The number formed by the combination of one with one is named two.
of two with one - - three.
of three with one - - four.
of four with one - - five.
of five with one - - six.
of six with one - - seven.
of seven with one - - eight.
of eight with one - - nine.

The series of numbers formed in this manner is indefinite, for, however far it be carried, one may always be combined with the last number, and another term obtained.

Since each number must have its distinguishing name, it becomes necessary to devise some means by which the inconvenience of an unlimited nomenclature may be avoided.

With this view, it has been agreed to consider the number resulting from the combination of nine with one (which is named ten) as a second kind, or new order, of unit; and as the units of the first order have been reckoned from one to nine, so, in accordance with the same convention, the tens, or units of the second order, are also reckoned from one to nine.

Thus, the number formed by
the combination of ten with ten is named twenty.
twenty with ten - - thirty.
thirty with ten - - forty.
forty with ten - - fifty.
fifty with ten - - sixty.
sixty with ten - - seventy.
seventy with ten - - eighty.
eighty with ten - - ninety.

Between each of these collections of tens there are nine numbers. Of the names of the nine between ten and twenty, two are irregular: the others are formed by combining the names of the simple units, from three to nine inclusive, with ten.

The number formed by the combination of ten with one is named eleven.
. . . . of eleven with one, or ten with two, twelve.
of twelve with one, or ten with three, thirteen.
of thirteen with one, or ten with four, fourteen.
of fourteen with one, or ten with five, fifteen.
of fifteen with one, or ten with six, sixteen.
of sixteen with one, or ten with seven, seventeen.
of seventeen with one, or ten with eight, eighteen.
of eighteen with one, or ten with nine, nineteen.

The names of the nine numbers between twenty and thirty are formed by annexing to the word twenty the names of the simple units from one to nine inclusive.

The names are, twenty-one.
twenty-two.
twenty-three.
twenty-four.
twenty-five.
twenty-six.
twenty-seven.
twenty-eight.
twenty-nine.

Similarly are formed the names of the numbers between thirty and forty; forty and fifty; . . . . eighty and ninety; ninety and ninety-nine; that is, by annexing one, two . . . . to the word expressing the number of tens.

Combining ninety-nine with one, *i.e.*, ninety with ten, or nine tens with one ten, the collection of ten times ten simple units is obtained for the number immediately following ninety-nine.

By convention, the collection of ten simple units has been made a unit of the second order. In like manner, the collection of ten tens is taken to form a third order of units, designated hundred or hundreds; and the name of the number next after ninety-nine is one hundred.

The hundreds, like the units and tens, are taken from one to nine. Their names are, one hundred.
two hundred.
three hundred.
. . . .
. . . .
nine hundred.

Intermediate between each of the successive hundreds are ninety-nine numbers. These obtain names compounded of the terms one hundred, two hundred . . . . . and the names already given to the series from one to ninety-nine. Thus, one hundred and one.

one hundred and two.
one hundred and three.
. . . . . .
one hundred and ten.
one hundred and eleven.
one hundred and twelve.
. . . .
one hundred and twenty.
one hundred and twenty-one.
. . . .
one hundred and ninety-nine.
two hundred.
two hundred and one.
. . . . .
. . . .
nine hundred and ninety-nine.

The next number, nine hundred and ninety-nine, increased by one, or nine hundred increased by one hundred, is named not ten hundred, but one thousand.

As the assemblage of ten individuals or units of the first order was, by convention, made a unit of the second order, and the assemblage of ten tens (or one hundred) a unit of the third, so the assemblage of ten hundreds (or one thousand) is made a unit of the fourth, that of ten thousands a unit of the fifth, and of ten tens of thousands a unit of the sixth order.

To avoid, however, the too great multiplication of names, a single word has not been chosen to signify either ten thousand, or ten times that number, but a new convention has been introduced.

Between simple units and thousands are interposed tens and hundreds. It has been agreed to consider one thousand as a new principal unit, and to continue, in like manner, the thousands through tens and hundreds.

Whence the names of the consecutive thousands are formed by placing successively the names in the series beginning with one, and ending nine hundred and ninety-nine, before the word thousand, thus,

one thousand.
two thousand.
. . . . . .
nine thousand.
ten thousand.
. . . . .
. . . . .
ninety-nine thousand.
one hundred thousand.
one hundred and one thousand.
. . . .
. . . .
nine hundred thousand.
. . . . .
nine hundred and ninety-nine thousand.

The names of the nine hundred and ninety-nine numbers between one thousand and two thousand, two thousand and three thousand, &c. are formed by writing the words from one to nine hundred and ninety-nine after each of the successive thousands, as,

one thousand and one.
one thousand and two.
. . . . .

one thousand and ninety-nine.
one thousand one hundred.
one thousand one hundred and one.
. . . .
one thousand nine hundred and ninety-nine.
two thousand.
two thousand and one.
. . . . . .
nine thousand nine hundred and ninety-nine
ten thousand.
ten thousand and one.
. . . . . .
ninety-nine thousand nine hundred and ninety-nine.
one hundred thousand.
one hundred thousand and one.
. . . .
one hundred thousand nine hundred and ninety-nine.
one hundred and one thousand.
one hundred and one thousand and one.
. . . .
one hundred and nine thousand nine hundred and ninety-nine.
one hundred and ten thousand.
one hundred and ten thousand and one.
. . . .
one hundred and ninety-nine thousand nine hundred and ninety-nine.
two hundred thousand.
two hundred thousand and one.
. . . .
. . . .
nine hundred and ninety-nine thousand nine hundred and ninety-nine.
ten hundred thousand, or
one thousand thousand.

The number, ten hundred thousand, is named one million.

One million is taken as a third principal unit, which, like the simple units and thousands, is reckoned by tens and hundreds.

Millions, from one to nine, form a seventh order of units.
Tens of millions - - an eighth.
Hundreds of millions - - a ninth.

From the number one million to the number nine hundred and ninety-nine millions nine hundred and ninety-nine thousand nine hundred and ninety-nine, the names are analogous to those of the numbers between one thousand and nine hundred and ninety-nine thousand nine hundred and ninety-nine.

But to denote the succeeding number, or ten hundred millions, the term employed is one billion.

In like manner, for ten hundred billions the term is trillion, &c. &c.

4. *Recapitulation.*

One, the element of number, is the first of the first order of units. The first order comprehends one, two, three . . . . . nine.

Ten is the first number of the second order. The second order comprehends ten, twenty, thirty . . . . . ninety.

One hundred is the first number of the third order. The third order comprehends one hundred, two hundred . . . . . nine hundred.

One thousand is the first number of the fourth order. The fourth order comprehends one thousand, two thousand, three thousand . . . . . nine thousand.

This convention, by which ten of any given order of units is constituted one of the next higher order, pervades the whole series of numbers.

By a term derived from the Latin equivalent of ten (decem) numbers thus reckoned are said to belong to the decimal scale.

Besides this rate or scale of progression by tens, a second contrivance is introduced; namely, to consider one as a first principal unit, to reckon it by tens and hundreds: to consider ten hundred times one, or one thousand, as a second principal unit; to carry it, in like manner, through tens and hundreds: to consider ten hundred times a thousand, or one million, as a third principal unit; to carry it through tens and hundreds: to consider ten hundred times a million, or one billion, as a fourth principal unit . . . .

In consequence of this arrangement, the names of the first four orders of units (thousands, hundreds, tens, and simple units) suffice to distinguish all numbers below the seventh order, or one million; the names of the first, second, third, fourth, and seventh, to distinguish numbers below the tenth order, or a billion, &c. &c.

Three orders of units being placed in the first principal order (that of units), three in the second principal order (that of thousands), &c. &c., the arrangement has obtained the name of a ternary division of the scale of numbers.

5. The nomenclature, which has been explained, is simple, and sufficiently comprehensive to embrace all finite numbers. Yet, if the names of two or more numbers consisting of several orders of units were written at length, it would be troublesome and difficult, on account of the tediousness of the expressions, to effect any combinations amongst them. Experience of this difficulty has suggested the expedient of an abridged notation.

One; ten; hundred; thousand . . . . . are the names of the successive orders of units; and the terms one, two, three, ... nine express the repetitions of each sort of unit in any given number.

If, therefore, convenient symbols or characters were employed to represent the nine simple units, and a means were also devised by which these characters could be rendered capable of representing different orders of units, the difficulty would be surmounted.

And it has been surmounted in the manner following:—

The numbers, one, two, three, four, five, six, seven, eight, nine are denoted thus, 1, 2, 3, 4, 5, 6, 7, 8, 9*; and, to express the different orders of units, this principle (conventional, of course) has been agreed upon, that every figure placed to the left of another shall be held to express units of the order immediately superior to those of that other figure.

6. As an illustration, let it be required to express, in figures, the number five hundred and forty-six. This number is composed of six (6) units, four (4) tens, and five (5) hundreds; and is, therefore, in accordance with the conventions already made, to be written 546.

Similarly, the number eight thousand six hundred and thirty-two, which is composed of 2 units, 3 tens, 6 hundreds, and 8 thousands, if translated into figures, will stand thus, 8632.

Let the number seven hundred and ninety be proposed. This contains 9 tens and 7 hundreds, but no units; whence arises the question, in what manner the tens and hundreds are to obtain their proper value by occupying the second and third places; for 79 signifies seventy-nine only, and not seven hundred and ninety.

That such a number may be capable of expression in figures, it is necessary that there be a figure without value to occupy the place of the units of any order which may be wanting. The character adopted is 0, or zero; and the number seven hundred and ninety is written 790.

Whence also the numbers

ten,	twenty,	thirty, ...	ninety are expressed thus,
10,	20,	30, ...	90

---

* These characters are termed figures or cyphers.

The numbers
one hundred, two hundred, three hundred, ... nine hundred, thus,
100, 200, 300, ... 900.

7. The ten characters, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0 are capable of being so combined as to express any whole number; for every number which can be named is composed of simple units, tens, hundreds, thousands, &c.; the order of each sort of unit is denoted by its place; the assemblage of units of each order is, of necessity, some number in the series 1, 2, 3 . . . . . 9; and the places of deficient units of any order are occupied by the character 0. In contra-distinction to the 0, which has no value when taken alone, 1, 2, 3, ... 9 are called significant figures. These have two values: the one belonging to the figure itself, and by virtue of which it is 2, and not 3 . . . . .; the other derived from the place, as ten, hundred . . . ., which the figure occupies. That is termed the absolute, this the relative, value of the figure.

8. The principle of a ternary division of the scale of numbers has been explained. (Art. 4.) Now, the characters 1, 2, 3, ... 9 are compendious representations of the names one, two, three, ... nine; and the place of each figure, be it second, third, ... from the right or place of simple units, indicates the particular order of units, whether second, third, ..., which that figure is employed to express. Whence each name, whether hundred, or ten, or one of simple units, of thousands, of millions, &c., is represented by a single figure holding its appropriate place; and consequently the three orders of units which, in the verbal enunciation of numbers are contained in each principal unit, are represented in figures by three cyphers, and no more.

Therefore, if any number expressed in figures be divided into periods of three cyphers each, beginning from the right, the division will correspond with the ternary arrangement of the names, as expressed in words at length.

The number twenty-eight millions five hundred and forty-seven thousand three hundred and nine, being composed of 9 units, 0 tens, 3 hundreds, 7 thousands, 4 ten-thousands, 5 hundred-thousands, 8 millions, 2 ten-millions, is written in figures thus, 28547309; and this expression may be divided into periods, as follows:

millions	thousands	units
28,	547,	309.

Each of the periods contains three figures, excepting the last, or that of the highest order of units; this may contain three, two, or only one.

Also, proceeding from right to left,
the first period is composed of hundreds, tens, and units of unity,
second - - - hundreds, tens, and units of one thousand,
third - - - hundreds, tens, and units of one million,
fourth - - - hundreds, tens, and units of one billion,
fifth - - - hundreds, tens, and units of one trillion.

The progression ascends regularly by units, tens, and hundreds of the successive principal units. The names appropriated to these, in order, after trillion, are quadrillion, quintillion, sextillion, septillion, octillion, nonillion, decillion . . . . . .

9. These remarks make it easy to translate into words the expression, in figures, of any number, however large; for, beginning at the right and pointing the number into periods of three figures, the first period on the right is that of units; the second, that of thousands; the third, that of millions . . . . . .

The hundreds, tens, and units; tens and units; or units (as the case may be), of the extreme period on the left are first to be read, making the name of the period to follow that of the units; the second period is to be proceeded with in the same manner, then the third, and so on to the last or extreme period on the right.

Take, as an example, the number 5738649879204651; this divided into periods is

5, 738, 649, 879, 204, 651.

writing over each period its denomination,

quadrillions	trillions	billions	millions	thousands	units
5,	738,	649,	879,	204,	651,

the number, expressed in words, is five quadrillions, seven hundred and thirty-eight trillions, six hundred and forty-nine billions, eight hundred and seventy-nine millions, two hundred and four thousand, six hundred and fifty-one.

As exercises in numeration, let it be required to express in figures the numbers following:—

1. Fifty thousand seven hundred and six.
2. Twenty-seven millions three hundred and ten thousand five hundred and twenty.
3. Nine thousand and ninety.
4. Thirty-five billions four hundred and seven millions one hundred and thirty thousand six hundred and sixty-four.
5. Ninety trillions five hundred and eighty billions seventy-six millions eight hundred and twelve thousand and three.
6. One hundred thousand and one.
7. One thousand one hundred.

Also, let it be required to express the following numbers in words:—

1. 3749584.
2. 501876308.
3. 2005809700106.
4. 1234567890.
5. 27358429060726.
6. 73458946537053096.

10. The contrivance to reckon one ten, then another ten, &c., and thence to arrange numbers in a decimal scale, was probably suggested to computists, in remote antiquity, by the number of the fingers of both hands. The ten numeral characters or cyphers, although of comparatively recent invention, are often, from the same cause, termed digits, that is, fingers.

11. Every combination of terms taken from the nomenclature of numbers, and, by consequence, every arrangement of figures, represents some particular assemblage of units. Yet, in reasoning upon number, it is often advantageous to employ considerations, and to establish principles, independent of particular quantity. In such cases the usual practice (based on convention) is to employ letters of the alphabet a, b, c, ... x, y, z, to represent numbers generally.

The investigation of the properties of numbers by means of general symbols belongs to the science of Algebra.

12. In treating of Numeration, it has been shown that numbers are formed by successive reunions of unity to itself, or to existing collections already formed in the same manner. There is, manifestly, no limit to the multitude of different assemblages of unity, that is, of different numbers which may be thus composed.

Now, it may become necessary to combine two or more such collections into a single number, or it may be required to find how many times oftener unity is repeated in one collection than it is in another.

The former object is accomplished by an operation named Addition;

The latter by another, Subtraction.

In addition the result is termed Sum; and in subtraction, Difference.

The numbers to be added together are sometimes called (by a Latin term) Addenda. In subtraction the greater number is the Minuend, the less the Subtrahend.

It is sometimes convenient to indicate, without performing, an addition or subtraction. In such cases the symbol +, named plus, is employed to signify that the sum of the numbers which it connects together is to be taken; and —, named minus, to denote that the number written after the symbol is to be subtracted from that placed before it.

SECTION II.

ADDITION.

13. In Addition certain numbers are given, to find a single number composed of as many units as are contained in all the given numbers.

If, to select a familiar instance, it were required to find the number of shells contained in two separate heaps, one would follow some such course as this: having reckoned the number in one heap (suppose it fifteen), the computer would transfer a shell from the other heap to this, saying sixteen; then another, saying seventeen; another, eighteen; and thus proceed until all the shells in the second heap were transferred to the first. The name corresponding to the last transfer would express the whole number of shells.

14. As the moving of that heap to this, or of this heap to that, cannot increase or diminish the absolute number of shells, it manifestly is indifferent which is added to the other.

If, instead of two heaps, there were three or more, the addition would be performed by transferring any one heap to any other, a third to these, or these to a third; and, in like manner, a fourth, a fifth .... to the last inclusive.

The reasoning from which it was inferred, in the case of two heaps, that the order of combination does not affect the result, may be extended from two to any greater number. Further, the same reasoning may be employed to prove that, whether several numbers (or heaps, considered as sensible representations of numbers,) are singly added to another number, and the successive results, or their sum is at once added to that other number, the last result, or absolute number of individuals, is the same.

In the example considered, the addition consists in the transfer, shell after shell, of the one parcel to the other; and a knowledge of the number of shells in the combined heap is attained by means of the name of the number pronounced simultaneously with the last transfer. This name is the last of a series setting out from unity, and increasing, by successive augmentations of one, to the inclusion, first, of every shell in one heap; and from this proceeding, still by augmentations of one, to the comprehension under one name of all the individuals in both heaps.

This process does not differ from that by which, in numeration, the series of numbers is formed by successive reunions of unity.

15. Addition, then, is simply an application of the principles of numeration. To add quantities together, the names of numbers and the order of their succession must be known; and more than this is not required.

The Addenda may be either abstract or (as in the example already given) concrete numbers. If the latter, the quantities must be of the same kind. Should they be of different kinds, as hours and pounds, the question is absurd, and the addition impracticable.

The name of the number of individuals resulting from the combination of several quantities into one is found by means of the names of number, which express the repetitions of unity in each of the several quantities. These names being independent of the particular kind of the unit, it follows that the addition of concrete numbers (provided they be of the same kind) and also of abstract numbers is performed in the same manner; that is, by a reduction of the names of the different numbers into the name of a single number composed of all the others, in conformity with the principles of numeration.

The addition of units can be made in this manner only, and the process is easily and (after a little practice) quickly executed.

16. If the addenda contain tens and units, the operation is tedious; if higher numbers, as thousands, the combination, one by one, of so many individuals becomes excessively laborious, or even impracticable. The conventional laws of numeration afford a means of overcoming this difficulty. By these the assemblage of units composing any large number is broken into parts, and distributed amongst other orders of units, each of which is ten times greater than that immediately succeeding it. A large number is thus to be regarded as composed of so many units of the first, so many of the second, so many of the third . . . . . so many of the *n*th order; that is, of so many units, tens, hundreds, &c.

When, therefore, the sum of two or more large numbers is required, each of the numbers is to be considered as decomposed into units, tens, hundreds . . . . . the sum of all the units of the first order contained in the given numbers is to be found; then that of all the units of the second order, or tens; next that of all the units of the third order, or hundreds, &c. &c. Then, since, in any quantity which is divided into parts, the sum of all the parts is equal to the whole quantity; since also the units, tens, hundreds . . . . . of any number compose that number; and consequently all the units, the tens, the hundreds of the given numbers, compose these numbers; it follows that the result obtained by adding the different collections of units into one sum, of tens into a second, of hundreds into a third . . . . must comprehend all the units, the tens, the hundreds . . . . . contained in the given numbers, and must, therefore, be equal to the sum of them all.

The addition of large numbers is thus reduced to the repetition, for each order of units, of the process employed in the addition of units of the first order; that is, of numbers each not greater than nine.

17. The method to be followed in the addition of simple units is the same as that explained in Article 13.

These additions may be at first performed by means of sensible objects. A little practice will fix the results in the memory, and render any subsidiary means unnecessary.

Or (the principles of addition being well understood) some labour may be saved by referring, for the sum of any two units, to the following

Table of Addition.

	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	10
2	3	4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10	11	12
4	5	6	7	8	9	10	11	12	13
5	6	7	8	9	10	11	12	13	14
6	7	8	9	10	11	12	13	14	15
7	8	9	10	11	12	13	14	15	16
8	9	10	11	12	13	14	15	16	17
9	10	11	12	13	14	15	16	17	18

An example will best show the manner of using this table: for which purpose, let it be required to find the sum of the two numbers 8 and 7. Find 8 in the first column on the left, and 7 in the uppermost column; trace the former to the right, and the latter downwards to the crossing of the two; 15 (the number filling the square formed by the intersection of horizontal column 8 with vertical column 7) is the sum of the numbers 8 and 7.

18. Before proceeding to the consideration of examples, it will be convenient to introduce a new symbol $=$, (equal to), which is employed to indicate that numbers between which it is interposed are equal. Thus, the expression, $2+1=3$[*], signifies that the sum of 2 and 1 is equal to 3.

19. This premised, let it be required to find the sum of the numbers 5 and 4.

Recurring to first principles (Art. 13.) the addition is performed thus:

$$5+4=6+3=7+2=8+1=9.$$

Therefore 9 is the sum of the numbers 5 and 4.

The same result, $5+4=9$, is obtained by a reference to the table of addition.

As a second example, let it be required to find the sum of the numbers 9, 7, and 8.

The operation at length is as follows:

$$9+7=10+6=11+5=12+4=13+3=14+2=15+1=16.$$

That is, $9+7=16$.

Next, $16+8=17+7=18+6=19+5=20+4=21+3=22+2=23+1=24$,

or, $16+8=24$.

But $16=9+7$, therefore $9+7+8=24$.

Employing the table to find the sum, $9+7+8$, it gives $9+7=16$: but how find the sum of 16 and 8, seeing the table contains none greater than $9+9$? Thus: 16 is composed of 1 ten and 6 units. Reserving the ten, the 6 units, added to the other 8, make 14 units, or 1 ten and 4 units; which, added to the ten reserved, make 2 tens and 4 units, or 24.

Required the sums of

(1)	(2)	(3)	(4)	(5)	(6)	(7)
2	6	2	3	5	9	8
3	7	4	5	8	7	9
4	8	6	7	7	4	6
5	9	8	9	6	8	9
14	30	20	24	26	28	32

20. The preceding examples contain units of the first order only. It appears, however (Art. 16), that to accomplish the addition of numbers containing units of several orders, no different method is required; but that the same process is to be repeated for each distinct order of units.

Suppose the numbers 32 and 57 to be given; and let it be required to find their sum.

Following the method pointed out in the preceding observations, the addition is to be effected as follows:

32 consists of 3 tens and 2 units.
57 consists of 5 tens and 7 units.
Adding the units, $2+7=9$ units,
And the tens, $3+5=8$ tens.

Whence $32+57=8$ tens$+9$ units$=89$.

The addenda and result may be put down more compactly thus:

Tens.	Units.
3	2
5	7
8	9

---

[*] An expression such as $2+1=3$, or $a+b=c$, is termed an equation.

Connected with the symbol $=$, are two other symbols, $>$ (greater than), $<$ (less than); thus $5>3$ signifies that 5 is greater than 3; and $a<b$, that the quantity represented by a is less than the quantity represented by b.

Let it next be required to find the sum of the numbers 29 and 68:

tens	units
2	9
6	8
8	17

This result, 8 tens and 17 units, rightly represents the sum of the units and tens contained in the given number; but it appears in a form requiring some modification to bring it into accordance with the decimal scale of notation; for it is a principle of this scale that ten units of any order make one unit of the order next higher; and here are seventeen units. The change to be made is this:—

Since 17 units=1 ten+7 units.

The number 8 tens+17 units=8 tens+1 ten+7 units.

That is, it equals 9 tens+7 units=97.

* ∴ 29+68=97.

Required the sum of 167, 386, 294, and 98?

hun.	tens	units
1	6	7
3	8	6
2	9	4
	9	8
9	4	5
	3	2

Explanation.—Having written the numbers under each other, in such manner that the units of the same order may occupy the same vertical column, the addition is made thus:—

1st, of the units: 8+4=12. 12+6=18. 18+7=25 units. 25 units=2 tens and 5 units. The 5 units are recorded under the column of units; and the two tens carried forward to the column of tens.

2d, of the tens: 2+9=11. 11+9=20. 20+8=28. 28+6=34 tens. 34 tens=3 hundreds and 4 tens. The 4 tens are written under the column of tens, and the 3 hundreds carried forward to the column of hundreds.

3rd, of the hundreds: 3+2=5. 5+3=8. 8+1=9 hundreds. This 9 being written in its proper place, the operation is completed, and the sum of the given numbers found to be 945.

21. Having found the sum of any column, and written the units contained in that sum in their proper place, the tens, reserved for combination with the next column, may be conveniently recorded by means of a small figure written under the units of that particular order from which they are carried. An advantage of this practice is, that if, after the sums of several columns have been found, any interruption occur, and the sum of the column under process of addition be lost, the number carried from the preceding column is not lost along with it.

Without some such expedient it might, in cases like the one imagined, be necessary to resume the work from the beginning.

In the last example the 2 written under the five units, and the 3 under the 4 tens, are instances of the precaution recommended.

The work and illustrations of the last example appear sufficient to direct the mode of procedure in the addition of any whole numbers whatever.

The words units, tens, hundreds, ... placed in the preceding examples over the different orders of units, were so placed to give distinctness to the explanations. The learner, after fully understanding the rationale of addition, may omit them from his work.

22. In finding the sum of several numbers, each composed of two or more orders of units, the addition is always commenced with the extreme column on the right, that is, with the simple units.

---

* The symbol ∴ is put for the word *therefore*.

The reason for so beginning is, that the tens, which may arise in the addition of the units, can thus be at once taken into account in adding the column of tens; the hundreds arising from the column of tens, in that of the hundreds, &c.

If the extreme left column be taken first, and the addition made from left to right, either new additions must be made to combine the tens arising from the units of any order with the units of the next higher; or the result must remain with more than nine units in, at least, some of the orders.

But for these circumstances, it would be matter of indifference whether, in adding numbers together, the process were performed from right to left or from left to right.

For the sake of illustration, the addition in the following example is made from left to right:—

th.	hun.	tens	units.
5	3	4	6
8	7	9	0
3	5	1	2
7	6	8	4
23	21	22	12

1st. $7+3=10$, $10+8=18$, $18+5=23$ thousands.
2d. $6+5=11$, $11+7=18$, $18+3=21$ hundreds.
3d. $8+1=9$, $9+9=18$, $18+4=22$ tens.
4th. $4+2=6$, $6+0=6$, $6+6=12$ units.

The sum is therefore made up of 23 thousands = 23000
21 hundreds = 2100
22 tens = 220
12 units = 12
Of which the total = 25332

To reduce the result to a single number, a second addition is necessary. As the sum of no column, in the present instance, exceeds nine, this second addition, whatever way begun, completes the work. But, supposing always the addition to be made from left to right, cases may occur in which three or more operations will be required to reduce tens of an inferior order into units of the next higher; that is, to reduce the sums of the different columns into one number.

The preceding remarks and demonstrations are, it is presumed, sufficient to render intelligible the principles and operations of addition.

A General Rule and Examples are annexed.

23. General Rule for the addition of whole numbers.

Write the numbers to be added together under each other in successive lines, so arranged that all the figures representing units of the same order may occupy the same vertical column.

Draw a line under the last number, to separate the addenda from the result. Then, beginning with the right column or units of the first order, find the sum of the figures composing this column: write the units of the sum under the column added, and reserve the tens, if any, for combination, as units of the second order, with the second column, or units of the second order. Add this column, in like manner: place the units of the sum below it, and reserve the tens for combination with the third column. In the same manner proceed with a third, a fourth .... to the extreme column on the left. The result of this operation is the sum of the given numbers.

24. Examples in the addition of whole numbers.

Required the sums of the following numbers:

1st. 5943, 7685, 2768, and 9374 ?..........................*Ans.* 25770.

2d. 38796, 67438, 54874, and 89657 ?....................*Ans.* 250765.

3d. 73564, 89753, 46586, 95845, and 38479 ?...........*Ans.* 344227.

4th. 4789, 6543, 2101, and 9876 ?..........................*Ans.* 23309.

5th. 123456, 789012, 345678, 901234, and 567890?...*Ans.* 2727270.

6th. 2574, 83965, 1781, 479, and 8698 ?..................*Ans.* 97497.

7th. 15893, 7586, 59948, 479, 3548, and 95 ?............*Ans.* 87549.

8th. 456, 7890, 12345, 67, 890123, and 4 ?..............*Ans.* 910885.

9th. 49876534, 15798249, 6789012, and 4789656?...*Ans.* 77253451.

10th. 7690458, 7690458, 7690458, and 7690458 ?...*Ans.* 30761832.

11th. 97, 764, 386, 17578, 8997, and 59604 ?.............*Ans.* 87426.

12th. 8759076, 605895, 961704, 87289, 743998, 6657, and 7985065 ?..*Ans.* 19149684.

13th. 8796, 5348, 284, 7963, 57, 9, 154, 9598, 358, and 85 ?
Ans. 32652.

14th. 574, 896, 78, 985, 347, 7584, 59, 870, 5794, 648, 1842, 31, 8628, and 779 ?..*Ans.* 29115.

15th. 689, 748, 986, 577, 854, 98, 379, 956, 28, 989, 497, 588, 867, 179, 996, 755, and 778 ?..*Ans.* 10964.

25. The addition of two numbers is indicated by interposing the symbol + between them. Thus $9+7$ is an expression signifying that the numbers 9 and 7 are to be combined into one number.

When general, instead of particular, expressions of number are employed (Art. 11) the same notation is used.

If a represent one number, and b another, the sum of a and b is written $a+b$.

In like manner, the sum of a, b, c, is denoted by the expression $a+b+c$.

If any quantity, a, is added to itself, the sum of the repetitions may be expressed in the same manner; as, $a+a$; $a+a+a$.

But it is to be observed that a signifies once the number a ;

that once a, + once a, = twice $a=2a$:

that once a, + once a, + once a, = three times $a=3a$, &c.

Hence the practice is, not to write the number as many times as it is repeated, but once only; and to express the number of repetitions by a figure prefixed to the a, or other general expression of number.

Numbers which (like 2 and 3 in the present instances) express the number of repetitions of a quantity, are termed co-efficients of that quantity.

Again, the sum of the quantities $2a$ and $3a$ may be indicated by the expression $2a+3a$.

But since $2a=a+a$

and $3a=a+a+a$.

Therefore $2a+3a=a+a$; $+a+a+a$.

or $2a+3a=5a$.

$2a$, $3a$, are expressions which signify that the quantity a is to be taken in the one instance two, and in the other three, times; that is, as often as the co-efficients contain simple units.

If the numeral co-efficients 2, 3, are replaced by the general expressions m, n (termed literal co-efficients), then, ma or m times a, indicates that the number a is to be repeated as often as the co-efficient m contains simple units; and na that the number a is again to be repeated as often as the co-efficient n contains simple units.

Whence, if a repeated m times is increased by a repeated n times, the sum is equal to a repeated as often as m and n together contain simple units.

But the number of simple units contained in m and n is indicated by the expression $m+n$. Consequently $m\,a+n\,a=(m+n)\,a$.

To signify that a is repeated $m+n$ times, and not m times only, or n times only, the sum of the co-efficients m, n, is written thus $(m+n)$.

26. Quantities expressed by the same letters, or by the same combinations of the same letters, are termed like or similar: quantities expressed by different letters, or by different combinations of the same letters, unlike or dissimilar.

From this definition, and the conclusions of Article 25, it follows that the addition of unlike quantities is indicated by connecting the quantities together with the symbol +, and the addition of like quantities, by adding the co-efficients together, and writing their sum before the common quantity.

26′. Each of the quantities composing an expression such as $a+b+c$, or $a+b-c$, is called a term.

A quantity composed of one term only is called a monomial; a quantity composed of two terms, a binomial; a quantity composed of three terms, sometimes a trinomial; but generally a quantity composed of more terms than two is called a polynomial.

Exercises in the addition of general expressions of quantity.

1st. Required the sum of $3a$, $5a$, a, $6a$, $17a$?……………………… *Ans.* $32a$.
2d. - - - ma, $2ma$, $5na$, and a?……… *Ans.* $(3m+5n+1)a$.
3d. - - - a, $5a$, ma ?………………………… *Ans.* $(6+m)a$.
4th. - - - a, mb, $6d$, $3a$, b, and $4g$?
Ans. $4a+6d+(m+1)b+4g$.
5th. - - - $a+b$, $ma+nc$, $3b+5c$?
Ans. $(1+m)a+4b+(n+5)c$.
6th. - - - $a+2ab$, and $ab+b$?
Ans. $a+2ab+b$, or $a+(2a+1)b$.
7th. - - - $a+2ab+db$, and $ab+2db+b$?
Ans. $a+3ab+3db+b$, or $a+(3a+3d+1)b$, or $a+\{3(a+d)+1\}b$.
8th. - - - $a+3b+5$, $b+5a+17$, and $2a+10$?
Ans. $8a+4b+32$.
9th. - - - $ma+md+1$, $3na$, and $7d+13$?
Ans. $(m+3n)a+(m+7)d+14$.
10th. - - - $a+d+e$, $b+c$, $5d$, $7c$, and $b+9e$?
Ans. $a+2b+8c+6d+10e$.

SECTION III.

SUBTRACTION.

27. Two unequal collections of units are given, and it is required to find how many more are contained in the greater collection than in the less: or, how many units of the greater collection are left when as many as are contained in the less have been taken from it. Such is the problem* of subtraction.

As a particular case, let it, for the sake of illustration, be supposed that a boy who had twelve apples gave five away; the question is, How many had he left?

Giving away one, he retained twelve less one, or eleven,
two, - - eleven less one, or ten,
three, - - ten less one, or nine,
four, - - nine less one, or eight,
five, - - eight less one, or seven.

Whence, having from twelve apples given away five, the boy had seven left.

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* A problem is a question which requires a solution.

In this example the subtraction is effected by taking from the whole number of apples, first one, then a second, a third, a fourth, a fifth. Also, the steps of the operation are traced and the final result enunciated by means of a series of names of numbers, commencing with that of the greater quantity, and descending by successive subtractions of unity until the name corresponding to the sum of the units taken away represents the less quantity. The name arrived at by these successive decompositions is that of the result sought.

28. This result is termed Remainder.

If two numbers are mentioned as being unequal, but without distinction of the greater, the term Difference is given to that number which constitutes the inequality. But if the greater be distinguished, instead of difference, the proper term is Excess.

29. Subtraction, like addition, is manifestly an application of the principles of numeration. It requires that the names of numbers be known in a descending series; or in the order contrary to that of their formation. Addition being performed by means of the ascending series of numbers, the one operation is the reverse of the other.

The minuend and subtrahend may be abstract or concrete numbers: if concrete, the numbers must express collections of individuals of the same kind; the operation, being conducted through the intervention of names, is the same for concrete as for abstract numbers.

30. The difference of small numbers is found by decomposing the minuend into two numbers, of which one is the subtrahend and the other the remainder; and this decomposition is effected by taking from the minuend successive units, until the number taken away is equal to the subtrahend.

But as, in finding the difference of large numbers, this method, by the continued subtraction of simple units, would be impracticable, the conventions of numeration which were employed to facilitate addition are in like manner rendered subsidiary to the operation of subtraction. Thus both the quantities are considered to be decomposed into collections of simple units, tens, hundreds, &c.; the units of the subtrahend are taken from the units of the minuend, the tens from the tens . . . . . . in fine, the units of every order contained in the former from the units of the correspondent order in the latter. The result of these partial subtractions is the remainder sought.

31. That a remainder, thus obtained, correctly expresses the difference of the given numbers, is proved by the considerations following:—

The whole being equal to the sum of all its parts, the minuend and subtrahend are respectively equal to the sum of the units, tens, &c. into which they are decomposed.

All the units, tens, &c. of the subtrahend are taken from the units, tens, &c. of the minuend, each from each; that is, all the parts of the subtrahend are taken from the minuend. But to take away all the component parts of any quantity, amounts to the subtraction of the whole of that quantity. Whence the whole of the subtrahend is taken from the minuend, and consequently the remainder obtained by taking the units of the former from those of the latter, the tens from the tens, &c., is, in effect, the correct expression of that number of units by which the greater quantity exceeds the less.

32. Reviewing the whole subject, it appears that the subtraction of one unit from another, or of one small number from another, is accomplished by the process of decomposition explained in Article 27; and that the subtraction of large numbers (which has been reduced to that of the successive orders of units in the subtrahend from the corresponding orders in the minuend) is, in effect, performed by repetitions of the same elementary process.

33. The subtraction of one small number from another may be, at first, effected by means of sensible objects, as was recommended in addition; or, they who prefer it, may use the following—

Table of Subtraction.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
1	0	1	2	3	4	5	6	7	8	9									
2		0	1	2	3	4	5	6	7	8	9								
3			0	1	2	3	4	5	6	7	8	9							
4				0	1	2	3	4	5	6	7	8	9						
5					0	1	2	3	4	5	6	7	8	9					
6						0	1	2	3	4	5	6	7	8	9				
7							0	1	2	3	4	5	6	7	8	9			
8								0	1	2	3	4	5	6	7	8	9		
9									0	1	2	3	4	5	6	7	8	9	
10										0	1	2	3	4	5	6	7	8	9

In this table, the uppermost horizontal column contains the greater numbers, and the left vertical column the less; also the excess of any number in the former over any number in the latter, is placed in the square formed by the intersection of a vertical column, descending from the larger numbers, with a horizontal column extending to the right of the smaller numbers.

Thus, to find the difference of the numbers 12 and 5: tracing the vertical column 12 downwards to its intersection with the horizontal column 5, the number 7 is found to fill the square formed by their intersection. Whence, according to the table, 7 is the difference of the numbers 12 and 5.

34. Applying the principles which have been established to the solution of questions, let it be required to find the difference of the numbers 9 and 4.

The subtraction of 4 from 9 may be made by the method of decomposition explained in Article 27 thus:—

$$9-1=8$$
$$9-2=8-1=7$$
$$9-3=7-1=6$$
$$9-4=6-1=5$$
$$\therefore 9-4=5, \text{ the difference sought.}$$

Otherwise, the subtraction may be made by successive abstractions of unity from both minuend and subtrahend until the latter be exhausted, thus:—

$$9-4=8-3=7-2=6-1=5-0=5$$

Referring to the table of subtraction, it is found that $9-4=5$.

As an example involving larger numbers, let it be required to find the difference of the numbers 57 and 43.

It is demonstrated (Art. 31) that the difference of large numbers is found by taking the units of the subtrahend from the units of the minuend, the tens from the tens, &c.; whence

$$7-3=6-2=5-1=4-0=4=\text{difference of the units,}$$
$$5-4=4-3=3-2=2-1=1-0=1=\text{difference of the tens.}$$

The difference of the numbers 57 and 43 is thus composed of 1 ten and 4 units, consequently it is 14.

The operation may be represented as follows:

		tens	units
Minuend	-	5	7
Subtrahend	-	4	3
Remainder	-	1	4

35. Other examples:

		h. t. u.
Given the minuend	-	2 3 7
And the subtrahend	-	9 4

Required the remainder?

Subtracting, as in the last example, the units of the subtrahend from the units of the minuend, and the tens from the tens, the results are

7—4=3=difference of the units,
3—9, the subtraction cannot be effected.

The method of subtracting a number by parts creates, in this instance, a difficulty; it requires 9 to be taken from 3, a greater number from a less; and yet, seeing that 237 exceeds 94, the subtraction of the whole number 94 from 237 can certainly be accomplished.

The difficulty is removed thus: a convention made to facilitate the expression of numbers by means of cyphers is, that any figure placed to the left of another shall be held to express units of the order immediately superior to those of that other figure (Art. 5).

Whence, in the example under consideration, one in the place of hundreds is equal to ten in the place of tens. Let one be taken, or (as the usual phrase is) borrowed, from the hundreds,

this 1 hundred=10 tens,
and 10 tens+3 tens=13 tens,
from these 13 tens take 9 tens,
then 13—9=4=difference of the tens.

The subtrahend contains no units of the 3rd order; but, on account of the 10 tens, or 1 hundred, borrowed from the hundreds of the minuend to render possible the subtraction of 9 from 3, the 2 must be diminished by 1;

∴ 2—1=1=difference of the hundreds.

Assembling the remainders of the partial subtractions, 3 units, 4 tens, and 1 hundred, or 143, compose the general result.

Whence, from 237 taking 94, the remainder is 143.

In figures the operation will stand thus,

		hund.	tens	units
Minuend	-	2	3	7
Subtrahend	-		9	4
Remainder	-	1	4	3

Given the minuend	-	-	5367
And the subtrahend	-	-	3278

Required the remainder?

In order that the subtraction of the 8 units of the subtrahend from the 7 units of the minuend may be rendered possible, it becomes necessary to borrow 1 ten=10 units from the 6 tens of the minuend. The 6 tens and 7 units are, in consequence, to be taken as 5 tens and 17 units.

Whence, proceeding with the subtraction of the units,

17—8=9=difference of the units.

To subtract 7 from 5 a like arrangement is requisite, by which means the 5 in the second place becomes 15, and the 3 in the third place is reduced to 2.

Then 15—7=8=difference of the tens,
2—2=0=difference of the hundreds,
5—3=2=difference of the thousands.

Consequently 9 units+8 tens+0 hundreds+2 thousands, that is 2089, is the remainder left by the subtraction of 3278 from 5367.

36. To accomplish, in the last example, the subtraction of 8 units from 7, 1 ten or 10 units are borrowed from 6, the next figure to the left; and the 6 is, in consequence, reduced to 5. But, since the absolute difference of two numbers is not changed by the addition of unity to each of them, the result will be the same, whether the 1 borrowed be taken from the succeeding figure in the minuend, and the corresponding figure in the subtrahend be suffered to remain as it is, or the minuend be left as it is and 1 be added to the figure in the subtrahend, for this amounts simply to the increasing of each by unity.

The second method, namely, that of carrying one, for ten borrowed, to the next higher figure of the subtrahend, instead of deducting one from the corresponding figure of the minuend, is usually adopted in practice, on account of its convenience.

Example: Given minuend - 3954
subtrahend - 1986
Required the remainder?

4 being less than 6, 10 are borrowed;
10+4=14 and 14—6=8=difference of the units;
carrying to 8, the next figure of the subtrahend, 1 for the ten borrowed, 8+1=9; the next figure to be subtracted, 5, being less than 9, 10 are borrowed,
10+5=15 and 15—9=6=difference of the tens;
carrying 1 for 10 borrowed, 9+1=10,
and borrowing again, to render the subtraction possible,
10+9=19 and 19—10=9=difference of the hundreds;
carrying 1 for 10 borrowed, 1+1=2,
2 being less than 3, the subtraction is made at once;
3—2=1=difference of the thousands.

Whence, if from - 8954
There be taken - 1986
The remainder is - 1968

By this example it appears that (carrying for loans to the subtrahend) it may be necessary to subtract 10 from 19; a contingency which has been provided for in the table of subtraction.

		h.m.	t.m.	m.	h.th.	t.th.	th.	h.	t.	u.
Example:	From	3	7	0	4	0	0	1	0	0
	Take	2	9	3	5	2	6	4	8	2
	Remainder -		7	6	8	7	3	6	1	8

Abridged detail of the operation:
10—2=8=units of the difference.
1+8=9 and 10—9=1=tens of the difference.
1+4=5 and 11—5=6=hundreds of the difference.
1+6=7 and 10—7=3=thousands of the difference.
1+2=3 and 10—3=7=ten thousands of the difference.
1+5=6 and 14—6=8=hundred thousands of the difference.
1+3=4 and 10—4=6=millions of the difference.
1+9=10 and 17—10=7=ten millions of the difference.
1+2=3 and 3—3=0=hundred millions of the difference.

37. In the preceding example, the places of several orders of units in the minuend are filled with zeros, the corresponding orders of the subtrahend being significant figures. If, instead of carrying one to the next figure of the subtrahend for every ten borrowed, the other method of taking one from the next figure of the minuend were followed, the details of the work would require some explanation.

For instance, 2 units cannot be subtracted from 0:

1 ten=10 units, ought therefore to be borrowed from the tens of the minuend. But the place of tens (or units of the 2d order), is also occupied

by 0; and from 0, 1 cannot be borrowed. In such a case it is necessary to proceed to the hundreds (or units of the 3d order) of the minuend, borowing 1 from the units of the 3d order:

1 hundred=10 tens=9 tens+10 units; whence
10—2=8=units of the difference,
and 9—8=1=tens of the difference.

The figure expressing units of the third order is 1; and 1 having been borrowed from it, this 1 is reduced to zero.

4, the corresponding figure of the subtrahend, cannot be taken from 0; therefore a necessity of borrowing recurs.

The places of the units of the 4th and 5th order being both filled with zeros, the loan must be taken from the units of the 6th, that is, the hundred thousands:

1 hundred thousand=10 ten thousands.
= 9 ten thousands+10 thousand.
= 9 ten thousands+9 thousand+10 hundred.

Whence, 10—4=6=hundreds of the difference.
9—6=3=thousands of the difference.
9—2=7=ten thousands of the difference.

The figure in the 6th place, from which 1 was taken, is 4: on account of the 1 borrowed it is reduced to 3. 3 being less than 5, the corresponding figure of the subtrahend, 1 must be borrowed, not from the 0, which fills the place of the units of the 7th order, but from 7 of the 8th order.

1 unit of 8th order=10 millions.
= 9 millions+10 hundred thousand.

Also 10 hundred thousand+3 hundred thousand=13 hundred thousand.

Whence, 13—5=8=hundred thousands of the difference,
and 9—3=6=millions of the difference.

The 7 in the 8th place is reduced to 6 by the loan taken from it, and 6 being less 9, 1 is borrowed from the 3 in the 9th place.

1 unit of 9th order=10 units of 8th order.

10+6=16 and 16—9=7=ten millions of the difference.

The 3 in the 9th place is reduced to 2 by the 1 borrowed, and 2—2=0. Whence, the remainder contains no units of the 9th order.

Collecting the partial differences, the result, as before, is 76873618.

38. If every figure of the subtrahend is less than the corresponding figure of the minuend, the subtraction may be performed from right to left, or from left to right, at pleasure. Thus, from 7854 to take 3721, the remainders, beginning with the highest figure, are,

7—3=4=thousands of the difference.
8—7=1=hundreds of the difference.
5—2=3=tens of the difference.
4—1=3=units of the difference.

And the whole remainder is composed of 4 thousands+1 hundred+3 tens +3 units, or 4133.

Subtracting from right to left, the same results are obtained, but in a contrary order.

When, however, a figure of any order of units in the subtrahend is greater than the corresponding figure in the minuend, and a necessity of borrowing arises, the course of the operation from left to right is encumbered with troublesome reductions.

As affording an instance of such, let it be required from 5748 to subtract 3876.

5—3=2=thousands of the difference.

7—8; without a power of borrowing, the subtraction is impossible. The loan, besides, must be taken from the next higher figure, that on the left; and the operation, so far as regards this figure, is already performed. Ten, however, must be borrowed, and the thousands of the difference made less by 1. Proceeding,

7+10=17 and 17−8=9=hundreds of the difference.

4−7 ; again the subtraction is not possible without a loan. One must be borrowed from the hundreds of the difference, and the proper reduction made; 4+10=14 and 14−7=7=tens of the difference, 8−6=2=units of the difference.

The result of the preceding operations may be put down as follows:

	m.	h.	t.	u.
Minuend -	5	7	4	8
Subtrahend -	3	8	7	6
Remainder, without reduction -	2	9	7	2
Loans to be taken from remainder -	1	1		
	1	8	7	2

If zero occupy any place in the unreduced remainder, and it be requisite, on account of a loan, to take 1 from this 0, a further reduction will be necessary.

The conventions, that ten units of any order shall be considered as one unit of the next higher order, and that this shall be written immediately to the left of those units, from the decomposition of which into tens and units it has arisen, fix the direction in which the progression of numbers by tens is carried, namely, from right to left, and confer a facility on the performance in the same direction of those arithmetical operations in which reductions between units of different orders may be necessary.

If in addition it were never requisite to carry, or in subtraction to borrow, the direction in which the work is performed would be of no importance. But when the units in the sum of a column exceed 10, the next figure to the left must be augmented; and when, in subtraction, a figure of the subtrahend is greater than the corresponding figure in the minuend, one must be borrowed from the next figure to the left. Proceeding from right to left, these operations fall into the work; but from left to right, they give rise to new additions and subtractions.*

39. To find the difference of two unequal numbers:

General Rule. Write the minuend, and underneath it the subtrahend, in such a manner that the figure expressing the units of any order in the latter shall be under the figure expressing units of the same order in the former.

Then, if all the figures of the subtrahend are less than the corresponding figures of the minuend, take the simple units of the subtrahend from the simple units of the minuend, and record the excess of the latter below the subtrahend, in the place of simple units. Proceed after the same manner to take the tens of the subtrahend from the tens of the minuend, the hundreds from the hundreds, &c.,

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* The tens arising in addition from the sum of any column might be disposed of by an arrangement expressing the deficit of the column from, or its excess over, certain repetitions of ten; and in subtraction, whenever a figure of the minuend is less than that figure of the subtrahend which is to be taken from it, a means could be devised to express the deficient units.

Thus in addition the operation could be performed from left to right, without a necessity of disturbing any part of the result by subsequent combination of the tens arising from collections of inferior units; and in subtraction, without borrowing from the units of that order immediately to the left of any deficient figure.

But such contrivances, without being of any real utility, infringe the principles of decimal notation, and the results expressed by their help would require some reduction before they could be made to enter into combination with numbers expressed in the usual manner.

and to write the partial remainders in their proper places; the result thus obtained is the difference sought.

But if any figure of the minuend is less than the corresponding figure of the subtrahend, add ten to the units expressed by that figure of the minuend, subtract the figure of the subtrahend from this sum; write the remainder in its proper place, carry 1 to the next figure of the subtrahend as an equivalent for the 10 combined with the next lower units of the minuend; take the augmented figure of the subtrahend from the corresponding figure of the minuend, borrowing and carrying, if requisite; and thus proceed till all the partial subtractions are effected.

The result is the difference of the given numbers.

40. Examples in the subtraction of whole numbers.

Required the differences of the following numbers:

1st. 1842 and 1769?..*Ans.* 73.
2d. 15760 and 9874?...*Ans.* 5886.
3d. 25040 and 18360?...*Ans.* 6680.
4th. 35417840 and 8709864?..........................*Ans.* 26707976.
5th. 104352576 and 96847986?.......................*Ans.* 7504590.
6th. 57984 and 38976?......................................*Ans.* 19008.
7th. 130462 and 92785?....................................*Ans.* 37677.
8th. 5807854 and 3918765?...........................*Ans.* 1889089.
9th. 743850261 and 683979548?....................*Ans.* 59870713.
10th. 100154798654 and 968839527?..........*Ans.* 99185959127.
11th. 5081347100 and 4998538605?...............*Ans.* 82808495.
12th. 9876543210 and 123456789?..............*Ans.* 9753086421.
13th. 987654321 and 123456789?.................*Ans.* 864197532.
14th. 40081306500 and 19987517864?.........*Ans.* 20093788636.
15th. 2507815384 and 1827938879?..............*Ans.* 679876505.
16th. 3020417000 and 1543868004?............*Ans.* 1476548996.
17th. 5320749813 and 257868754?..............*Ans.* 5062881059.
18th. 1958700401 and 986943684?................*Ans.* 971756717.
19th. 10370986451 and 971883569?............*Ans.* 9399102882.
20th. 30507406854 and 29735849673?..........*Ans.* 771557181.

41. In the preceding examples it is required to subtract one number only from any given minuend. It may be required, however, to determine the remainder which is left after several numbers have been taken from a given minuend, and the successive remainders. The most obvious manner of proceeding in such a case is, to take one subtrahend from the minuend, a second from the 1st remainder, a third from the 2d remainder, &c., till all the numbers are subtracted.

The order of making the subtractions cannot, it is evident, affect the last remainder so long as the absolute number of individuals taken away is the same. Whence the subtractions may be made in any order.

Since the last remainder is the difference between the minuend and the sum of all the subtrahends, if the latter are added together, and their sum at once taken from the minuend, the result must be equal to that obtained by the method of repeated subtractions; for thus also the absolute number of individuals subtracted is the same.

42. Any given number can be subtracted from 1, followed by as many zeros as the given number contains figures. The remainder is termed the Arithmetical Complement of the given number.

Thus the arithmetical complement of 7 is $10-7=3$;
of 84 is $100-84=16$;
of 576 is $1000-576=424$;

Whence, generally, the sum of a number and its arithmetical complement is equal to 10, considered as having the relative value of the highest figure of the number.

When the difference of two numbers is to be found, if the less is taken from the greater, the remainder is the difference required. But if, instead of taking the less number from the greater, the arithmetical complement of the less is added to the greater, the result must contain ten units, of the same order as the highest figure of the subtrahend, more than the remainder obtained by the common process of subtraction. Taking away these ten units, the results are equal. Whence the arithmetical complement of a number supplies the means of changing subtraction into addition.

Examples of subtraction made by adding the arithmetical complement of the subtrahend to the minuend:

Example 1. From 795484 take 389685.

Solution by subtraction.		Solution by addition of arith. compt.	
From	795484	To the minuend - -	795484
Take	389685	Add arith. compt. of 389685	610315*
Rem.	405799	The sum is - - -	1405799
		From this sum subtract	- 10
		The rem. is, as before - -	405799

Example 2. From 854097 take 97548.

Solution by subtraction.		Solution by addition of arith. compt.	
From	854097	To the minuend - -	854097
Take	97548	Add arith. compt. of 97548	2452
Rem.	756549	The sum is equal to - -	856549
		From this sum subtract	- 10
		The rem. is - - -	756549

When in the same calculation it is requisite to add several numbers and to subtract several others, the arithmetical complements of the latter may sometimes be employed with advantage.

But it is chiefly in calculating with logarithms that this employment of the arithmetical complement is of much utility.

43. The subtraction of one number from another is indicated by writing the minuend to the left and the subtrahend to the right of the symbol—. Thus, the expression $25-18$ indicates that 18 is to be subtracted from 25.

The actual subtraction may be expressed thus,

$$25-18=7.$$

Replacing particular by general expressions of number, the subtraction of b from a is indicated by the expression $a-b$.

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* If the subtraction of any given subtrahend from 10, followed by the proper number of zeros, is made from right to left, it is evident that ones, for tens borrowed, must either be carried to the figures of the subtrahend or taken from the corresponding figures of the minuend; since the figures of the minuend are zeros, excepting the last, if the second method is adopted (see Article 35) the last significant figure of the subtrahend must be taken from 10 and the others from 9. Whence the subtraction, which gives the arithmetical complement of a number, may be performed from left to right, by taking the figures of the subtrahend each from 9, excepting the last significant figure, and this from 10.

The arithmetical complement of the subtrahend in Example 1. may be found thus: $9-3=6$: $9-8=1$: $9-9=0$: $9-6=3$: $9-8=1$: $10-5=5$; and 610315, the number composed of these remainders, is the arithmetical complement required. With a little practice the arithmetical complement of any number may be found in this manner mentally, and with very little trouble.

If two quantities, b, c, are both to be taken from a, the expression of the remainder is

$$a-b-c$$

$$\text{or, } a-(b+c) \text{ Art. 41.}$$

If the excess of a over the sum of b and c is equal to d, the subtraction is expressed thus :—

$$a-(b+c)=d.$$

When the quantities whose difference is required are similar, that difference is expressed in terms of the common quantity.

Thus $5a-3a=2a$: And, $ma-na=(m-n)a$.

For $5a$ and $3a$ express respectively 5 repetitions and 3 repetitions of the common quantity a.

From $5a=a+a+a+a+a$.
Take $3a=a+a+a$.
The remainder$=a+a=2a$.

Therefore $5a-3a=2a=(5-3)a$.

Again, ma is equal to a repeated as often as m contains simple units ; and na is equal to a repeated as often as n contains simple units.

Therefore, if n repetitions of a are taken from m repetitions of a, the remainder must contain a as often as the excess of m over n contains simple units. This excess is represented by the expression $m-n$. Whence $ma-na=(m-n)a$.

Exercises in the subtraction of general expressions of quantity :

1st.	from	$4a$	subtract	a	rem.	$=3a$.
2nd.	-	ma	-	$12a$	-	$(m-12)a$.
3d.	-	$9a+6b$	-	$5a+4b$	-	$4a+2b$.
4th.	-	$ma+nb$	-	$n(a+b)$	-	$(m-n)a$
5th.	-	$15x$	-	ax	-	$(15-a)x$.
6th.	-	$a+b$	-	$x+y$	-	$a+b-x-y$.
7th.	-	nx	-	$x+y+z$	-	$(n-1)x-y-z$.
8th.	-	$ab+12$	-	$ac+d+5$	-	$a(b-c)-d+7$.

Verification of Addition and Subtraction.

44. The composition and decomposition of numbers, as explained under the titles of Addition and Subtraction, depending partly on the conventions of numeration, and partly on axioms or first principles of indisputable certainty, the accuracy of the methods can require no further proof.

But as those combinations of number which constitute the operations of Addition and Subtraction are made, chiefly or altogether, through means of the memory, and are on this account liable to the suspicion of error, it becomes necessary to find some *test*, by the application of which the accuracy of any given result may be tried. Such process may, with more propriety, be termed verification than (what it is commonly called) proof.

The process of verification ought, for an obvious reason, to be of, at the utmost, not greater difficulty than that of the operation whose accuracy is tested ; for if the verification be the more difficult, so also will the chance of error in performing it be greater, and the coincidence of result less probable.

45. In addition the accuracy of the result may be tried in several ways.

1st. If the operation has been performed by commencing with the last figure of each column and adding upwards, a repetition of the process, only beginning with the first figure of each column and adding downwards, will constitute a verification.

For, the combinations being made in an order one the reverse of the other, it is most unlikely that the same error can affect both results. Whence, if they agree, a strong probability arises that both are accurate.

2d. If the addenda are, one after another, subtracted from the result, and the subtraction of the last leaves no remainder, the addition may be presummed correct. The reason is plain: the result arises from the combination of all the addenda into one number; and if the whole of these be again taken away there can be no remainder.

3d. Another verification is obtained by combining the preceding methods. The manner of effecting this may be, perhaps, most clearly explained through the medium of an example.

Let, therefore, the numbers to be added together be,

$$\begin{array}{lcccccc}
 & & 3 & 7 & 8 & 6 & 5 \\
 & & 4 & 9 & 7 & 3 & 6 \\
 & & 8 & 4 & 2 & 9 & 7 \\
 & & 5 & 8 & 6 & 4 & 1 \\ \cline{2-7}
\text{The sum is} \quad - & 2 & 3 & 0 & 5 & 3 & 9 \\ \cline{2-7}
 & 2 & 0 & & & & \\ \cline{2-3}
 & & 3 & 0 & & & \\
 & & 2 & 8 & & & \\ \cline{3-4}
 & & & 2 & 5 & & \\
 & & & 2 & 3 & & \\ \cline{4-5}
 & & & & 2 & 3 & \\
 & & & & 2 & 2 & \\ \cline{5-6}
 & & & & & 1 & 9 \\
 & & & & & 1 & 9 \\ \cline{6-7}
\end{array}$$

To verify the result, begin with the left column, and proceed thus:

$3+4=7$; $7+8=15$; $15+5=20$.

From the two highest figures in the sum, *i. e.* 23, subtract 20, the sum of the left column, and to the remainder 3 annex 0, the next figure of the sum.

The sum of the second column $=7+9+4+8=28$. Take this from 30; and to the remainder, 2, bring down 5, the next figure of the sum.

The sum of the third column is 23, and $25-23=2$. To this remainder annex 3, and from the 23 subtract the sum of the fourth column, which is 22.

To the remainder $=1$, annex 9, the last figure of the sum, and from 19, subtract the sum of the last column; this sum is also 19. $19-19=0$.

The last subtraction leaves no remainder, from which the inference is that the addition is correct.

The subtraction of the sum of the numbers in each column from the corresponding figures in the entire result may be made mentally; in which case the work may stand thus,

$$\begin{array}{cccccc}
 & 3 & 7 & 8 & 6 & 5 \\
 & 4 & 9 & 7 & 3 & 6 \\
 & 8 & 4 & 2 & 9 & 7 \\
 & 5 & 8 & 6 & 4 & 1 \\ \hline
2 & 3 & 0 & 5 & 3 & 9 \\ \hline
 & \cancel{3} & \cancel{2} & \cancel{2} & \cancel{1} & 0 \\
\end{array}$$

3, 2, 2, 1, 0, are the remainders from the partial subtractions. As soon as each figure is cancelled, in succession, by these subtractions, a stroke is drawn through it to record the fact.

These remainders, it will be observed, are the tens which in the course of the addition were carried from the sums of the lower columns respectively, and combined as units with the next higher.

46. The most simple and obvious verification of subtraction is made by reversing the operation.

The remainder being obtained by taking the subtrahend from the minuend, it follows that, adding together the subtrahend and remainder, their sum ought to reproduce the minuend. If not, the subtraction ought to be revised as probably erroneous.

Example:

From	517846
Take	348978
Remainder -	178868

$$\left.\begin{array}{r}\text{Adding together the remainder}\\ \text{and subtrahend, their sum}\end{array}\right\}=517846=\text{minuend.}$$

Whence, since the sum of the remainder and subtrahend is equal to the minuend, the subtraction is to be presumed correct.

Another mode of verifying subtraction is to take the remainder from the minuend. The number left ought in this case to be the subtrahend.

Example:

From	736942
Take	547986
Remainder -	188956

$$\left.\begin{array}{r}\text{Subtracting the remainder from the}\\ \text{minuend, their difference is}\end{array}\right\}=547986=\text{the subtrahend.}$$

Remark.—These verifications, as well as the results which they are employed to corroborate, are in a great measure performed mentally, and their accuracy must depend upon the strength of memory and closeness of attention which the computer brings to the performance of his task.

It is unlikely that, the work and the verification starting from different points and following different routes, they should both agree, and yet be both in error. Such a coincidence is not, however, impossible. Hence the utmost which one is entitled to infer from the agreement of a verification with a given result is, that a strong probability, but still not an absolute certainty, of its accuracy exists.

Multiplication and Division.

47. Addition and subtraction are, in effect, the only elementary operations of arithmetic.

But when, in a case of addition, the addenda, instead of being different numbers, are all equal, the work admits of considerable abbreviation. The methods by which this is accomplished are constituted into an operation, also generally regarded as elementary, and which has been named Multiplication.

The terms Addenda and Sum, respectively employed to distinguish the numbers which are to be added together, and the result of such addition are, like the name of the process itself, replaced by others. The addendum, or single number which is to be repeated a given number of times, is termed Multiplicand; the number which expresses the times of repetition, Multiplier; and the sum or result of these repetitions, Product.

The multiplicand and multiplier are also termed Factors (that is, makers) of the product.

The symbol, × (into, or multiplied by), placed between two numbers, indicates that the one is to be multiplied by the other.

48. A similar arrangement is made with respect to a particular case of subtraction.

If, instead of taking the subtrahend only once from the minuend, it were required to take the subtrahend again from the first remainder, then from the 2d, 3d, ... until a remainder either less than the subtrahend or equal to zero were attained, for the purpose of ascertaining the exact number of subtractions which must be made in order to exhaust the minuend the repeated subtractions would be tedious. To avoid this an abridged process has been devised, and this is called Division.

The number, from which the repeated subtractions are made, is termed Dividend; that by the repeated subtraction of which the dividend is exhausted, Division; and the number expressing the times of subtraction possible, Quotient.

The symbol ÷ (divided by) is employed to denote that the number placed before is to be divided by that which comes after it. Thus, $8 \div 2$ signifies that 8 is to be divided by 2. $2\underline{)8,}$ 2)8(and $\frac{8}{2}$ are expressions having the same import.

SECTION IV.

MULTIPLICATION.

49. In multiplication the quantities which are given are,

1st. The multiplicand, or number which is to be repeatedly added to itself.

2d. The multiplier, or number expressing the times of repetition of the multiplicand.

And the thing required is, to find the sum of these repetitions of the multiplicand, or the product of the two given numbers.

50. If the multiplicand is repeated but a few times, the product is readily found by addition. But when the times of repetition are many, the addition becomes exceedingly tedious; and hence arises the necessity of investigating a more compendious method. This is inferred from a consideration of what is actually done when a number is repeatedly added to itself.

Taking a few examples of such addition; let it be required to find the sum of three times the number 8.

$$8+8=16 \text{ and } 16+8=24, \text{ that is } 8+8+8 \text{ or } 8\times 3=24.$$

In this instance there is no room for simplification.

Let it be required, next, to find the sum of 4 times the number 358, or the product of 358 by 4.

Writing 358 the required number of times, and performing the addition, as under,

$$\begin{array}{r}358\\358\\358\\\underline{358}\\\underline{1432}\end{array}$$

the sum is found to be 1432, that is, $358\times 4=1432$.

Regarding the 358 of this example as composed of 8 units+5 tens+3 hundreds, it is evident, on inspection of the columns to be added, that the sum 1432 is made up of 4 times 8 units+4 times 5 tens+4 times 3 hundreds, that is, of four repetitions of the units of each order contained in the multiplicand.

Now, the process would, manifestly, be the same whether the multiplicand contained more or fewer digits than three, and the multiplier were 4, or any other figure of the series 1, 2, 3 . . . . . 9.

Whence, generally, the repetition 1, 2, 3, ... 9 times of any number is reduced to the repetition 1, 2, 3, ... 9 times of the successive digits of that number.

Otherwise, the multiplication of any number by a single figure is reduced to the multiplication of the successive digits of that number by the said figure.

51. The multipliers 1, 2, 3 . . . . . . 9 having been considered, the next in order is 10.

Now, to find the product of any number by 10, the multiplicand may be written ten times, and the result found by addition. But such labour is not necessary, since, by the principles of numeration (Art. 5), the same result is obtained by moving every figure of the multiplicand one place to the left.

This is done by writing a zero to the right of the number.

Similarly, the multiplication by 100 is performed by annexing two zeros to the multiplicand; by 1000, three zeros, &c.

If instead of 1 ten it is required to multiply by

2 tens, 3 tens, ... 9 tens, or by
20, 30 90,

the operation manifestly is the same as to multiply 10 times the multiplicand by 2, 3, ... 9*, and this is accomplished by first annexing 0 to the multiplicand, and then proceeding to multiply by the figure expressing the number of tens, precisely as was done in the last example by the figure expressing the number of units.

To multiply by 2 hundreds, 3 hundreds, ... 9 hundreds, two zeros are annexed to the multiplicand, and this result multiplied by 2, 3, ... 9.

The multiplication by thousands, ten thousands, &c., is performed in a similar manner.

52. As examples of the application of these principles, let it be required,

1st. To multiply 427 by 35.

This multiplier is composed of 3 tens and 5 units, or of 30+5. Whence if to 30 times the multiplicand 5 times the multiplicand be added, the sum will consist of 35 repetitions of the multiplicand, and, consequently, be the result required.

The calculation may be made as follows:

Multiplication by the tens.	Multiplication by the units.
427 × 10=4270	427
	427
4270	427
4270	427
4270	427
12810	2135

that is, 427 × 10=4270 and 4270 × 3=12810,

$$\begin{aligned} \text{or, } 427\times30&=12810\\ \text{also, } 427\times\ 5&=\ \ 2135\\ \text{whence, by addition, } 427\times35&=14945 \end{aligned}$$

14945 is, therefore, the product required.

2d Example. Let it be required to multiply 354 by 245; the multiplier may be resolved into 2 hundreds+4 tens+5 units, and the calculation performed thus,

Multiplication by the hundreds of the multiplier,

$$354\times100=35400, \text{ and } 35400+35400=70800,$$
$$\therefore 354\times200=70800:$$

Multiplication by the tens,

$$354\times10=3540, \text{ and } 3540+3540+3540+3540=14160,$$
$$\therefore 354\times40=14160:$$

---

* To multiply a number by 20, 30, ... 90, and to find the sum of 20, 30, ... 90 repetitions of that number, amounts to the same thing. Let the multiplicand be written 20, 30, ... 90 times, in the form of an example in addition, and let the sum be found, first, in the usual manner, next separate the vertical column into parts, each containing ten repetitions of the multiplicand, find the sum of these parts separately, (or of one of them, for they are all equal,) and add the sums together; the first and second methods must afford the same result, since each comprehends all the repetitions of the number.

But, by the first method, the given number is multiplied by 20, 30, ... 90, and by the second it is multiplied by 10, and afterwards by 2, 3, ... 9. Whence, in multiplying a number by 20, 30, ... 90, the product is the same, whether the number be at once multiplied by 20, 30, ... 90, or first by 10, and then by 2, 3, ... 9.

Multiplication by the units;

$$354+354+354+354+354=1770,$$
$$\therefore\ 354\times 5=1770:$$

Collecting the partial products;

$$\begin{array}{r} 354\times200=70800 \\ 354\times\ \ 40=14160 \\ \underline{354\times\ \ \ 5=\ \ 1770} \end{array}$$

and adding $354\times245=86730$=product required.

53. From the preceding remarks and examples it follows,

That the product of a number less than 10 by another number less than 10 is found by repeating the figure representing the multiplicand as many times as the multiplier contains unity, and taking the sum; or, in other words, by adding to itself the multiplicand as many times, less one, as there are units in the multiplier;

That the product of a number not less than 10 by another number less than 10 is found by applying, successively, the same method of repeated additions to each of the orders of units comprehended in the multiplicand; to the units, to the tens, to the hundreds, &c.;

And that the product of a number not less than 10 by another number not less than 10, is found by resolving the multiplier into the several orders of units of which it is composed, finding the product arising from the multiplication of the multiplicand by each of these units, and then collecting all the partial products into one result.

Also, the product of the multiplicand by the units of the first order, or simple units, is found as in Article 50.

The product of the multiplicand by the units of the second order, or tens, is found by annexing zero to the multiplicand, (that is, multiplying it by 10,) and then multiplying this result by the figure expressing the number of units of the second order, which, again, is effected as in Article 50.

The product of the multiplicand by the units of the third order, or hundreds, is found by annexing two zeros to the multiplicand, and multiplying this result by the units of the third order, also as in Article 50; and similarly for the higher orders of units.

Wherefore, universally, the multiplication of any number by any other number is reduced to a series of partial multiplications, in performing which it is never necessary to find the mutual product of two numbers exceeding 9.

54. Knowing the results of all the combinations of the numbers 1, 2, 3, ... 9, with the numbers 1, 2, 3, ... 9, the learner will be in a condition to find the product of any numbers whatever.

These results are obtained by adding each of the simple units to itself 1, 2, 3, ... 9 times, thus,

$$\begin{array}{lll} 1+1=2. & 1+1+1=3. & 1+1+1+1=4,\ \&c. \\ 2+2=4. & 2+2+2=6. & 2+2+2+2=8,\ \&c. \\ 3+3=6. & 3+3+3=9. & 3+3+3+3=12,\ \&c. \\ \dots & \dots & \dots \\ \dots & \dots & \dots \\ 9+9=18. & 9+9+9=27. & 9+9+9+9=36,\ \&c. \end{array}$$

55. The sums of these nine repetitions of each of the 9 digits being computed in this manner, the whole may be arranged in the following—

Table of Multiplication.

1	2	3	4	5	6	7	8	9
2	4	6	8	10	12	14	16	18
3	6	9	12	15	18	21	24	27
4	8	12	16	20	24	28	32	36
5	10	15	20	25	30	35	40	45
6	12	18	24	30	36	42	48	54
7	14	21	28	35	42	49	56	63
8	16	24	32	40	48	56	64	72
9	18	27	36	45	54	63	72	81

The principle of arrangement is the same here as in the tables of addition and subtraction, the product of any two numbers being placed in the square formed by the intersection of the vertical and horizontal columns in which these numbers are situated.

Thus the product of 4 by 8 is found, by the table, to be 32; and that of 9 by 7, 63.

In these instances 4 and 9 are multiplicands, 8 and 7 multipliers; but if the product of 8 by 4 be sought, that also is 32; and the product of 7 by 9 is 63, as before.

56. So far, then, as can be inferred from the table, it appears that if two numbers are multiplied together, then interchanged so as to render the multiplier multiplicand, and the multiplicand multiplier, and again multiplied together, the two products are equal.

Inference, however, from particular cases cannot be admitted as general proof; and since, in finding the mutual product of two numbers, it is convenient to employ as multiplier that of the two which contains the fewer digits (which may as often be the multiplicand as the multiplier), it becomes of importance to demonstrate generally that the mutual product of any two numbers is not altered by an interchange of the factors.

To multiply one number by another is, in effect, to repeat each unit of the multiplicand as many times as the multiplier contains unity, and the product is composed of the whole assemblage of these units.

Writing the units of the multiplicand, one after another, in a horizontal line, and placing as many such lines, one under another, as the multiplier contains units,

Thus, 1, 1, 1, 1, 1. . . . .
1, 1, 1, 1, 1, ...
1, 1, 1, 1, 1, ...
1, 1, 1, 1, 1, ...
. . . .
. . . .

it is manifest that the sum of all these units is the product of the numbers.

Now this sum may be found by beginning with the uppermost horizontal line; embracing the units contained in it under one name; combining the successive units of the second horizontal line with the sum of the units in the first, and so proceeding with a 3d, a 4th, ... an nth line, until all the units are comprehended under one name of number. This name (to reach which the whole intermediate series from unity has been traversed) expresses the sum of all the units contained in the table, *i.e.*, the product of the numbers.

But the sum of all the units in the table may be found in another manner, namely, by commencing with a vertical column; combining into one number all the units contained in it; joining, successively, to this number the units of a 2d, a 3d, ... an *m*th column, until the whole are again included in one number.

Now, in the first instance, the horizontal line is made multiplicand and the vertical column multiplier; while in the second the vertical column is multiplicand, and the horizontal line multiplier; and in each case the result or product contains all the units in the table. Consequently the products are equal.

But the reasoning is general, for its evidence does not depend on any position that the multiplicand is some one and the multiplier some other particular number. Whence the product of any two numbers is independent of the order in which these numbers are multiplied together.

This proposition may, by means of general symbols, be expressed thus,

$$a\times b=b\times a.$$

57. Examples of multiplication:

1st Example. Multiply 72 by 4.

The multiplicand being greater than 10, and the multiplier less, this example falls under Article 50.

Resolving the multiplicand into 2 units and 7 tens, the partial multiplications are

$$\overset{\text{units}}{2}\times 4=\overset{\text{units}}{8}\quad - \quad - \quad - \quad - \quad 8$$

$$\overset{\text{tens}}{7}\times 4=\overset{\text{tens}}{28}=\overset{\text{tens}}{20}+\overset{\text{tens}}{8}=\overset{\text{hund.}}{2}+\overset{\text{tens.}}{8}=\quad - \quad - \quad \underline{280}$$

$$\text{Whence, adding the partial products, } 72\times 4=\quad \underline{\underline{288}}$$

2d Example. Multiply 49 by 8.

The multiplicand being resolved into 9 units and 4 tens, the details of the multiplication are,

$$\overset{\text{units}}{9}\times 8=\overset{\text{units}}{72}=\overset{\text{units}}{70}+\overset{\text{units}}{2}=\overset{\text{tens}}{7}+\overset{\text{units}}{2}=\quad - \quad - \quad - \quad 72$$

$$\overset{\text{tens}}{4}\times\overset{\text{tens}}{8}=\overset{\text{tens}}{32}=\overset{\text{tens}}{30}+\overset{\text{tens}}{2}=\overset{\text{hund.}}{3}+\overset{\text{tens}}{2}=\quad - \quad - \quad - \quad \underline{320}$$

$$\text{Adding the products, } 49\times 8=\quad - \quad - \quad - \quad \underline{\underline{392}}$$

In combining the partial products, the 7 tens arising from the multiplication of 9 units by 8 are added to the 2 tens resulting from the multiplication of 4 tens by 8. But this addition may be made and the combination of the partial products effected simultaneously with the multiplication, thus,

$$\overset{\text{units}}{9}\times 8=\overset{\text{units}}{72}=\overset{\text{tens}}{7}+\overset{\text{units}}{2}.$$

Let the 2 units be written under the units of the multiplicand, and the 7 tens reserved for addition to the product of the tens.

$$\overset{\text{tens}}{4}\times 8=\overset{\text{tens}}{32},\text{ and }\overset{\text{tens}}{32}+\overset{\text{tens}}{7}\text{ carried, }=\overset{\text{tens}}{39}=\overset{\text{tens}}{30}+\overset{\text{tens}}{9}=\overset{\text{hds}}{3}+\overset{\text{tens}}{9}.$$

Therefore the product is composed of 3 hundreds, 9 tens, and 2 units or it is equal to 392.

The calculation may be written as under,

$$\begin{array}{lcr}\text{Multiplicand} & = & 49\\ \text{Multiplier} & = & \underline{8}\\ \text{Product} & = & \underline{\underline{392}}\\ & & \cancel{7}\end{array}$$

3rd Example. Multiply 3854 by 7,

$$\begin{array}{lcr}\text{Multiplicand} & = & 3854\\ \text{Multiplier} & = & \underline{7}\\ \text{Product} & = & \underline{\underline{26978}}\\ & & \cancel{528}\end{array}$$

Detail of the operation,

$$\overset{\text{units}}{4}\times 7=\overset{\text{units}}{28}=\overset{\text{units}}{20}+\overset{\text{units}}{8}=\overset{\text{tens}}{2}+\overset{\text{units.}}{8};$$

the 8 units are written under the units of the multiplicand, and the 2 tens reserved for addition to the product of the tens;

$$\overset{\text{tens}}{5}\times 7=\overset{\text{tens}}{35},\text{ and }\overset{\text{tens}}{35}+\overset{\text{tens}}{2}\text{ carried }=\overset{\text{tens}}{37}=\overset{\text{tens}}{30}+\overset{\text{tens}}{7}=\overset{\text{hds.}}{3}+\overset{\text{tens.}}{7};$$

the 7 tens are written under the tens of the multiplicand, and the 3 hundreds reserved for addition to the product of the hundreds;

$$\overset{\text{hds.}}{8}\times 7=\overset{\text{hds.}}{56},\text{ and }\overset{\text{hds.}}{56}+\overset{\text{hds.}}{3}\text{ carried }=\overset{\text{hds.}}{59}=\overset{\text{hds.}}{50}+\overset{\text{hds.}}{9}=\overset{\text{ths.}}{5}+\overset{\text{hds.}}{9};$$

the 9 hundreds are written under the hundreds of the multiplicand, and the 5 thousands reserved for addition to the product of the thousands;

$$\overset{\text{ths.}}{3}\times 7=\overset{\text{ths.}}{21},\text{ and }\overset{\text{ths.}}{21}+\overset{\text{ths.}}{5}\text{ carried }=\overset{\text{ths.}}{26};$$

there being no higher unit to the product of which by 7 the 2 ten thousands can be carried, the 6 is written under the thousands of the multiplicand, and the 2, expressing ten thousands, in the next place to the left.

The result of this detail is that $3854\times 7=26978$.

58. 4th Example. Multiply 8762 by 47.

The multiplicand and multiplier being both greater than 10, this example falls under Article 51.

Whence, resolving the multiplier into 7 units and 4 tens, it follows that 8762×7 units $+\,8762\times 4$ tens $=$product required.

Proceeding as in Example 3, $8762\times\overset{\text{units}}{7}=61334=$product of multiplicand by units of multiplier, and $8762\times\overset{\text{tens}}{4}=$ten times 8762×4 (Art. 51)$=87620\times 4$ $=350480=$product by the tens of the multiplier.

Taking the sum of the partial products,

$$\begin{array}{rcr} 8762\times 7 & = & 61334 \\ 8762\times 40 & = & 350480 \\ \hline \therefore\ 8762\times 47 & = & 411814 \end{array}$$

The calculation is conveniently represented thus,

$$\begin{array}{lcr} \text{Multiplicand} & = & 8762 \\ \text{Multiplier} & = & 47 \\ \hline & & 61334 \\ & & 350480 \\ \hline \text{Product} & = & 411814 \end{array}$$

5th Example. Multiply 684 by 35742.

Employing 35742 as multiplier, five partial products must be found; whereas if 684 be taken as multiplier, the number of partial products to be found is only three.

Now (Art. 56) $684\times 35742=35742\times 684$; therefore it is permitted to employ 684 instead of 35742 as multiplier; and since 684 is composed of 4 units, 8 tens, and 6 hundreds (Art. 51).'

35742×4 units $+\,35742\times 8$ tens $+\,35742\times 6$ hundreds $=$product required.

$$\begin{array}{llr} 35742\times 4\text{ units} & =142968\ \ -\ \ -\ \ - & 142968 \\ 35742\times 8\text{ tens} & =357420\times 8\text{ units}=2859360\ \ - & 2859360 \\ 35742\times 6\text{ hundreds} & =3574200\times 6\text{ units}=21445200\ \ - & 21445200 \\ \hline \therefore\ 35742\times 684 & =\ \ -\ \ -\ \ -\ \ -\ \ - & 24447528 \end{array}$$

Otherwise,

$$\begin{array}{lcr} \text{Multiplicand} & = & 35742 \\ \text{Multiplier} & = & 684 \\ \hline & & 142968 \\ & & 2859360 \\ & & 21445200 \\ \hline \text{Product} & = & 24447528 \end{array}$$

6th Example. Multiply 13607 by 5578.

$$\begin{array}{r@{\qquad\qquad\qquad}r}
13607 & 13607 \\
5578 & 5578 \\
\hline
108856 & 108856 \\
952490 & 95249 \\
6803500 & 68035 \\
68035000 & 68035 \\
\hline
75899846 & 75899846
\end{array}$$

It seems unnecessary to repeat the details of the multiplication.

59. The calculation of the last example is exhibited in two forms, slightly differing the one from the other. In the first the zeros arising from the multiplication of the multiplicand by the tens, the hundreds, &c. of the multiplier, are retained, and in the second omitted.

This omission may always be made, provided care is taken to place the units of the lowest order in each partial product directly under that figure by which this partial product arises. The reason of so arranging the partial products is obvious, for the product of simple units by simple units is simple units, of simple units by tens it is tens, &c.

60. When a multiplier contains the same digit oftener than once, the significant figures of the partial product must, for every repetition, be the same, but their relative values different. It is only necessary, therefore, for every such repetition, to write the partial product, already found, taking care to annex in each case the proper number of zeros, or to place the lowest figure under the partial multiplier.

7th Example. Multiply 13256 by 6080.

$$\begin{array}{r}
13256 \\
6080 \\
\hline
1060480 \\
795360 \\
\hline
80596480
\end{array}$$

This example affords the occasion to observe, that when the last figure of a multiplier is zero, or when zeros are interposed between significant figures, the preceding remarks with regard to placing the last figures of the partial products under those figures of the multiplicand to which they respectively belong, apply as well to a multiplier such as this, as to one wholly composed of significant figures.

61. Any necessity, except as an exercise, of finding the product of two very large numbers, can seldom occur. If, however, it were required to find the product of a number expressed by, *ex. gr.*, 20 digits, by another containing as many, the labour might be considerably abridged by the following means.

How many soever the number of digits in the multiplier, it is certain that there can be only 1, 2, 3, ... 9 distinct figures; and that the excess, at the very least, of the whole number of digits over nine is composed of repetitions of some of the figures.

Whence, if once, twice, three times, ... nine times the multiplicand be computed, all the partial products will be known; and it will further be only necessary to write in its proper place the product of the multiplicand by each digit of the multiplier, and to add these partial products together, in order to obtain the result sought.

The product of the multiplicand by 1, 2, 3, ... 9, is readily found by repeated additions.

$$\begin{array}{r@{}l}
\text{Let the multiplicand}= & 7385498067132548 7963 \\
 & \underline{7385498067132548 7963} \\
\text{Multiplicand}\times 2= & 14770996134265097 5926 \\
 & \underline{7385498067132548 7963} \\
\text{Multiplicand}\times 3= & 22156494201397646 3889 \\
 & \underline{7385498067132548 7963} \\
\text{Multiplicand}\times 4= & 29541992268530195 1852 \\
 & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\
 & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot
\end{array}$$

The table being completed to multiplicand × 9, the partial products are copied from it in their proper order, and added together: their sum, thus obtained, is the product of the given numbers.

62. The multiplication of a large number by a single figure is performed from right to left, to facilitate the combination of the tens produced by the multiplication of units of any order with the units of the product of the order which is next higher; and in the multiplication of one large number by another the partial products are, for the same reason, found by following the same course. The digits of the multiplier may, however, be taken from right to left, left to right, or in any order whatever; for, since the sum of any given numbers is independent of the order in which they are added together, the sum of the same partial products, in whatever order taken, must give the same result.

In the following example the figures of the multiplier are first taken from left to right, and second from right to left:

$$\begin{array}{r}
537 \\
\underline{463} \\
2148 \\
3222 \\
1611 \\
\hline
\underline{248631}
\end{array}
\qquad\qquad
\begin{array}{r}
537 \\
\underline{463} \\
1611 \\
3222 \\
2148 \\
\hline
\underline{248631}
\end{array}$$

The order of the partial products in the first operation is the reverse of that in the second. Their sum is, of course, the same in both.

Although the partial products are found most conveniently by taking the figures of the multiplicand from right to left, it is not impossible to perform the multiplication, in respect of both factors, from left to right, as the following instance may show:

$$\left.\begin{array}{r@{}r}
\text{Multiply} & 537 \\
\text{By} & \underline{463} \\
500\times400= & 200000 \\
30\times400= & 12000 \\
7\times400= & 2800 \\
500\times\ 60= & 30000 \\
30\times\ 60= & 1800 \\
7\times\ 60= & 420 \\
500\times\ \ 3= & 1500 \\
30\times\ \ 3= & 90 \\
7\times\ \ 3= & 21 \\
\hline
\multicolumn{2}{r}{\underline{537\times463=248631}}
\end{array}\right\}\text{or, omitting the zeros}
\left\{\begin{array}{l@{}r}
 & 537 \\
 & \underline{463} \\
5\times4=20 & \\
3\times4=\ \ 12 & \\
7\times4=\ \ \ \ 28 & \\
5\times6=\ \ 30 & \\
3\times6=\ \ \ \ 18 & \\
7\times6=\ \ \ \ \ \ 42 & \\
5\times3=\ \ \ \ 15 & \\
3\times3=\ \ \ \ \ \ \ 9 & \\
7\times3=\ \ \ \ \ \ \ 21 & \\
\hline
 & \underline{248631}
\end{array}\right.$$

Upon the whole, the usual practice, by which the figures as well of the multiplier as of the multiplicand are taken from right to left, seems that which is most convenient.

63. As the quantity repeatedly added to itself may be of any kind, so the number expressing it (that is, the multiplicand) may be concrete or abstract. But the multiplier, expressing only times of repetition, must be always an abstract number.

The product consists of units of the same kind as the multiplicand.

In performing the multiplication, it is usual, on account that the factors are sometimes interchanged, to regard both as abstract numbers, and afterwards to restore to the product the denomination of the multiplicand.

64. To find the product of any two numbers,

General Rule. Write the greater of the two numbers as multiplicand, and the less as multiplier, placing the units of each order in the latter under the corresponding units of the former.

Take as a partial factor the figure expressing the units of the first order, or simple units of the multiplier; multiply the units of the first order in the multiplicand by this figure; write the units of the product in the same vertical column with the partial factor, and reserve the tens (if there be any) for combination with the product of the units of the second order of the multiplicand by this partial factor.

Multiply the units of the second order of the multiplicand by the partial factor, and to the product add, as units of this order, the tens reserved from the product of the simple units; write the units of the sum on the immediate left of that figure of the product which is already found, and reserve the tens, as before, for combination with the product of the partial factor into the units of the third order of the multiplicand.

Proceed in the same manner with the units of the 3d, the 4th, ... the *n*th order, so as to form the partial product of the multiplicand by the simple units of the multiplier.

Next, take as partial factor the figure expressing the units of the second order, or tens of the multiplier; multiply the units of the first order of the multiplicand by this figure; write the units of the partial product in the same vertical column with this partial factor, and reserve the tens for combination with the product of the units of the second order of the multiplicand by the partial factor.

Continue to multiply, one after another, the remaining figures of the multiplicand by the figure expressing the tens of the multiplier, and dispose of the units and tens of the successive products as in the case of the partial factor expressing units of the first order; the result is the product of the multiplicand by the tens of the multiplier.

Find, in like manner, the product of the multiplicand by the units of the 3d, the 4th, . . . the *m*th order of the multiplier, and write the successive products, one below another, in horizontal lines, placing always the lowest figure of each partial product in the same column with that partial factor from the multiplication of the multiplicand by which it has been obtained.

Then add together all these partial products. Their sum is the product of the given numbers.

65. Examples in the multiplication of whole numbers.

Required the products of the following numbers?

1st. 765498 and 4? .... *Ans.* 3061992.
2d. 350486 and 5? .... *Ans.* 1752430.
3d. 83795602 and 6? .... *Ans.* 502773612.
4th. 479685036 and 7? .... *Ans.* 3357795252.
5th. 153879864 and 8? .... *Ans.* 1231038912.
6th. 123456789 and 9? .... *Ans.* 1111111101.
7th. 987654321 and 9? .... *Ans.* 8888888889.
8th. 742 and 68? .... *Ans.* 50456.
9th. 853 and 720? .... *Ans.* 614160.
10th. 6730 and 89? .... *Ans.* 598970.
11th. 87694 and 358? .... *Ans.* 31394452.
12th. 59387 and 796? .... *Ans.* 47272052.
13th. 78464 and 4207? .... *Ans.* 330098048.
14th. 58470 and 90070? .... *Ans.* 5266392900.
15th. 12345 and 6789? .... *Ans.* 83810205.
16th. 123456 and 708009? .... *Ans.* 87407959104.
17th. 8214356 and 132? .... *Ans.* 1084294992.
18th. 8210075 and 420306? .... *Ans.* 3450743782950.
19th. 821436 and 672576? .... *Ans.* 552478139136.
20th. 870497 and 500407? .... *Ans.* 435602792279.
21st. 4700 and 290? .... *Ans.* 1363000.
22d. 5704 and 4870? .... *Ans.* 27778480.
23d. 944784 and 972? .... *Ans.* 918330048.
24th. 46090521 and 6789? .... *Ans.* 312908547069.
25th. 594786043 and 64074958? .... *Ans.* 38110890724211194.
26th. 3529987504 and 7985463? .... *Ans.* 28188584603654352.
27th. 179865380024 and 90730856? . *Ans.* 16319339894342820544.
28th. 405970382 and 470098435? ...... *Ans.* 190846041234552170.

66. The quantities a, b, may be considered as general expressions of number. Making them the factors of a product, that product is indicated in any of the three forms following:

$a \times b$; $a.b$; ab; each of which is a general expression of the product of two factors.

It is in the first form only that the product of two numerical factors can be expressed; 5×7 signifies 5 multiplied by 7; but 5·7 is not distinguishable from the expression of a decimal fraction; or 57 from the number fifty-seven.

If one factor is a particular, and the other a general expression of number, as 5 and a, the product is written in any of the forms $5 \times a$; $5.a$; $5\,a$.

67. Hitherto attention has been given to the formation of such products only as result from the combination of two factors.

But it is often necessary to find the product of more than two numbers.

To afford a distinct notion of this subject, it may be of use to take a particular case requiring the multiplication of more than two factors.

One pound sterling contains 20 shillings; and one shilling contains 12 pence: let it be required to find the number of pence in three pounds.

To find the number of pence in 3£, it is obviously necessary to determine the product of the three factors 12, 20, and 3. Now, granting that the whole sum is given in penny-pieces, the number of these may be determined as follows, by a recurrence to the first principles of multiplication:—

Let twelve pence, the number contained in 1 shilling, be placed in a horizontal line or row, and 20 such rows arranged one beneath another.

This collection of 20 rows, each containing 12 pence $=12\times20=240=$ the number of pence in 1 pound.

A. 12

1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1.
1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1.
1
1
1
1
1 $12\times20=240$
1
1
1
.
.
.
.
.
.
.
.
.
.
20

B.

1, 1, 1, 1, ... 240.
1, 1, 1, 1, ... 240.
1, 1, 1, 1, ... 240.
$240\times3=720.$

Let now the 240 pence of this collection be arranged anew in one horizontal row; and a second, and after that a third row, each containing 240 pence, be arranged successively underneath.

This new collection of three rows, each containing 240 pence $=240\times3=720=$ the number required.

In this instance the result has been obtained by multiplying together two of the factors, and then employing the product of these two as a single factor to be multiplied by the third; and since the manner of performing the operation depends in no respect on the particular numbers 12, 20, 3, it is to be concluded that the product of any three numbers whatever is found in the same manner.

68. Again, since no change of arrangement can affect the absolute number of individuals in the collections A, B, it follows that the product of three numbers is the same, whatever be the order in which the factors are multiplied together. Representing the factors by a, b, c, the general expression of this proposition is,

$$abc=acb=bac=bca=cab=cba.$$

69. Further, (for the same reason that changes of arrangement cannot increase or diminish the number of units composing a product,)

$$720\times3 \text{ or } 3\times720=3\times12\times20$$

otherwise, more generally, $a\times(bc)=a\times b\times c$.

Whence the result arising from the multiplication of a number by the product of two factors is the same as that obtained by multiplying the number successively by these factors.

70. Returning to diagram A, in which the horizontal line is 12, the vertical column 20, and the product 240; let it be intersected by a horizontal line separating the vertical columns into (for example) the parts 13 and 7.

The number of pence contained in one part is $12\times13=\ 156$
in the other $12\times\ 7=\ \ 84$

But the sum of these parts makes the whole of A.

Whence $12\times13+12\times7=12\times(13+7)=12\times20=240$.

or, more generally, $a\times b+a\times c=a\ (b+c)$*

The same reasoning is applicable to cases in which the divided number is separated into more parts than two. Whence, if a multiplier is divided into any number of parts, the sum of the partial products of the multiplicand by these parts is equal to the product of the multiplicand by the undivided multiplier.

Since the multiplicand and multiplier may be interchanged, this proposition must be true in the case also of a divided multiplicand and undivided multiplier.

71. When the product of four factors is to be found, the product of three is obtained by Article 67, and this result, multiplied by the fourth factor, gives the product required.

A sensible illustration of the process could be obtained by arranging the units composing the product of three of the factors in one line (for instance, the units of diagram B), and repeating this line as often as the fourth factor contains unity.

If the number of factors multiplied together exceeds two, the final result is termed the continual product of these factors.

The continual product of five factors may be found from that of four; of six, from five; of n, from $n-1$; precisely as in the last article the product of four factors was obtained from that of three.

And since, however many the number of factors, the process can always be reduced to a series of additions in which the absolute number of units must remain the same, whatever may be the order in which they are combined together, it follows that the continual product of 3, 4, 5, . . . n factors is obtained by multiplying together two of the factors; the product by a third; this by a fourth; and so on to the inclusion of the nth factor; and that the result is the same, in whatever order the factors are multiplied together.

72. The product of 2, 3, ... n, factors is termed a multiple of each factor, or of any combination of them, not comprehending the whole; reciprocally, the factors are said to be sub-multiples, or parts of the product.

If the 2, 3, ... n, factors are all equal, the products are termed the 2d, 3d, ... nth powers of the single factor; and this the 2d, 3d, ... nth root of the corresponding power or product.

The second and third powers and roots have also particular names, being termed respectively Square and Cube, Square Root and Cube Root.

73. The product of any number less than 10 by any number less than 10 is composed of not more than two figures; for $10\times10=100$ is the lowest number expressed by three; and since the product of some two numbers, as 4 by 2, for example, which is 8, contains but one figure, it follows that the product of any number less than 10 by any number less than 10 is expressed by two figures, or by one only.

The product of any number greater than 10 by any number less than 10 is composed of as many figures as are contained in both factors, or of that number less one.

For by the multiplication of each figure of the greater number up to the last on the left, one figure is given to the product; by that of the last, two figures may be given, or one.

If two, the product contains as many figures as both factors; if one only, it contains that number of figures less one.

The product of any number greater than 10 by any number greater than 10 is composed of as many figures as are contained in both factors, or of that number less one.

---

* Quantities inclosed within a parenthesis are taken collectively; thus $a\ (b+c)$ indicates that the sum of b and c is multiplied by a.

For the absolute product of the multiplicand by that figure which expresses the units of the highest order of the multiplier is composed of as many figures as there are in both factors, or of that number of figures less one; and to give this product its relative value, one zero must be annexed to it for every remaining figure of the multiplier.

Whence the product of the multiplicand by the units of the highest order of the multiplier is composed of as many figures as there are in both factors, or of that number of figures less one.

When this partial product is composed of as many figures as both factors, the number of figures cannot be increased by the tens which, in the process of adding together the several partial products, it may be requisite to carry from lower to higher orders of units.

Let that figure of the multiplicand which expresses the units of the highest order be replaced by 10; the 10 is greater than the whole of the multiplicand. Let 10 replace the highest figure also of the multiplier.

The product of these tens, followed by zeros equal in number to the remaining figures respectively of the multiplicand and of the multiplier, must exceed the product of the multiplicand and multiplier. The product of the tens is 100; annexing to this as many zeros, less one, as there are figures in the multiplicand, and then as many, less one, as there are figures in the multiplier, the total number must be one greater than the number contained in both factors. But this product, which is 1 followed by zeros, is the least quantity which can be expressed by the number of figures composing it.

Therefore the smaller quantity resulting from the multiplication of the two factors cannot be composed of more figures than are contained in the multiplicand and multiplier.

Wherefore, universally, the product of any two factors is composed of as many figures as are contained in both factors, or of that number less one.

Let the multiplicand contain m figures, and the multiplier n, the product contains $m+n$ figures, or $m+n-1$.

74. The general expression of the continual product of many factors is $a\times b\times c\times d\times \ldots$, or $a{\cdot}b{\cdot}c{\cdot}d{\cdot} \ldots$, or $abcd \ldots$, of which forms the last is that which is most commonly used. The continual product of numbers expressed by figures can be indicated in the first manner only. (See Article 66.)

It has been already proved, by referring to the principles of addition (of which multiplication is only an abridged form), that the product of any number of factors is the same, whatever be the order in which these factors are multiplied together.

In effect, the product ab expresses some assemblage of individuals or simple units whose absolute number can be rendered neither greater nor less by changes of arrangement. The same is true of $ab\times c$, or abc; of $abc\times d$, or $abcd$, $abcde$. . .

If any of the literal factors of a product have numeral co-efficients, or if numeral and literal factors are combined in the same product, the numeral co-efficients or factors may be multiplied together, and the result prefixed to the expression of the product of the literal factors.

Supposing that it is required to find the product of the factors $8a$, b, $7c$, 9, and $5d$, it follows from this article that,

$$8a\times b\times 7c\times 9\times 5d=8\times 7\times 9\times 5\times a\times b\times c\times d;$$
or since $8\times 7\times 9\times 5=2520$,
that, $8a\times b\times 7c\times 9\times 5d=2520abcd$.

75. When a quantity is added to itself the number of repetitions is expressed by a numeral or literal co-efficient, which is prefixed to the quantity (Art. 25). Thus $a+a+a$ is abridged into $3a$.

By a similar convention, when a quantity is multiplied by itself, the times of repetition of the individual factor are expressed by a small figure or character written to the right of that factor, and above the horizontal line in which it is placed, thus,

$a \times a$ - - - - - - is written a^2.
$a \times a \times a$ - - - - - - - - a^3.
$a \times a \times a \times a$ - - - - - - - a^4.
$a \times a \times a \times a \times a \times \ldots$ to n repetitions of the factor a - - a^n.

The figure or character which expresses the times of repetition of the single factor is termed an index or exponent. The co-efficient, then, expresses the repeated additions of the quantity to which it is prefixed, and the exponent the repeated multiplications of the quantity to which it is annexed. This distinction it is important to keep in mind.

A co-efficient or exponent $=1$ is not expressed. Thus $1 \times a^1$ is written a.

The product of a^3 by a^4 may be expressed in the usual form, $a^3 \times a^4$.

But since $a^3 = a \times a \times a$, and $a^4 = a \times a \times a \times a$;

$\therefore a^3 \times a^4 = aaa \times aaaa = a^7 = a^{3+4}$.

Generally the product $a^m \times a^n$ is equal to a^m or $aaaa \ldots$ to m factors;
multiplied by a^n or $aaaaaa \ldots$ to n factors.

That is, $a^m \times a^n = aaaaaaaa \ldots\ldots$ to $m+n$ factors.

But a, repeatedly multiplied to $m+n$ factors, is written a^{m+n},

Whence, $a^m \times a^n = a^{m+n}$.

Consequently the product of two powers of the same root is equal to that root raised to a power indicated by the sum of the exponents of the two factors:

If $m=n$, $a^m \times a^n$ becomes $a^m \times a^m = a^{m+m} = a^{2m}$.

76. $a+b$ is a general expression for the sum of two quantities.

Twice this quantity is $2(a+b)$ or $2a+2b$. (Art. 70).

Three times $a+b$ is $3(a+b)$ or $3a+3b$.

And, generally, n times $a+b = n(a+b) = na+nb$.

If it is required to multiply the sum of two general expressions of quantity, $a+t$, by the sum of two other general expressions of quantity $a'+t'$,

Make $a'+t'=n$, and therefore $(a+t)(a'+t') = (a+t)n = an+tn$.

Then, since $n = a'+t'$, $an = a(a'+t') = aa'+at'$,
and $tn = t(a'+t') = a't+tt'$.

therefore (Art. 25) $an+tn = aa'+at'+a't+tt'$;

or, $(a+t)(a'+t') = aa'+at'+a't+tt'$.

Whence the product of one binomial expression by another is equal to the sum of the partial products of each term of the multiplicand by each term of the multiplier.

a. To find the expression of the product of one trinomial quantity by another, make t equal to $b+u$, and $t' = b+u'$,

$\therefore a+t = a+b+u : a'+t' = a'+b'+u'$;

And $aa'+at'+a't+tt'$ becomes

$aa'+a(b'+u')+a'(b+u)+(b+u)(b'+u')$.

Or, $aa'+ab'+au'+a'b+a'u+bb'+bu'+b'u+uu'$,

$\therefore (a+b+u)(a'+b'+u') = aa'+ab'+a'b+bb'+au'+a'u+bu'+b'u+uu'$.

b. Making $u = c+v$, $u' = c'+v'$, and reducing as before, it is found that

$(a+b+c+v)(a'+b'+c'+v') = aa'+ab'+a'b+bb'+ac'+a'c+bc'+b'c$
$+cc'+av'+a'v+bv'+b'v+cv'+c'v+vv'$.

Whence, also, when the multiplicand and multiplier are each composed of three terms, or of four terms, the product is equal to the sum of the partial products of each term of the multiplicand by each term of the multiplier.

By making $v = d+x$, $x = e+y$, &c., the number of terms of the polynomial factors may be increased indefinitely, and with the same result.

c. From the rule observed in forming these products, it is evident that if the multiplicand and multiplier are not composed of the same number of terms, the complete product is still equal to the sum of the partial products of each term of the multiplicand by each term of the multiplier.

d. If a, b, c, d, e, . . . represent the figures of a multiplicand, taken with respect to their relative values, and $a', b', c' \ldots$ the figures of a multiplier taken also with respect to their relative values, the result of this investigation may be taken as a general proof of the proposition of Article 53.

e. Let $a=a'$, $b=b'$, $c=c'$ ..... then,

$$(a+b)\ (a'+b')=aa'+ab'+a'b+bb', \text{ becomes}$$
$$(a+b)\ (a+b)=aa+ab+ab+bb, \text{ or}$$
$$(a+b)^2=a^2+2ab+b^2.$$

$$(a+b+c)\ (a'+b'+c')=aa'+ab'+a'b+bb'+ac'+a'c+bc'+b'c+cc',$$

becomes

$$(a+b+c)\ (a+b+c)=aa+ab+ab+bb+ac+ac+bc+bc+cc,$$
$$\text{or } (a+b+c)^2=a^2+2ab+b^2+2ac+2bc+c^2,$$
$$\text{or } (a+b+c)^2=a^2+2ab+b^2+2(a+b)c+c^2.$$

Similarly,

$$(a+b+c+d)^2=a^2+2ab+b^2+2(a+b)c+c^2+2(a+b+c)d+d^2,$$
$$(a+b+c+d+e)^2=a^2+2ab+b^2+2(a+b)c+c^2+2(a+b+c)d+d^2+2(a+b+c+d)e+e^2.$$

The square of a polynomial expression is consequently composed of the square of the first term, plus twice the product of the first term by the second, plus the square of the second term, plus twice the product of the sum of the first and second terms by the third, plus the square of the third term, &c. &c.

f. $(a+u)^2=a^2+2au+u^2$. Let $a+u=m$: then

$$(a+u)^2\times m=a^2m+2aum+u^2m,$$
$$\text{or } (a+u)^2(a+u)=a^2(a+u)+2au(a+u)+u^2(a+u),$$
$$\text{or } (a+u)^3=a^3+a^2u+2a^2u+2au^2+au^2+u^3,$$
$$\text{or } (a+u)^3=a^3+3a^2u+3au^2+u^3.$$

Make $u=b+v$, and therefore $(a+u)^3=(a+b+v)^3$, then

$$(a+b+v)^3=a^3+3a^2(b+v)+3a(b+v)^2+(b+v)^3,$$
$$\text{or } (a+b+v)^3=a^3+3a^2b+3a^2v+3ab^2+3a\times 2bv+3av^2+b^3+3b^2v+3bv^2+v^3,$$
$$\text{or } (a+b+v)^3=a^3+3a^2b+3ab^2+b^3+3a^2v+3a\times 2bv+3b^2v+3av^2+3bv^2+v^3,$$
$$\text{or } (a+b+v)^3=a^3+3a^2b+3ab^2+b^3+3(a^2+2ab+b^2)v+3(a+b)v^2+v^3,$$
$$\text{or } (a+b)+v)^3=a^3+3a^2b+3ab^2+b^3+3(a+b)^2v+3(a+b)v^2+v^3.$$

Making $v=c+x$, and reducing, it is found that $(a+b+c+x)^3=a^3+3a^2b+3ab^2+b^3+3(a+b)^2c+3(a+b)c^2+c^3+3(a+b+c)^2x+3(a+b+c)x^2+x^3$, and generally, that $(a+b+c+d+e\ldots)^3=a^3+3a^2b+3ab^2+b^3+3(a+b)^2c+3(a+b)c^2+c^3+3(a+b+c)^2d+3(a+b+c)d^2+d^3+3(a+b+c+d)^2e+3(a+b+c+d)e^2+e^3+\ldots\ldots$

Hence the cube of a polynomial expression is equal to the cube of the first term of the expression, plus three times the square of the first term into the second, plus three times the first term into the square of the second, plus the cube of the second term, plus three times the square of the sum of the first and second terms into the third, plus three times the sum of the first and second terms into the square of the third, plus the cube of the third term, plus three times the square of the sum of the first, second, and third terms into the fourth term, plus, &c. &c.

The multiplication of $a+b$ by $a+b$, and of the product by $a+b$, may be performed as follows:

$$\begin{array}{l}
a+b \\
\underline{a+b} \\
a^2+ab \\
\underline{\qquad ab+b^2} \\
a^2+2ab+b^2=(a+b)^2 \\
\underline{a\ +b} \\
a^3+2a^2b+ab^2 \\
\underline{\qquad a^2b+2ab^2+b^3} \\
a^3+3a^2b+3ab^2+b^3=(a+b)^3
\end{array}$$

Making $b=1$ in the expanded expressions of $(a+b)^2$ and $(a+b)^3$,

$$(a+1)^2=a^2+2a+1, \text{ and } (a+1)^3=a^3+3a^2+3a+1.$$

Exercises in the multiplication of general expressions of quantity:

1. Multiply abc by abd ?..*Ans.* a^2b^2cd.
2. - - $3a^2b$ by $5ab^4$? ....*Ans.* $15a^3b^5$.
3. - - a^mc^4 by ac^n ? ....*Ans.* $a^{m+1}c^{n+4}$.
4. - - $a+b$ by $c+d$? ....*Ans.* $ac+ad+bc+bd$.
5. - - $a+b+c$ by d ? ....*Ans.* $ad+bd+cd$.
6. - - a^2+b^2 by a^2+b^2 ? ....*Ans.* $a^4+2a^2b^2+b^4$.
7. - - a^m+b^n by a^m+b^n ? ....*Ans.* $a^{2m}+2a^mb^n+b^{2n}$.
8. - - $a+b+c$ by $b+c$? ....*Ans.* $ab+ac+b^2+2bc+c^2$.
9. - - $a+b+c$ by $d+e$? ....*Ans.* $ad+ae+bd+be+cd+ce$.
10. Required the continual product of $4ab$, a^2c, 6, $2c^3$, and b^4 ?
Ans. $48a^3b^5c^4$.
11. - - - - - - - $2a^m$, $3a^nb^4$, $2b^3c^2$, and 24 ?
Ans. $2^5\times3^2\times a^{m+n}b^7c^2$.
12. - - - - - - - $a+b$, $a+c$, $a+d$?
Ans. $a^3+a^2(b+c+d)+a(bc+bd+cd)+bcd$.

SECTION V.

DIVISION.

77. In Article 48 of this treatise, definitions are given of the terms Division, Dividend, Divisor, and Quotient.

The process of division consists in the decomposition of one quantity into parts each equal to another quantity; the result sought being the number of these equal parts.

78. Since the decomposition of the dividend into parts equal to the divisor can be effected by subtracting the divisor successively from the dividend and remainders until either no remainder is left, or one less than the divisor, and the number of parts equal to the divisor can be ascertained by reckoning *one* for each subtraction, and combining the *ones* into a single number, it is evident that subtraction and addition afford the means of resolving questions in division.

79. But when the dividend is large as compared with the divisor, the process by individual subtractions of the latter is laborious. Whence it becomes requisite to investigate a more compendious method.

For this purpose let D denote the dividend, d the divisor, and q the quotient of an example in division; q being found in the manner pointed out in the preceding Article.

q being obtained by reckoning 1 for each subtraction of the divisor from the dividend, it follows that the quotient contains unity as many times as the dividend contains the divisor.

If, instead of subtracting d q times in succession from D, the sum of q repetitions of d are at once subtracted from D, the result is the same (Art. 41). But the sum of q repetitions of d is the product $d\times q$.

When the last subtraction leaves no remainder, q times the divisor is equal to the dividend; that is, $d\times q=\mathrm{D}$.

Whence, in such cases, the dividend may be regarded as a product of which the divisor and the quotient are the factors, and division is reduced to this problem: given a product and one of its factors, to find the other factor.

In discussing this problem, it will be convenient to consider,

1st. Cases in which d is less than 10, and D less than 9×10.

2d. Cases in which d is less than 10, and D any number whatever.

3d. Cases in which both d and D are any numbers whatever.

80. First, let the dividend be 36, and the divisor 9. Adopting the elementary process by subtraction,

$36-9=27$ - - 1 subtraction.
$27-9=18$ - - 1 „
$18-9=9$ - - 1 „
$9-9=0$ - - 1 „
4=number of subtractions.

The dividend being exhausted by 4 subtractions of the divisor, it follows that the quotient is 4, a result which may be expressed thus (Art. 48)

$$36 \div 9=4$$

Since (Art. 79) $D=d \times q$, and (in the preceding example) $36=9 \times 4$; it is evident that division may be performed by means of the multiplication table.

To find the product of 9 by 4, horizontal column 9 and vertical column 4 are traced to their intersection. The square in which they meet is occupied by their product, 36. Changing this order of research, the divisor 9 is found, the eye carried along horizontal column 9 to the dividend 36, and then to the number at the top of the vertical column in which 36 is placed. This number (4) is the factor which multiplied by 9 produces 36, and it is therefore the quotient required.

Similarly it is found that $56 \div 8=7$
- - - - $54 \div 6=9$
- - - - $72 \div 9=8$.

81. In these examples both dividend and divisor are numbers contained in the multiplication table.

Let it next be required to divide 43 by 8.

Consulting horizontal column 8, it is found not to contain the number 43.

As 43 is comprehended between $40=8 \times 5$, and $48=8 \times 6$, it follows that 43 contains 8 more than 5, and fewer than 6, times; and since $8 \times 5=40$, and $43-40=3$, it is to be concluded that the quotient is 5, and the remainder 3.

The dividend and divisor of an unexecuted division are sometimes written, the first over and the second under a horizontal line, as $\frac{3}{8}$. In this manner the result of the preceding example may be expressed thus,

$$43 \div 8=5\tfrac{3}{8}$$

By Article 79. it appears that when the division is exact (or leaves no remainder), $D=d \times q$.

It now becomes necessary to extend this principle to cases in which the operation terminates in a remainder less than the divisor.

If the principle is true in such cases, it must follow that

$$43=8 \times 5\tfrac{3}{8}$$
$$\text{or } 43=5\tfrac{3}{8} \times 8 \text{ (Art. 56).}$$

In multiplying $5\frac{3}{8}$ by 8, the products 5×8 and $\frac{3}{8} \times 8$ may be formed separately, and the results added together (Art. 70).

Now $\frac{3}{8}$ is a form of writing 3 divided by 8.

Whence $\frac{3}{8} \times 8$ signifies 3 divided by 8, and multiplied by 8. But if a quantity is both divided and multiplied by the same number, it cannot be changed by the process, which (because $\frac{8}{8}=1$, and generally $\frac{d}{d}=1$) amounts merely to the multiplication of the quantity by 1.

$$\begin{array}{ll} \text{Wherefore} & \frac{3}{8} \times 8=3 \\ \text{And - -} & \underline{5 \times 8=40} \\ \text{Consequently} & \underline{5\frac{3}{8} \times 8=43} \end{array}$$

This reasoning being of general application, it follows that in any case the dividend is equal to the product of the divisor and quotient.

82. Between any multiple of a divisor and the multiple following there are as many numbers, less one, as the divisor contains units. But no number between a multiple of the divisor and the multiple next following can be exactly divided by the divisor; whence a remainder greater than zero and less than the divisor must, most frequently, terminate the operation of division.

A remainder written as in the last example (namely, as $\frac{3}{8}$) may be read, three divided by eight, three over eight, or three eighths. Numbers so written are named fractions.

83. Let, next, *d* be less than 10, and D any number whatever:

Since the product of the divisor and quotient is equal to the dividend, and the divisor contains one figure, the quotient must contain as many as the dividend, or that number less one (Art. 73); and the dividend must be composed of the partial products of the divisor by these quotient figures taken individually, and with regard to their relative values (Art. 50).

The divisor being some number in the series,

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ \text{(A)}$$

and the figures of the quotient (regard being had to their relative values) numbers in the series,

$$\left.\begin{array}{llllll} 0, & 1, & 2, & 3, & 4, \ldots\ldots & 9, \\ 0, & 10, & 20, & 30, & 40, \ldots\ldots & 90, \\ 0, & 100, & 200, & 300, & 400, \ldots\ldots & 900, \\ 0, & 1000, & 2000, & 3000, & 4000, \ldots\ldots & 9000, \\ & \&c. & \&c. & \&c. & \&c. & \&c. \end{array}\right\}\text{(B)}$$

it follows that the dividend is equal to the sum of the partial products of some number in series A by one term in each of the series B; and this sum is the same, whatever be the order in which the partial products are added together; thus it is evident that if these very products are again taken away, the order of subtraction can have no effect upon the final result.

It is also evident that the product of the divisor by one term in any of the series B cannot contain more than two significant figures, and may contain only one (Art. 73);

That the zeros which follow such significant figures or figure indicate merely the relative value of the partial product;

And that in subtracting the partial products, that of the divisor by the tens of the quotient cannot tend to diminish the units of the dividend; that of the divisor by the hundreds cannot tend to diminish the tens of the dividend; and, generally, that the product of the divisor by the highest figure of the quotient cannot affect those units of the dividend which are of a lower order than the highest figure of the quotient.

From these principles it follows, that the partial product of the divisor by the highest quotient figure may be subtracted first, and the others by the quotient figures following in succession; that the absolute product of the divisor by the highest quotient figure can affect only the first, or the first and second, figures of the dividend; and that if the divisor is contained in the first figure, the quotient contains as many figures as the dividend; if in the first and second, it contains one figure fewer than the dividend.

The highest figure of the quotient is consequently obtained by dividing the first, or the first and second, figures of the dividend by a number less than 10; hence it is given by the multiplication table.

After subtraction of the product of the divisor by the first quotient figure, the remainder may be considered as a new dividend, and the same process repeated; and this, it is evident, may be done with the successive remainders until the operation is brought to an end.

84. Let it now be required to divide 59213 by 7.

Since 7 is not contained in 5, the first partial dividend is 59, and the quotient is composed of 4 figures:

$59 \div 7 = 8 +$ a remainder; the 1st quotient number is, therefore, 8000;
$8000 \times 7 = 56000$; and $59213 - 56000 = 3213 =$ 1st remainder.
$32 \div 7 = 4 +$ a remainder; the 2d quotient number is, therefore, 400;
$400 \times 7 = 2800$; and $3213 - 2800 = 413 =$ 2d remainder.
$41 \div 7 = 5 +$ a remainder; the 3d quotient number is, therefore, 50;
$50 \times 7 = 350$; and $413 - 350 = 63 =$ 3d remainder.
$63 \div 7 = 9 + 0$ remainder; the 4th quotient number is, therefore, 9;

$$\begin{array}{lr}
\text{Whence the partial quotients are} \; - & 8000 \\
 & 400 \\
 & 50 \\
 & \underline{9} \\
\text{And } 59213 \div 7 \quad - \quad - \quad - & = \underline{\underline{8459}}
\end{array}$$

The calculation may be represented thus,

$$\begin{array}{rrl}
\text{divisor} & \text{dividend} & \text{partial quotients} \\
7\;) & 59213 & \left(\begin{array}{r} 8000 \\ 400 \\ 50 \\ 9 \end{array} \right\} \begin{array}{l} \text{quotient} \\ 8459 \end{array} \\
 & \underline{56000} & \\
 & 3213 & \\
 & \underline{2800} & \\
 & 413 & \\
 & \underline{350} & \\
 & 63 & \\
 & \underline{63} & \\
 & \underline{\underline{0}} &
\end{array}$$

85. In the preceding calculation the relative value of every figure of the quotient is put in evidence, and the product of each by the divisor subtracted from the whole of the dividend and remainders. The subject being understood, such detail is not necessary.

The division may be performed thus,

$59 \div 7 = 8 + 3$ remainder,
$32 \div 7 = 4 + 4$ remainder,
$41 \div 7 = 5 + 6$ remainder,
$63 \div 7 = 9 + 0$ remainder.

and the calculation written as under,

$$\begin{array}{rl}
7\;) & 59213 \; (\; 8459 \\
 & \underline{56} \\
 & \;\;32 \\
 & \;\;\underline{28} \\
 & \;\;\;\;41 \\
 & \;\;\;\;\underline{35} \\
 & \;\;\;\;\;\;63 \\
 & \;\;\;\;\;\;\underline{63} \\
 & \;\;\;\;\;\;\;\;\underline{\underline{0}}
\end{array}$$

In explanation of this abridged process it is to be observed, that the remainder being less than the divisor, the third figure (2) of the dividend must be annexed to the first remainder (3) before a second partial division can be made. In like manner, the fourth figure of the dividend must be annexed to the second remainder; the fifth to the third, &c. The figure brought down is annexed, not added to, the remainder, because it expresses units of the order immediately inferior to those of the last figure of the remainder.

Since each quotient figure has the same relative value as the lowest significant figure of the corresponding partial dividend*, it follows that if the quotient figures obtained in this manner are written each to the right of those previously obtained, the relative values of the whole will be properly determined.

After annexing the proper figure of the dividend to the remainder left by a preceding division, it may happen that the divisor is greater than the partial dividend. In this case it becomes necessary to write zero in the quotient, for the purpose of preserving to the preceding quotient figures their proper relative values.

For the same reason, when the last figure of a dividend is 0, and the remainder to which this 0 is to be annexed happens to be also equal to zero, 0 must be written to the right of the quotient figures previously obtained; and when, under the same circumstances, there are several zeros terminating the dividend, one 0 for each must be annexed to the significant figures of the quotient.

It is only in the abridged process of division that these precautions, for the purpose of preserving to each figure of the quotient its proper relative value, become necessary.

As a means of illustrating these remarks, let it be required to divide 70825536 by 9, and 36863680 by 8.

2d Example.

```
9) 70825536 (7869504
   63
   --
    78
    72
    --
     62
     54
     --
      85
      81
      --
       45
       45
       --
         36
         36
         --
```

The subtraction of the partial product of 9 by the fifth quotient figure 5 leaving no remainder, the sixth partial dividend is composed of only the figure brought down. Thus, figure 3 is less than 9, by which it cannot be divided. Zero is written in that place which, had the division been possible, some significant figure must have filled; the next figure of the dividend is then brought down, and the process continued.

3d Example.

```
8) 36863680 (4607960
   32
   --
    48
    48
    --
      63
      56
      --
       76
       72
       --
        48
        48
        --
         0
        --
```

---

* The first quotient figure has the same relative value as the lowest figure of the first partial dividend, for each remaining figure of the dividend gives one figure to the quotient.

For the same reason, any other figure of the quotient has the same relative value as the lowest figure of the corresponding partial dividend.

In this example the third partial division being impossible, 0 is made to occupy the place of the deficient figure; also 0 is annexed to the last significant figure of the quotient. Restoring the detailed process, as explained in Example 1, the reason becomes manifest, or it may be inferred from the considerations following: had a remainder been left from the preceding subtraction, this remainder, with the 0 of the dividend annexed, must have been divisible by 8; another quotient figure must have been obtained, and the 6 made to occupy (its proper place) the place of tens.

86. When the divisor is a factor contained in the multiplication table, the successive quotient figures may be found, and the partial products formed and subtracted, mentally. In this manner the calculation is made to appear in a very abridged form; the results only, or results and partial remainders, being registered thus:

Example 1 repeated:

$$\begin{array}{rl} 7)\ \underline{59213} & \\ 8459 & \text{quotient.} \\ \underline{\cancel{3}\cancel{4}\cancel{6}\cancel{0}} & \end{array}$$

Example 2 repeated:

$$\begin{array}{rl} 9)\ \underline{70825536} & \\ 7869504 & \text{quotient.} \\ \underline{\cancel{7}\cancel{6}\cancel{8}\cancel{4}\cancel{0}\cancel{3}\cancel{0}} & \end{array}$$

Example 3 repeated:

$$\begin{array}{rl} 8)\ \underline{36863680} & \\ \underline{4607960} & \text{quotient.} \end{array}$$

87. Lastly, let D and d be any numbers whatever, with this limitation, however, that D is greater than d.

Since $D = d \times q$, the dividend is composed of the partial products of the divisor by the figures of the quotient taken singly, and with regard to their relative values (Art. 53). The relative values of these figures may be determined from the principles established in Articles 73 and 85; and when the absolute values also have been found, the products by the divisor may be subtracted from the dividend and remainders, as has been done in the case of Example 1, Art. 84.

When the divisor is less than 10, the quotient figures are obtained individually from the multiplication table; but when greater than 10, the manner of discovering them is not at first so obvious. It is certain, however, that the significant part of the partial product of the divisor by any figure of the quotient must be the product of the divisor by some number in the series 1, 2, 3, 4, 5, 6, 7, 8, 9. Whence, if the products of the divisor by these numbers are computed and arranged in a tabular form, it will be as easy from this table to determine the quotient figures in succession as it is, in the case of a divisor less than 10, to find the quotient figures by means of the multiplication table.

88. The process of dividing by a divisor greater than 10 being thus rendered identical with that of dividing by a divisor less than 10, the calculation in the former case may be abridged as it has been in the latter. (See Article 85.) Provided always, that attention is given to the explanations and directions contained in Article 85.

Example:—Divide 640791247 by 74693.

The product of 74693 by each of the nine digits may be formed by repeated additions, as follows:

$$\begin{array}{rr} 74693 \times 1 = & 74693 \\ & \underline{74693} \\ 74693 \times 2 = & 149386 \\ & \underline{74693} \\ & 224079 \end{array}$$

74693 × 3 = 224079
74693
74693 × 4 = 298772
74693
74693 × 5 = 373465
74693
74693 × 6 = 448158
74693
74693 × 7 = 522851
74693
74693 × 8 = 597544
74693
74693 × 9 = 672237

divisor dividend quotient
74693) 640791247 (8579
597544
432472
373465
590074
522851
672237
672237

Detail of the operation:

64079 ÷ 74693; this division gives no whole number to the quotient.
640791 ÷ 74693 = 8[*] + 43247 remainder.
432472 ÷ 74693 = 5 + 59007 remainder.
590074 ÷ 74693 = 7 + 67223 remainder.
672237 ÷ 74693 = 9 + 0 remainder.

89. The labour of computing such a table for every example more than counterbalances the advantage to be derived from its use. In practice the successive quotient figures are obtained by trial, and a little experience renders it easy to select the proper figures. Usually the first, or first and second figures of the dividend are divided by the first figure of the divisor. The result (given by the multiplication table) is taken as a first quotient figure. If the product of the divisor by this quotient figure is greater than the partial dividend from which it is to be subtracted, the figure next lower in the series, 1, 2, 3, ... 9 must be taken, and tried in the same manner; on the other hand, if the remainder is greater than the divisor, the figure next higher is to be taken. The subtraction being possible, and the remainder less than the divisor, the quotient figure chosen is the proper one.

To apply these remarks, let it be required to divide 2577922208 by 538976.

538976)2577922208(4783
2155904
4220182
3772832
4473500
4311808
1616928
1616928

---

[*] The product of the first quotient figure by the units of the divisor falls under the thousands of the dividend. Whence the quotient must contain four figures.

Detail of the operation:

$2 \div 5$; the division cannot be made. Whence the first partial dividend contains 7 figures, and the quotient 4 figures (Art. 85).

$$25 \div 5 = 5, \text{ and } 538976 \times 5 = 2694880.$$

This product is greater than the partial dividend, 2577922; the quotient figure 5 is therefore too great.

Trying 4, the next lower figure,

$$538976 \times 4 = 2155904, \text{ and } 257792 - 2155904 = 422018;$$

the remainder, 422018, being less than the divisor, 4 is the proper quotient figure.

Annexing 2, the next figure of the dividend, to the remainder 422018, the second partial dividend is 4220182,

$$42 \div 5 = 8, \text{ and } 538976 \times 8 = 4311808.$$

This number exceeds the dividend; the quotient figure 8 is therefore too great. Trying 7,

$$538976 \times 7 = 3772832, \text{ and } 4220182 - 3772832 = 447350;$$

whence 7 is the proper quotient figure, and 447350 the remainder.

Annexing to this remainder the next figure of the dividend, which is 0, 4473500 is obtained for the third partial dividend;

$44 \div 5 = 8$, and $538976 \times 8 = 4311808$, a number less than the partial dividend;

$$4473500 - 4311808 = 161692;$$

8 is the proper quotient, and 161692 the remainder.

Annexing to this remainder 8, the last figure of the dividend, 1616928 is the fourth partial dividend.

$16 \div 5 = 3$, and $538976 \times 3 = 1616928$, a number equal to the partial dividend.

Whence the division is exact, and the quotient $= 4783$.

In the first and second partial divisions it is found, that, although $25 \div 5 = 5$, and $42 \div 5 = 8$, yet 5 and 8 are not the proper quotient figures. This arises from the circumstance, that when the product of the remaining part of the divisor (namely, 38976) by 5 or 8 is added to the product of 500000 by 5 or 8, the sum of both exceeds the partial dividend.

90. After finding a quotient figure in the manner prescribed, if its product by the first and second figures of the divisor is formed mentally, and compared with the corresponding figures of the partial dividend, it will in general be possible, without further trial, to ascertain whether the figure found is or is not too great.

In the preceding instances

53×5, or 265, is greater than 257;

and 53×8, or 424, is greater than 422.

The preceding remarks will be found applicable to the following example: Divide 303374185950 by 4985607.

4985607)303374185950(60850
29913642
42377659
39884856
24928035
24928035
0

It does not seem necessary to go at length through the details of the process in this example also. Why zeros are written in the quotient in the places of thousands and simple units is explained in Article 85.

91. An abridged form of division in the case of a divisor less than 10 has been noticed in Article 86. When the divisor is greater than 10, the successive multiplications and subtractions may, in like manner, be made mentally, and the quotient figures and remainders alone recorded.

In division by this method the divisor seems most conveniently placed over the figures composing the first partial dividend.

As this abridged method may be most intelligibly explained through the medium of an example, let it be required to divide 945945 by 2457.

The calculation is represented thus,

2457
945945(385
20884
12285
0000

And the details of the process are,

1st. To find the highest quotient figure,

9+2=4. But since 24×4=96, and 96 is greater than 94, this result is too great.

The first quotient figure cannot, therefore, be greater than 3; and since 24×3=72, a number considerably less than 94, it may be presumed that 3 is the proper figure.

2d. To find the remainder left by the subtraction of 3 times the divisor from the first partial dividend. In the ordinary process the divisor, 2457, is multiplied by 3, and the product subtracted from the partial dividend 9459; but in this abridged method the multiplication of each figure of the divisor by 3, and the subtraction of this product from the corresponding figure of the partial dividend, are performed simultaneously thus,

The last figure of the first partial dividend is 9, and 7×3=21; therefore 21 is to be taken from 9.

To render this subtraction possible 2 is borrowed from 5, the next higher figure of the dividend; regard being had to the relative value of the 5 as compared with that of the 9, the 2 borrowed are equal to 20 units of the order of the 9.

Whence 20+9=29, and 29−21=8.

This 8 is written under the figure 9 of the dividend, and the 2 borrowed is either taken from the next higher figure of the dividend, 5, or reserved for combination with the product of the tens of the divisor by the quotient figure 3. The latter course, which is found the more convenient, is generally adopted in practice.

Proceeding to the second figures of the divisor and partial dividend,

5×3=15, and 15+2 borrowed=17.
5−17, subtraction not possible: borrow 20,
20+5=25, and 25−17=8.

The remainder 8 is written, and the two tens borrowed are carried, as before.

Third figures of the divisor and partial dividend,

4×3=12, and 12+2 borrowed=14.
4−14, subtraction not possible: borrow 10,
10+4=14, and 14−14=0.

The 0 is written, and the 10 carried, as before.

Fourth figures of the divisor and partial quotient,

2×3=6, and 6+1 borrowed=7.
9−7=2.

The 2 being written in its proper place, it is found that the first figure of the quotient is 3, and the first remainder 2088: annexing the next figure of the dividend to this remainder, 20884 is obtained as the second partial dividend.

The quotient evidently contains two figures in addition to that already found. For each the process is the same as that which has been developed at so great length.

To find the second figure of the quotient.

20÷2=10. If the quotient really is 10, the preceding quotient figure has been taken too low, for it is evident that this 10 in the second place is equal to 1 in the first place, and that the 3 must become 3+1, or 4.

Applying the practical test of Article 90,

24×10=240: but the corresponding number of the dividend is only 208; therefore the quotient is less than 10. In the same manner it is found to be less than 9; to be 8.

8 is written in the quotient; and the multiplication, by 8, of the figures of the divisor, and the subtraction of the uncombined products continued as before,

7×8=56. To render the subtraction possible 60 is borrowed,
60+4=64, and 64—56=8.

The 8 remainder is written under the 4 of the partial dividend, and 6 carried for the 60 borrowed,

5×8=40, and 40+6=46.
8—46. To render the subtraction possible 40 is borrowed,
40+8=48, and 48—46=2.

The 2 remainder is written under the proper figure, and 4 is carried.

4×8=32, and 32+4=36.
8—36. Borrow 30,
30+8=38, and 38—36=2.

Write 2 remainder, and carry 3 for 30 borrowed.

2×8=16, and 16+3=19.

20—19=1. Writing 1 under the 0, the second remainder is found to be 1228.

If to this remainder the last figure of the dividend is annexed, the third partial dividend is 12285.

To find the last figure of the quotient, and the remainder, if any.

12÷2=6; but since 24×6=144, and the corresponding figures of the partial dividend are 122, the quotient figure must be less than 6.

24×5=120, a number less than 122; therefore 5 may be tried.

Writing 5 in the quotient, and proceeding as before,

7×5=35. 5—35: to render the subtraction possible borrow 30,
30+5=35, and 35—35=0.

Write 0 under 5 of the partial dividend, and carry 3,

5×5=25, and 25+3 borrowed=28.
8—28. Borrow 20.
20+8=28, and 28—28=0.

Write 0 under 8, and carry 2.

4×5=20, and 20+2 borrowed=22.
2—22. Borrow 20.
20+2=22, and 22—22=0.

Write 0 under 2, and carry 2.

2×5=10, and 10+2 borrowed=12.
12—12=0.

Whence the quotient=385, and the remainder=0.

Second example of calculation by this method.

7584
3548728(467
51512
60088
0000

Third example.

$$\begin{array}{l}
\underline{2789} \\
\underline{9639475}(3456\tfrac{691}{2789}. \\
\underline{12724} \\
\underline{15687} \\
\underline{17425} \\
\underline{691}
\end{array}$$

92. In the preceding operations of arithmetic it has been shown that the calculations may be made either from left to right, or from right to left; but that the latter course is the more convenient.

The calculation in division may, in like manner, be carried from right to left; for the divisor may be subtracted from the dividend and remainders time after time till the latter are exhausted, and the times of subtraction assembled to form the quotient; or the divisor, and its multiples by 10, 100, 1000, &c. &c. may be subtracted as often as possible, and the ones, tens, hundreds, &c. &c. combined into a quotient number. In either case the calculation may proceed from right to left.

But the calculation by the rule of division, properly so called, is carried from the left to the right of the dividend. The reason for so doing is this: the dividend is the sum of the partial products of the divisor by the units, the tens, the hundreds, &c. &c. of the quotient. In the addition of these partial products, tens arising from the product of units of the first order are combined as ones with units of the second order, tens of the second with units of the third, &c. Now, in the process of decomposition the tens carried must be restored to the units of that order from which they came; and it is only by commencing the decomposition with the highest figures of the dividend that these tens can be disengaged in such a manner as to admit at once of combination with the units from which they are derived, and to which they belong.

93. To divide the greater of two whole numbers by the less.

General Rule. Arrange the numbers thus,

Divisor) Dividend (Quotient.

Or thus,

Divisor.
Dividend (Quotient.

Take from the left of the dividend as many figures as are contained in the divisor, or (when the absolute number expressed by these figures is less than the divisor) one figure more. These form the first partial dividend, of which, also, the last figure has the same relative value as the first figure of the quotient.

Find by trials (or by a table as in Article 88) how often this partial dividend contains the divisor: the figure expressing the number of times is the first partial quotient. Write this figure in its proper place; multiply the divisor by it, and subtract the product from the first partial dividend.

Annex the next figure of the dividend to the remainder, to form a second partial dividend.

If the second partial dividend is greater than the divisor, find, as before, how often it contains the divisor.

Annex to the first partial quotient the figure expressing the number of times; multiply the divisor by it, subtract the product from the second partial dividend, and to the remainder annex the next figure of the dividend. The third partial dividend is thus obtained.

But if the second partial dividend is less than the divisor, annex to the first partial quotient a zero, to preserve the place of that order of units which is not expressed by a significant figure; and bring down the next figure of the dividend, as if the second partial dividend were a remainder.

In this manner, whether the second partial division gives to the quotient a significant figure or a zero, the third partial dividend is attained.

The process with the third partial dividend, the fourth . . . . the last, is the same as with the second.

When the last remainder is zero the division is exact; when it is composed of one or more significant figures, these are written over the divisor, and annexed to the quotient, in order to complete it, as in the example of Article 81.

94. Exercises in the division of whole numbers:

1st. 409248÷ 3= 136416.
2d. 234787÷ 4= 58696$\frac{3}{4}$.
3d. 478594÷ 5= 95718$\frac{4}{5}$.
4th. 3957468÷ 6= 659578.
5th. 58696358÷ 7= 8385194.
6th. 56032897÷ 8= 7004112$\frac{1}{8}$.
7th. 7854954÷ 9= 872772$\frac{6}{9}$.
8th. 5847865÷ 11= 531624$\frac{1}{11}$.
9th. 17598576÷ 13= 1353736$\frac{8}{13}$.
10th. 75840629÷ 17= 4461213$\frac{8}{17}$.
11th. 112918539÷ 19= 5943081.
12th. 173606070÷ 23= 7548090.
13th. 89296336÷ 29= 3079184.
14th. 89925606÷ 31= 2900826.
15th. 152024453÷ 37= 4108769.
16th. 405779296÷ 41= 9897056.
17th. 46672071÷ 43= 1085397.
18th. 102027318÷ 47= 2170794.
19th. 444948224÷ 49= 9080576.
20th. 810208÷ 112= 7234.
21st. 381060÷ 87= 4380.
22d. 1068912÷ 234= 4568.
23d. 3326689÷ 347= 9587.
24th. 47218592÷ 7408= 6374.
25th. 653520081752÷ 8678= 75307684.
26th. 963259373376÷ 9876= 97535376.
27th. 574063384231÷ 8311= 69072721.
28th. 882250012071÷ 9591= 91987281.
29th. 10477691340÷ 27964= 374685.
30th. 905353634024÷ 4837=187172552.
31st. 711165446776÷ 8926= 79673476.
32d. 334544448888÷ 6942= 48191364.
33d. 464584018625÷ 7745= 59985025.
34th. 1016263909833÷ 6971=145784523.
35th. 651037378878÷ 6879= 94641282.
36th. 4178379461760÷ 7476=558905760.
37th. 19548607332474÷ 45678=427965483.

38th. 274616097456 ÷ 579486 = 473896.
39th. 3811089072421194 ÷ 594786043 = 64074958.
40th. 29857547202 ÷ 498673 = 59874.
41st. 962674279624 ÷ 24373969 = 39496.
42d. 66275122223520 ÷ 97486353 = 679840.
43d. 8907343771535 ÷ 29601847 = 300905.
44th. 2498145110262 ÷ 3785946 = 659847.
45th. 3424123359360 ÷ 97888032 = 34980.

95. If two factors are multiplied together, and the product is divided by one of the factors, the result is the other factor.

Whence, if any number of factors are employed to form a product, and this product is divided by these factors one after another in a reverse order, it is evident that the products in a reverse order, and the original multiplicand, must be in succession recovered.

Now, to form the product of a multiplicand by 2, 3, 4, ... factors, either the multiplicand is multiplied by one factor, the product by a second, this by a third, &c. &c., in any order; or the product of all the factors is formed, and the multiplicand multiplied by this product (Art. 71).

Consequently, if the product of any multiplicand by several factors is divided by these factors one after another, the last quotient is the multiplicand; or, if the same product is at once divided by the product of all the factors, the quotient is also the multiplicand.

Whence, it is indifferent to divide a number by several divisors in succession, or at once by the product of these divisors.

96. It is sometimes convenient to break down a large divisor into factors each less than 10, and to divide by these factors one after another, instead of by the whole divisor. It follows from the last article that the results given by the two methods are the same. But when the calculation by the unbroken divisor ends in a remainder, that by the factors of the divisor must exhibit remainders terminating some or perhaps all the divisions. From these remainders it may be necessary to find the whole remainder left when the division is performed at once.

The remainder from the division of the dividend by the first factor of the divisor is evidently composed of units of the dividend; and the remainder from the division of the first quotient by the second factor of the divisor is, in like manner, composed of units of the first quotient. Similarly the remainder from the third division is composed of units of the second quotient, &c. &c.

The manner, therefore, of reducing the remainder of the second division into units of the first dividend being discovered, the other reductions must be made in the same way. Whence it is sufficient to investigate the means of reducing this remainder.

Let $\frac{D}{d}=q$ (Art 79).

Then, since D is composed of d repeated as often as there are units in q, it follows that

to 1 unit in q correspond $1\times d$ units in D;
to 2 units in q correspond $2\times d$ units in D;
to n units in q correspond $n\times d$ units in D;

consequently, from any number in q to recover the corresponding number in D, it is necessary to multiply the former by d.

97. As the whole calculation may be best explained through the medium of an example, let it be required to divide 359482 by 105, which is equal to the continual product of the factors 3, 5, and 7.

Calculation:

$$\begin{array}{l} 3)\underline{359482} \\ 5)\underline{119827}+1 \text{ first remainder.} \\ \;\;7)\underline{23965}+2 \text{ second remainder.} \\ \;\;\;\;\underline{3423}+4 \text{ third remainder.} \end{array}$$

Explanation. The quotient resulting from the division of 359482 by 3 is 119827, and the remainder 1. This remainder (Art. 96) is 1 unit of the dividend.

The quotient resulting from the division of 119827 by 5 is 23965, and the remainder 2. This remainder is composed of units of the first quotient; and 1 unit of the first quotient corresponding to 3 units of the dividend, the second remainder, 2, corresponds to 2×3=6 units of the dividend.

The quotient resulting from the division of 23965 by 7 is 3423, and the remainder 4. This remainder is composed of units of the second quotient.

Now 1 unit of the second quotient corresponds to 5 units of the first:

Therefore 4 units of the second quotient correspond to 4×5= 20 units of the first.

Again, 1 unit of the first quotient corresponds to 3 units of the dividend:

Therefore 20 units of the first quotient correspond to 20×3=60 units of the dividend.

Whence 4 units of the second quotient correspond to 60 units of the dividend.

The 1st remainder is equal to 1 unit of the dividend.
2d - - - 6 " "
3d - - - 60 " "
The sum of the three remainders is 67

The whole remainder which would be obtained if the division were performed at once is therefore 67:

$$\text{And } 359482 \div 105 = 3423\tfrac{67}{105}.$$

The several remainders may be reduced into one number briefly, as follows:

$$\overset{\text{3d rem.}}{4} \times \overset{\text{2d div.}}{5} = 20, \text{ and } 20 + \overset{\text{2d rem.}}{2} = 22.$$

$$22 \times \overset{\text{1st div.}}{3} = 66, \text{ and } 66 + \overset{\text{1st rem.}}{1} = 67.$$

98. If any of the divisions are exact, let 0 be written as the remainder. This done, the following practical rule for the reduction of all the remainders into one is perfectly general:

Multiply the last remainder by the last divisor but one, and to the product add the preceding remainder; multiply this sum by the next preceding divisor, and to the product add the next preceding remainder; repeat this process until all the divisors and remainders are included. The last result is the number sought.

99. It is only when a number can be broken into factors, each not greater than the greatest number contained in the first horizontal or first vertical column of a multiplication table, that the employment of this method can bring any practical advantage.

Whether a given number can be decomposed into such factors may be ascertained by trials, thus: it may be divided by 2 as often as the division can be made exactly; then by 3, 5, 7, &c. in the same manner. If a last remainder equal to zero is found, the product of the divisors is equal to the given number.

Some numbers cannot be broken into factors, being divisible by themselves only, and by unity, which is a divisor of every whole number.

100. The division of one general expression of quantity, a, by another, b, is indicated thus,

$$\frac{a}{b} \text{ or } a \div b.$$

Supposing the division exact, and the quotient c, the preceding expressions become $\frac{a}{b}=c$, or $a \div b=c$.

The division being inexact, the quotient c, and the remainder d, they become $\frac{a}{b}=c+\frac{d}{b}$, or $a \div b=c+\frac{d}{b}$.

When the quantities composing the dividend and divisor are unlike, the division and its result must be represented in one or other of the preceding forms; but when they are composed of similar factors, the quotient is expressible in terms of the common quantity.

For $\frac{a^5}{a^3}=\frac{a\times a\times a\times a\times a}{a\times a\times a}$ (Art. 75) $=\frac{a}{a}\times\frac{a}{a}\times\frac{a}{a}\times a\times a$.

But $\frac{a}{a}=1$ $\therefore \frac{a^5}{a^3}=1\times 1\times 1\times a\times a=a\times a=a^2$, or a^{5-3}.

$\frac{a^6}{a^2}=\frac{aaaaaa}{aa}=aaaa=a^4=a^{6-2}$.

$\frac{a^m}{a^n}=\frac{aaa \ldots \text{ to } m \text{ repetitions of the factor } a}{aaa \ldots \text{ to } n \text{ repetitions of the factor } a}$.

$\therefore \frac{a^m}{a^n}=aaa$ to $m-n$ repetitions of the factor a, that is, $\frac{a^m}{a^n}=a^{m-n}$.

When the dividend, therefore, and divisor are powers of the same root, the quotient is obtained by giving that root an exponent equal to the difference between the exponents of the dividend and divisor.

a. If the dividend is composed of the product of two or more factors, and the divisor involves one or more factors which are either similar to the factors of the dividend or are powers of the same root, the fractional expression of the quotient admits of the same kind of simplification:

Thus, $a^2b^2 \div a^2=\frac{a^2b^2}{a^2}=\frac{a^2}{a^2}\times b^2=1\times b^2=b^2$.

$$a^5bc \div a^2b=\frac{a^5}{a^2}\times\frac{b}{b}\times c=a^3\times 1\times c=a^3c.$$

$$\frac{a^mb^2c}{a^nbd}=\frac{a^m}{a^n}\times\frac{b^2}{b}\times\frac{c}{d}=a^{m-n}\times b^{2-1}\times\frac{c}{d}=\frac{a^{m-n}bc}{d}.$$

For the continuation of this subject see Article 154, on the reduction of fractions to the lowest terms.

b. The product of a multiplicand composed of several terms by a multiplier consisting of one term is obtained by multiplying each term of the multiplicand by the multiplier (Art. 76 c).

Consequently, in division (Art. 79), the dividend being expressed by several terms, and the divisor by one term, the quotient is a quantity consisting of as many terms as the dividend; and the product of the first term of the quotient by the divisor is the first term of the dividend; the product of the second term of the quotient by the divisor is the second term of the dividend, &c. &c.

Whence, the divisor and dividend being given, the quotient in this case is obtained by dividing each term of the dividend by the divisor, and taking the sum of the partial quotients for the result required:

$$\therefore (a^3+a^2b+ac) \div a=\frac{a^3}{a}+\frac{a^2b}{a}+\frac{ac}{a}=a^2+ab+c.$$

$$(a^5+a^2b^2+ab) \div ab=\frac{a^5}{ab}+\frac{a^2b^2}{ab}+\frac{ab}{ab}=\frac{a^4}{b}+ab+1.$$

For the division of one polynomial expression by another see Part II.

Exercises in the division of general expressions of quantity:

1st. Divide a^6 by a? .... *Ans.* a^5.
2d. - - a^m by a^3? .... *Ans.* a^{m-3}.
3d. - - b^c by b^e? .... *Ans.* b^{c-e}.
4th. - - a^4 by a^3? .... *Ans.* a.
5th. - - a^4 by a^4? .... *Ans.* 1.
6th. - - $a^6b^4c^2$ by a^2bc? .... *Ans.* a^4b^3c.
7th. - - $a^3b^2+a^2bc+ab^2c^2$ by ab? .... *Ans.* $a^2b+ac+bc^2$.
8th. - - $ax^m+bx^{m-1}+cx^{m-2}+nx^2+px+q$ by x?

Ans. $ax^{m-1}+bx^{m-2}+cx^{m-3}+nx+p+\frac{q}{x}$.

Verifications of Multiplication and Division.

101. The product of two factors having been found, the accuracy of the result may be tested by making the multiplicand multiplier, and the multiplier multiplicand, and repeating the multiplication. Then, if both products agree, it is to be presumed that the result is accurate; for, as the combinations effected are different, it is unlikely that the same error can occur in both calculations.

Another verification of the accuracy of a product is supplied by division, for the product of the divisor and quotient is the dividend. Whence, if the product of two factors is made dividend, and one of the factors divisor, the quotient must be equal to the other factor.

As the process of division is more complicated than that of multiplication, this verification is liable to the objection of Article 44.

102. A convenient, though not an indispensable, preliminary to the verification of a quotient is the subtraction of the remainder (if there is one) from the dividend; the division is thus rendered exact.

Then, since the dividend is equal to the product of the divisor and quotient, if the divisor is multiplied by the quotient, and the product resulting from this multiplication is equal to the dividend (or the dividend less the remainder), it is to be presumed that the division has been accurately performed.

Another verification, without subtraction of the remainder, may be obtained by making the quotient divisor, and repeating the division. Then the number which before was divisor ought to be quotient now, and the remainder the same as before.

Another verification of multiplication and division is given, Article 131.

SECTION VI.

DIVISIBILITY OF NUMBERS. GREATEST COMMON MEASURE AND LEAST COMMON MULTIPLE OF TWO OR MORE NUMBERS.

103. A tentative method of decomposing a large number into factors has been explained in a preceding Article. The possibility of exactly dividing one number by another can always be determined by trial, and it is often the easiest way of ascertaining whether the division can be effected or not.

There are, however, tests by which the divisibility of numbers by certain divisors can be readily ascertained. Some of the most simple and elementary shall be given in the course of this Section.

As the research of the divisors of one number, the common divisors of two numbers, and the least common multiple of two or more numbers, depends on a common principle, it seems useful to take these subjects in connection with each other. Some new terms are necessary, of which the following are definitions:

a. One whole number is said to be exactly divisible by another whole number when there exists a third whole number, which, multiplied by the second, produces the first.

b. Every whole number which exactly divides another whole number is called a factor, divisor, measure, sub-multiple, or aliquot part of that number.

c. Every whole number which has no divisor, excepting unity, is called an absolute prime number, or, simply, a prime number.

d. Every whole number which exactly divides each of two other whole numbers is called a common divisor or common measure of these numbers.

e. Two whole numbers are said to be prime to each other when they have no common measure but unity.

f. From definitions c, e, it follows that a prime number, which is not a sub-multiple of another whole number, is prime with that other number, for they can have no common measure but unity.

g. Two whole numbers not prime to each other may have several common divisors. Taking, for example, 12 and 18, the numbers 2, 3, 6, are divisors common to both. Of these common divisors, or common measures, if there are several, one must be greater than the others. This is termed the Greatest Common Measure; which expression, for brevity, is sometimes written, g. c. m.

h. A whole number which is exactly divisible by another whole number is called a multiple of that by which it is divided; and a whole number which is exactly divisible by two or more numbers, a common multiple of these numbers.

i. The same divisors have many common multiples. For example, 12, 24, 36, ... are all common multiples of the numbers 2, 3, 4. Of these common multiples one must be the least of those which are exactly divisible by the given divisors; this is called their Least Common Multiple. The expression, least common multiple, is sometimes written in the abridged form, l. c. m.

k. To avoid prolixity of expression and ambiguity, the following abbreviations are also employed; viz.

N, N', N'' . . .	for numbers (N' is read N. one, &c.)
d, d', d'' . . .	for divisors or measures.
P, P', P'' . . .	for the parts into which a number is divided.
r, r', r'' . . .	for remainders.
q, q', q'' . . .	for quotients or factors of a number.
m, m', m'' . . .	for coefficients of a number.

104. Every number which measures another number measures also any multiple of that other number.

Let N be a multiple of d; mN, any multiple of N, is also a multiple of d.

Since N is a multiple of d, let $N=qd$;

Whence $mN=mqd=mq\times d$; that is, if d is contained q times in N, it is contained mq times in mN. Consequently, as m, q, are by hypothesis whole numbers, the proposition is established.

105. Every number which is decomposed into two parts, each divisible by a second number, is itself divisible by this second number.

Let a number, N, be divided into the parts P, P′; and let P, P′ be both divisible by d; N shall be divisible by d.

Since P is divisible by d, let $P=qd$.

„ P' „ d, „ $P'=q'd$.

Therefore $P+P'=qd+q'd=(q+q')d$ (Art. 70).

But $P+P'=N$; therefore $N=(q+q')d$.

Now q, q' are, by hypothesis, whole numbers; therefore the sum of q, q' is a whole number, and consequently d is a measure of N.

106. If a number is decomposed into two parts, every number which measures the whole and one of the parts must also measure the other part.

Let $N=P+P'$, and let N, P be each divisible by d; P' shall be divisible by d.

By hypothesis, $N=qd$ (q being a whole number),
and $P=q'd$ (q' being also a whole number);
let $P'=q''d$ (q'' being any quotient, whole or not);
then, since $N=P+P'$,

$$qd=q'd+q''d=(q'+q'')d.$$

Now q is a whole number; therefore the sum of q', q'' is a whole number.

But q' is also a whole number; therefore q'' must be a whole number; otherwise one whole number must be equal to another whole number and a fraction, which is absurd.

Therefore q'' is a whole number, and consequently P' is divisible by d.

107. The preceding principles are sufficient for the investigation of the greatest common measure of two numbers.

To apply them, let it be required to find the greatest common measure of the numbers 637 and 143.

This greatest common measure cannot exceed the less number 143; and seeing that 143 divides itself, if it also divides 637, it must be the greatest common measure sought.

Dividing 637 by 143, the quotient is 4, and the remainder 65.

Or, $637=143\times4+65$. Wherefore 143 is not the greatest common measure.

The greatest common measure of the numbers 637 and 143 is therefore less than 143.

Every number which divides 143 divides also 143×4, or 572 (Art. 104); and since the greatest common measure divides 637, it must also divide $637-143\times4$ or 65 (Art. 106). Whence the greatest common measure of 637 and 143 cannot be greater than that of 143 and 65.

Again, the greatest common measure of 143 and 65 dividing the two parts of 637 (viz. 4×143 and 65) must divide the whole number 637 (Art. 105); and being an exact divisor of 637 and 143, it cannot be greater than the greatest common measure of 637 and 143. Whence, since the greatest common measure of 637 and 143 is not greater than that of 143 and 65, and the greatest common measure of 143 and 65 is not greater than that of 637 and 143, it follows that they are equal.

Following with 143 and 65 the process and reasoning employed with 637 and 143, it is found that $143\div65=2+13$ remainder, and that the greatest common measure of 143 and 65 must be the same as that of 65 and 13.

The question is therefore reduced to the investigation of the greatest common measure of 65 and 13.

Dividing 65 by 13, an exact quotient is found, namely 5.

Whence, as 13 divides itself and 65, it is a common measure of 65 and 13.

Being also the greatest number by which 13 can be divided, it is the greatest common measure of 65 and 13.

But it has been shown that the greatest common measure of 65 and 13 is also the greatest common measure of 143 and 65, and that the greatest common measure of 143 and 65 is the greatest common measure of 637 and 143.

Therefore 13 is the greatest common measure of 637 and 143.

The calculation may be made as under:

```
143)637(4
    572
    ---
     65)143(2
        130
        ---
         13)65(5
            65
            --
             0
```

108. The following practical rule is deduced from Article 107.

To find the greatest common measure of two numbers, Divide the greater number by the less; if there is no remainder, the less number is the greatest common measure of the two numbers.

If there is a remainder, divide the less number by this remainder. If the division is exact, the first remainder is the greatest common measure.

But if the second division gives a remainder, divide the first remainder by the second. If this division is exact, the second remainder is the greatest common measure: if not, and so long as the division is not exact, continue to divide the last remainder but one by the last.

An exact quotient being obtained, the last divisor is the greatest common measure of the two given numbers.

If the last divisor is unity, the two numbers, having no common measure but unity, are prime to each other.

109. If the process for finding the greatest common measure be employed upon two numbers which are prime to each other, the last remainder found must be unity; for, from the nature of the process, each remainder is less than that which preceded it; and a remainder equal to zero cannot be obtained before a remainder equal to 1, for the divisor greater than 1 which gives a remainder $=0$, must be a common measure of the numbers; that is, of two numbers prime to each other, which is absurd.

Whence, in the case of two numbers prime to each other after more or fewer divisions, a remainder equal to unity must always be found.

110. When any remainder is a prime number, and not a measure of the preceding remainder, a remainder equal to 1 must be found (Art. 103 f), and the numbers proposed have no greatest common measure.

Note.—Two numbers may have only one common measure greater than unity. This is called their greatest common measure.

111. Every number which measures the product of two factors, and which is prime with one factor, must measure the other factor.

Let $NN'=md$ (m representing a whole number), and let N and d be prime to each other; then N' is measured by d.

To establish this principle, let it be supposed, first, that N[*] is greater than d, and let the process for finding the greatest common measure of N and d be employed.

Since N and d are, by hypothesis, prime to each other, this process must end in a remainder $=1$ (Art. 109).

The detail is contained in the first column of the subjoined table.

In a second column the equal quantities of the first are multiplied by the other factor N'. The products are equal; for if equal quantities are multiplied by the same quantity (or have the same quantity added to them) the results must be equal.

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[*] If N is less than d, the application of the greatest common measure is made by dividing d by N; in other respects the process does not differ from that given.

In a third column the equal quantities of the second are divided by the measure d. The results again are equal, for if equal quantities are divided by the same quantity (or have the same quantity subtracted from them) the results are equal.

$$\begin{array}{l|l|l}
N=qd+r & N'N=N'qd+N'r & \frac{N'N}{d}=N'q+\frac{N'r}{d} \\
d=q'r+r' & N'd=N'q'r+N'r' & N'=\frac{N'q'r}{d}+\frac{N'r'}{d} \\
r=q''r'+r'' & N'r=N'q''r'+N'r'' & \frac{N'r}{d}=\frac{N'q'r'}{d}+\frac{N'r''}{d} \\
r'=q'''r''+r''' & N'r'=N'q'''r''+N'r''' & \frac{N'r'}{d}=\frac{N'q'''r''}{d}+\frac{N'r'''}{d} \\
\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot
\end{array}$$

Considering now the quantities in the third column;

$\frac{N'N}{d}$ is a whole number, by hypothesis.

$N'q$ is a whole number; since N' and q are both whole numbers, therefore (Art. 106) $\frac{N'r}{d}$ is also a whole number.

Again, N' is a whole number.

$\frac{N'q'r}{d}$ is a whole number, being the product of $\frac{N'r}{d}$ by q', a whole number; therefore (Art. 106) $\frac{N'r'}{d}$ is a whole number.

In the same manner it may be shown that the quantities $\frac{N'r''}{d}, \frac{N'r'''}{d}$ . . . . are whole numbers.

And since in the succession of remainders, r, r', r'', r''', ... a remainder$=1$ must be found, (Art. 109); it follows that $\frac{N'\times 1}{d}$ is a whole number, and, by consequence, that N' is measured by d.

112. Every absolute prime number which measures the product of two factors must measure one of the factors.

Let d be an absolute prime number which divides the product NN'; d must divide either N or N'.

For if d does not divide N it is necessarily prime with that factor (Art. 103 e), and therefore it must divide the other factor N' (Art. 111).

113. Every absolute prime number which divides N^2, or in general any power, N^m, of N must divide N.

For $N^2=N\times N$. Now, every absolute prime number which measures $N\times N$ must divide one of the factors (Art. 112). For the same reason every absolute prime number which divides N^3 must divide N or N^2; and to divide N^2, it must divide N.

This proof may be extended to any power of N.

114. Every number which is prime with both the factors of a product is also prime with the product.

Let d be prime to each of the numbers N, N'; d shall be prime to the product N, N'.

For any absolute prime number which divides d and $N\ N'$ must divide either N or N' (Art. 112). Whence d and N, or d and N', cannot be prime to each other, which is contrary to the hypothesis.

115. The product of many factors, N, N', N''. . . . . can contain no prime factors which are not contained in some of the numbers N, N', N''. . . . .

For every prime number which divides $N\ N'\ N''$, but does not divide N'', must divide $N\ N'$ (Art. 112), and every prime number which divides $N\ N'$ must divide N or N'.

Whence, if a number is formed by the continual multiplication of many factors, it cannot be equal to the product of other numbers containing prime factors different from those which enter into the numbers already multiplied together.

116. Every number which can be divided by two or more numbers prime to each other is also divisible by the product of these prime numbers.

Let N be divisible by d, d', d''. . . . which are prime to each other; N is divisible by the product $dd'd''$. . . . .

Since d divides N, $N=dq$ (q being a whole number); but d' divides N; therefore d' divides dq, and d', d are prime to each other; therefore d' divides q (Art. 111).

Again, since d' divides q, let $q=d'q'$ (q' being a whole number); therefore N, which is equal to dq, is also equal to $dd'q'=dd'\times q'$; whence N is divisible by dd'.

Similarly, since d'' divides N, it therefore divides $dd'q'$; but d'' is prime with d, d'; and therefore with $d\times d'$; whence d'' divides q'.

Let $q'=d''q''$ (q'' being a whole number); therefore $N=dd'd''q''=dd'd''\times q''$; whence N is divisible by $dd'd''$.

The proof is the same when the divisors are more in number than 3.

117. Let d, d', d''. . . . . be numbers which are prime to each other.

Let $d\times d\times d$ . . . . or d^n
$d'\times d'\times d'\times d'$ . . . or $d'^{n'}$
$d''\times d''\times d''\times d''\times d''$ . . . or $d''^{n''}$ be factors which are multiplied together; and let their product be N.

The number N is divisible by $d^n\times d'^{n'}\times d''^{n''}$ . . . .; it is also divisible by all the products which can be formed by multiplying together two by two, three by three, &c., the different powers of d, d', d''. . . . between the first power and that marked by n for d, by n' for d', by n'' for d'', &c.

Since d, d', d'' . . . . . are prime to each other.
d^2, d'^2, d''^2 . . . . .
d^3, d'^3, d''^3 . . . . .
. . . . .
d^n, $d'^{n'}$, $d''^{n''}$. . . . . are prime to each other (Art. 113); consequently N, which is the product of d^n, $d'^{n'}$, $d''^{n''}$. . . . must be divisible by these numbers, or by the products of any of them taken two by two, three by three, &c., in manner aforesaid; for these products are evidently submultiples of the entire product $d^n\times d'^{n'}\times d''^{n''}$ . . . ., which is equal to N.

118. The possibility of exactly dividing one number by another can always be determined by trial. It may, however, in the case of some numbers, be ascertained more easily by certain tests than by actual division.

To investigate these tests this principle (established Art. 105) is required, namely,

Every number which is decomposed into two parts, each divisible by a second number, is itself divisible by the second number.

To which must be subjoined the following:

That if $N=P+P'$, of which d divides P, but not P', then d does not divide N; and the remainder from the division of N by d is equal to that from the division of P' by d.

Since $N=P+P'$; $\frac{N}{d}=\frac{P}{d}+\frac{P'}{d}$ (see Art. 111).

Now, by hypothesis $\frac{P}{d}$ is a whole number; but

$\frac{P'}{d}$ is not a whole number.

Whence $\frac{P}{d}+\frac{P'}{d}$ or $\frac{N}{d}$ cannot be a whole number.

Next, to prove that the remainder from the division $\frac{P'}{d}$ is equal to that from the division $\frac{N}{d}$.

P being divisible by d, $P=dq$ (q being a whole number).

P′ being not divisible by d, $P=dq'+r$ (q' being a whole number).

Therefore $P+P'=dq+dq'+r=d(q+q')+r$

But $P+P'=N$.

Whence $N=d(q+q')+r$.

That is, N divided by d gives for quotient $q+q'$, and for remainder r; which r is the remainder from the division of P′ by d.

Tests of the divisibility of numbers by certain divisors.

119. Every number whose last figure is 0, 2, 4, 6, or 8, is divisible by 2.

Separating the number into two parts, namely, the figure expressing simple units, and the number to the left of that figure, the second part (regard being had to its relative value) is some multiple of 10, and therefore divisible by 2; and since each of the figures 0, 2, 4, 6, 8, is also divisible by 2, it follows (Art. 105) that the whole number can be divided by 2.

A number ending with one of the figures 1, 3, 5, 7, 9, is not divisible by 2.

For the number to the left of the last figure, considered with respect to its relative value, is divisible by 2; but the other part, 1, 3, 5, &c. is not (Art. 118).

Numbers divisible by 2 are named even; and numbers which cannot be divided by 2, odd.

120. A number can be divided by 3 when the sum of the digits of the number is divisible by 3.

1 in the place of tens $=$ $9+1$
1 in the place of hundreds $=$ $99+1$
1 in the place of thousands $=$ $999+1$

And, generally, 1 in any place (except the place of simple units) is equal to $1+$ a number expressed by 9 repeated as many times as the 1 has zeros after it.

Whence any other figure, m, $=m\times$ the proper number of nines $+m\times 1$, or the figure m itself regarded as simple units.

Now, any number 9, 99, 999 . . . is a multiple of 3.

Therefore $m\times 999$ . . . . is a multiple of 3 (Art. 104).

This reduction into a number expressed by a multiple of 9, 99, 999, . . . + the significant figure of the order, considered as simple units, may be made for every figure which is contained in a number.

Whence any number can be decomposed into two parts; the one consisting of multiples of 9, 99, 999, . . . ; the other of the sum of the significant figures of the number, regarded as simple units.

The multiples of 9, 99, 999 . . . . are divisible by 3.

Wherefore if the sum of the digits can be divided by 3, the number is itself divisible by 3; but if not, not (Art. 118).

121. Every number is divisible by 4 when the number composed of its two last figures can be divided by 4.

For the number may be separated into two parts; one, in which the places of tens and simple units are occupied by zeros; the other, the number composed of the tens and simple units.

The first part is a multiple of 100, and it is divisible by 4, since 100 is so (Art. 104).

Whence, if the second part, composed of the two last figures taken with their relative values can also be divided by 4, the whole number must be divisible by 4 (Art. 105).

122. Every number of which the last figure is 0 or 5, is divisible by 5.

Let the number be decomposed into parts, the first a multiple of 10, the second a figure expressing simple units.

The part which is a multiple of 10 is divisible by 5; and the other part being 0 or 5, the division of this other part by 5 leaves no remainder.

Whence (Art. 105) every number ending in 0 or 5 is divisible by 5.

If the last figure of a number is not 0 or 5 the number cannot be measured by 5 (Art. 118).

123. Every even number which can be divided by 3 is divisible by 6.

For every even number is divisible by 2 (Art. 119).

The even number which can be divided by 3 is therefore divisible also by 2; and since the divisors 2, 3, are prime numbers, the number which can be divided by 2 and by 3 is divisible by 2×3, or 6 (Art. 117).

124. To find the conditions of divisibility of any number by 7.

Any number, for example 13875964, may be decomposed into the several orders of units of which it consists.

And the several orders of units may be again decomposed into the product, for each, of the significant figure of the order by unity followed by the number of zeros required to give that significant figure its relative value. Thus,

$$\begin{aligned} 4&=4\times 1\\ 60&=6\times 10\\ 900&=9\times 100\\ 5000&=5\times 1000\\ 70000&=7\times 10000\\ 800000&=8\times 100000\\ 3000000&=3\times 1000000\\ 10000000&=1\times 10000000 \end{aligned}$$

The 1, 10, 100, 1000, &c. represent 1 of each order of units contained in the proposed number, and the whole number is composed of multiples of these different orders of units. Wherefore, if the conditions of divisibility of these units can be determined, the same conditions will be applicable to the whole number (Art. 118).

Now, a unit of each order may be represented by some multiple of 7, augmented or diminished by the quantity required to reproduce this unit by its combination with the multiple of 7.

Whence again, if the conditions of divisibility of the quantities by which the multiples of 7 are to be augmented or diminished can be determined, those of the whole number will be found.

				A	B
Now	1=	0×7+1	. and 4=4×	0×7	+4×1
	10=	1×7+3	. . . 60=6×	1×7	+6×3
	100=	14×7+2	. . . 900=9×	14×7	+9×2
	1000=	143×7−1	. . 5000=5×	143×7	−5×1
	10000=	1429×7−3	. 70000=7×	1429×7	−7×3
	100000=	14286×7−2	. 800000=8×	14286×7	−7×2
	1000000=	142857×7+1	. 3000000=3×	142857×7	+3×1
	10000000=	1428571×7+3	. 10000000=1×	1428571×7	+1×3

In the first part of the preceding table 1 of each order of units contained in the given number is divided by 7, and in the second part the results are multiplied by the significant figures of the respective orders of units.

0, 1, 14, the quotients of 1, 10, 100 by 7 are taken, so that the products 0×7, 1×7, 14×7 are less than 1, 10, 100; whence, in reconstructing the original number by the combination of all its parts, the remainders 1, 3, 2, or rather their products by the significant figures 4, 6, 9, must be added to the sum of the other parts.

But the quotients 143, 1429, 14286 resulting from the division of 1000, 10000, 100000 by 7 are taken in such manner that the products 143×7, 1429×7, 14286×7 are greater than 1000, 10000, 100000, respectively.

Whence, in recomposing the original number, the remainders —1, —3, —2, or their products by the corresponding significant figures, must be subtracted from the sum of the other parts.

Proceeding in this manner, it is found that a ternary arrangement of the remainders can be made, and that before multiplication by the significant figures of the respective orders of units the remainders are 1, 3, 2 to be added; 1, 3, 2 to be subtracted; 1, 3, 2 to be added; 1, 3, 2, to be subtracted, and so on alternately.

Now, any number decomposed as the preceding, may be considered as composed of two parts represented in the last table by columns A, B.

Column A, being composed of multiples of 7, can be divided by 7.

In column B let the numbers which are to be added to the other parts be formed into one sum, and those which are to be subtracted from the other parts into another sum. Then, if the difference of these two sums is either equal to zero, or is a multiple of 7, the whole number is divisible by 7 (Art. 105).

Whence, in conclusion, if the numbers 1, 3, 2 are multiplied respectively by the 1st, 2d, 3d; 7th, 8th, 9th; 13th, 14th, 15th . . . . digits of a given number, and the products are added together; also, if the same numbers 1, 3, 2 are multiplied by the 4th, 5th, 6th; 10th, 11th, 12th; 16th, 17th, 18th . . . . digits of the same number, and these products are added together; then, if the difference of these results is either 0, or a multiple of 7, the number proposed is divisible by 7.

It seems scarcely necessary to add, that this investigation is more curious than useful, since in practice it is much easier to divide any number by 7 than to apply the test.

125. If the three last figures of a number, considered with respect to their relative value, can be divided by 8, the number is itself divisible by 8.

For the number may be broken into two parts; the one, that in which the three last figures are replaced by zeros; the other, the number expressed by the three last figures.

The first part is a multiple of 1000; and since 1000 (=125×8) is divisible by 8, the multiple of 1000 is divisible by 8 (Art. 104).

Whence, if the second part, the number expressed by the three last figures, can be divided by 8, the whole number must be divisible by 8 (Art. 105).

Since 1000 is a multiple of 25, if the number expressed by the three last figures can be divided by 25, the whole number is divisible by 25.

126. Every number, the sum of whose digits can be divided by 9, is itself divisible by 9.

For every number can be decomposed into two parts, the one composed of multiples of 9, 99, 999, ... and the other, of the sum of the digits of the number.

The multiples of 9, 99, 999, ... are divisible by 9. Whence, if the sum also of the digits can be divided by 9, the whole number is divisible by 9 (Art. 105).

127. Since any number can be divided into two parts of which the one is divisible by 9, if the other part, which is the sum of the digits of the number, cannot be divided by 9, the whole number is not divisible by 9, and the remainder from the division of the sum of the digits by 9 is equal to the remainder from the division of the whole number by 9 (Art. 118).

128. Every number which ends in 0 is divisible by 10. This is self-evident.

The tests of the divisibility of numbers by the prime numbers 11, 13, 17, &c., might be in like manner investigated; but the investigations depend on principles which are not elementary, and (as in the case of the prime number 7) the application of the test is more tedious and difficult than the actual trial by division.

129. The following are more simple:

a. A number is divisible by 18, when that number is even, and the sum of its digits can be divided by 9; for under these conditions the number can be divided by 2 and by 9, and consequently by 2×9, or 18 (Art. 117).

b. A number is divisible by 12 or by 36 when the two last figures, taken with respect to their relative value, can be divided by 4, and the sum of the digits of the number by 3 or by 9.

For in the one case the number is divisible by 4 and by 3, and therefore by 4×3, or 12; in the other it is divisible by 4 and by 9, and therefore by 4×9, or 36.

c. A number is divisible by 15 or by 45 when the last figure is 0 or 5, and the sum of the digits of the number is divisible by 3 or by 9.

For, in the first instance, the number can be divided by 5 and by 3, and consequently by their product, 15; in the second it is divisible by 5 and by 9, and therefore by their product, 45.

130. The prime numbers between 1 and 100 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

131. From the properties of the number 9, established in Articles 126 and 127, are derived easy verifications of multiplication and division.

Let N, N′ be two numbers whose product is to be found.

Let N be divided by 9, and let the quotient be q, and the remainder r.
N' . 9 . . q' . . r'.

$$\text{Whence } \left.\begin{array}{l} N=9\times q+r \\ \underline{N'=9\times q'+r'} \end{array}\right\} \text{multiplying these quantities (Art. 76).}$$

$$N N'=9\times9\times q\times q'+9\times q'\times r+9\times q\times r'+r\times r'.$$

The product NN′ is composed of four partial products. Three of these products are multiples of 9, therefore their sum is a multiple of 9 (Art. 105).

Whence NN′ is composed of two parts, of which the one is divisible by 9. If the other part, $r\times r'$, can be divided by 9, then NN′ is also divisible by 9 (Art. 126); but if $r\times r'$ is not divisible by 9, neither is NN′; and the remainder from the division of $r\times r'$ by 9 is equal to the remainder from the division of NN′ by 9. Also, the remainders from the division of NN′ and rr' by 9 are equal respectively to the remainders left from the division of the sum of the digits of NN′, rr' by 9 (Art. 127).

Whence N, N′, being numbers which are to be multiplied together, if the sum of the digits of N, N′ are respectively divided by 9, and if the remainders r, r' are multiplied together,' and if the sum of the digits of the product $r\times r'$ is divided by 9, the remainder from this division of the sum of the digits of $r\times r'$ by 9 is equal to the remainder obtained from the division of the sum of the digits of the product NN′ by 9.

When r, or r', or $r\times r'$, is exactly divisible by 9, then the product NN′ is also exactly divisible by 9.

These principles afford a simple means of verifying the results of multiplication and division.

The manner of conducting the verification will be most easily understood from an instance of its application:

Let the factors and product of an example in multiplication be 378482, 3548, and 1342854136.

The sum of the digits is formed and the division by 9 effected simultaneously thus, beginning at the left of the multiplicand, 378482;

Division of the sum of the digits of the multiplicand by 9,

$3+7=10$, $10+9=1+1$ rem.

The quotients resulting from the division by 9 are omitted, and the remainders alone preserved:

1 rem.$+8=9$, $9\div 9=1+0$ rem.
0 rem.$+4=4$, $0+4+8=12$, $12\div 9=1+3$ rem.
3 rem.$+2=5$, $5\div 9=0+5$ rem. Whence $r=5$.

Division of the sum of the digits of the multiplier by 9 ·

$3+5=8$, $8+4=12$, $12\div 9=1+3$ rem.
$3+8=11$, $11\div 9=1+2$ rem. Whence $r'=2$.
Therefore $r\times r'=5\times 2=10$
And $1+0=1$, $1\div 9=0+1$ rem.

The remainder from the division of the sum of the digits of $r\times r'$ is consequently 1.

Division of the sum of the digits of the product by 9:

$1+3+4+2=10$, $10\div 9=1+1$ rem.
1 rem.$+8=9$, $9\div 9=1+0$ rem.
0 rem.$+5+4=9$, $9\div 9=1+0$ rem.
0 rem.$+1+3+6=10$, $10\div 9=1+1$ rem.

The remainder from the division of the sum of the digits of NN′ is 1; but the remainder from the division of the sum of the digits of $r\times r'$ is also 1; whence the product is to be presumed accurate.

When the remainder from the division of the sum of the digits of the product $r\times r'$ is not the same as the remainder from the division of the sum of the digits of the product $N\times N'$, the calculation ought to be revised, being probably erroneous.

In the case of division; when the quotient is exact, that is, when $D=d\times q$, d, q, D, are the representatives of N,N′ and $N\times N'$ respectively, and the process of verification is precisely like that already detailed.

But when the quotient is not exact, that is, when $D=d\times q+R$, before proceeding with the verification, it is necessary to subtract the remainder R from the dividend D.

The dividend is thus rendered the exact product of the divisor and quotient, and this case reduced to the preceding.

It is evident that the figure 9 may hold the place of 0, and conversely; also that the order of the figures in NN′ may be changed without changing the sum of the digits, and consequently without changing the remainder left from the division of that sum by 9.

For these reasons (and others might be adduced) the verifications of multiplication and division by this method are not so deserving of confidence as those already given in Articles 101 and 102.

132. A number which is exactly divisible by any other number greater than unity is said to be a composite number. The terms Prime and Composite are, hence, opposed to each other.

133. From the preceding remarks on the conditions of the divisibility of numbers, it is certain that no even number, excepting 2, can be a prime number (Art. 119). Among odd numbers the multiples of 3 and 5 are discovered by the tests of Articles 119 and 120. Therefore, in the research of prime numbers, it is not necessary to try any excepting such odd numbers as have none of these characters of divisibility.

The limits within which it is necessary to discover by trial whether a given number is prime or composite admit of another contraction.

If any number is divided by 6, a certain quotient, q, is found, and a remainder of 1, 2, 3, 4, 5, or 0; therefore all numbers may be considered as multiples of 6 with one of the remainders 1, 2, 3, 4, or 5.

Now, a multiple of 6 with the remainder 2 or 4 is divisible by 2, and a multiple of 6 with the remainder 3 is divisible by 3. Whence the prime numbers must all be found amongst multiples of $6+1$ and multiples of $6+5$.

These multiples of 6 may be represented by the expressions,

$$\left.\begin{array}{r}6\times q+1\\ \text{and }6\times q+5\end{array}\right\}\text{A.}$$

But 5 is equal to 6 diminished by 1, or $5=6-1$. Therefore $6\times q+5$ may be otherwise expressed,

$$6\times q+6-1$$
$$\text{or } q \text{ times } 6+1 \text{ time } 6-1$$
$$\text{or } (q+1)\times 6-1.$$

And since q denotes a number, that number and 1 may be added together. Let $q+1=q'$,

then $6\times q+5$ becomes

$$6\times q'-1.$$

And the expressions A become

$$\left.\begin{array}{r}6\times q+1\\ 6\times q'-1\end{array}\right\}\text{B.}$$

Since the conclusions deduced in this Article depend on the condition that $6\times q$ and $6\times q'$ are multiples of 6, without reference to the particular values of q, q', these conclusions must be true when q, q' are equal. Whence both of the expressions B may be put under the form,

$$6q\pm 1.^*$$

This conclusion, expressed in words, is: All prime numbers greater than 3 have this property, that if they are augmented or diminished by unity, the results are divisible by 6.

133. But although all prime numbers are comprehended in the formula $6q\pm 1$ (q being any whole number whatever), yet all the numbers expressed by $6q\pm 1$ are not prime numbers.

For the expressions, $6\times q+1$, $6\times q+3$, $6\times q+5$, evidently comprehend all odd numbers greater than 5. Of these, $6\times q+3$, which expresses multiples of 3, is not included in the general formula $6\times q\pm 1$. This formula, therefore, comprehends all odd numbers which are not multiples of 3.

Now, the product of any two prime numbers greater than 2 or 3 is an odd number which cannot be divided by 3 (Art. 114). This product being an odd number, not a multiple of 3, is expressed by the formula $6\times q\pm 1$, and it is not a prime number.

134. To discover whether a number expressed by the formula $6\times q\pm 1$ is an absolute prime number, it is necessary to attempt its division by all the prime numbers, from 7 inclusive to that which is immediately inferior to the half of the given number. Beyond half the number it is useless to extend the trial, as no number can be exactly divided by a divisor greater than its half.

135. The manner of distinguishing prime from composite numbers having been explained, let it be next required to find all the prime factors of any given number.

To give generality to the results, let N represent the given number and a the least prime number by which N is divisible; and let the division of N and the successive quotients by a be repeated until after n divisions a quotient, N′, not divisible by a is found.

$$\text{Then } N=a^n\times N'.$$

Since every prime number, other than a, which divides N, must divide N′ (Art. 111), the investigation of the prime factors of N, different from a, is reduced to that of the prime factors of N′.

Let b represent the least prime number by which N′ can be divided, n' the number of divisions possible by b, and N″ the quotient not divisible by b.

$$\text{Then } N'=b^{n'}\times N'',$$
$$\text{and } N=a^n\times N'=a^n\times b^{n'}\times N''.$$

Again, every prime factor, except a, b, which divides N must divide N″.

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[*] A general expression such as $6\times q\pm 1$ is termed a Formula.

Let c be the prime number which divides N''; let the division of N'' by c be n'' times possible; and the quotient not divisible by c, N'''.

$$\text{Then } N''=c^{n''}\times N''',$$

$$\text{and } N=a^{n}\times b^{n'}\times N''=a^{n}\times b^{n'}\times c^{n''}N'''.$$

Continuing this process, a prime number, or some power of a prime number, must be at length obtained for quotient. Dividing as often as possible by this prime number, a last quotient equal to unity is found.

Let it be supposed that $N'''=d^{n'''}$.

$$\text{Then } N=a^{n}\times b^{n'}\times c^{n''}\times N'''=a^{n}\times b^{n'}\times c^{n''}\times d^{n'''}.$$

And a, b, c, d are the only prime factors which can enter into the number N (Art. 115). This being accomplished, N is said to be decomposed into its simple factors.

136. As a particular example of the decomposition of a number into simple factors, let it be required to find the simple factors of the number 105840.

Calculation:

$$\begin{array}{r@{\,)\,}r}
2 & \underline{105840} \\
2 & \underline{52920} \\
2 & \underline{26460} \\
2 & \underline{13230} \\
3 & \underline{6615} \\
3 & \underline{2205} \\
3 & \underline{735} \\
5 & \underline{245} \\
7 & \underline{49} \\
7 & \underline{7} \\
\multicolumn{2}{r}{1}
\end{array}$$

Explanation: the given number, 105840, ends in 0, therefore it is divisible by 2 (Art. 119).

For the same reason the first, second, and third quotients are divisible by 2. The fourth quotient, 6615, being an odd number, is not divisible by 2.

Whence 105840 is divisible by $2\times2\times2\times2=2^4$,

And $105840=2^4\times6615$.

The sum of the digits of 6615 being divisible by 3, 6615 is divisible by 3 (Art. 120).

For the same reasons the fifth quotient, 2205, and the sixth, 735, are divisible by 3; but the seventh, 245, is not.

Whence 6615 is divisible by $3\times3\times3=3^3$.

And $105840=2^4\times6615=2^4\times3^3\times245$.

The last figure of 245 being 5, 245 is divisible by 5 (Art. 122); but the 8th quotient, 49, ending neither in 5 nor 0, is not divisible by 5.

Whence $245=5\times49$,

And $105840=2^4\times3^3\times245=2^4\times3^3\times5\times49$.

The next prime number is 7, attempting the division of 49 by 7; $49\div7=7$, and $7\div7=1$.

Whence $49=7\times7=7^2$,

And $105840=2^4\times3^3\times5\times49=2^4\times3^3\times5\times7^2$.

2, 3, 5, 7, are therefore the simple factors of the given number, 105840.

137. When two numbers, N, N′, have been reduced to expressions composed of the products of their simple factors, if none of the prime factors which enter into the one number are found in the other, the numbers are prime to each other (Art. 114).

If one prime factor, a, is common to N, N′, the numbers are both divisible by this factor, which is the only, and therefore the greatest, common measure of them both.

If the prime factor, a, is contained 2, 3, 4 . . . . or m times in each, then a^2, a^3, a^4. . . . . or a^m is the greatest common measure of N, N′.

But if a is contained m times in N and n times in N′, n being less than m, a^n is the greatest common measure of N, N′.

For, let $N=a^m\times q$, and $N'=a^n\times q'$. Then, since a, q, q' are prime to each other, the numbers N, N′ can contain no common prime factor excepting a.

Now $a^m\times q$ is divisible by a^n, since by hypothesis the factor a enters into the product expressed by a^m oftener than into that expressed by a^n; and $a^n\times q'$ is a multiple of a^n. Whence $a^m\times q$ or N, and $a^n\times q'$ or N′, are both multiples of a^n; but a^n is the greatest number by which a^n can itself be divided: whence it is evident that a^n is the greatest common measure of N, N′.

If $N=a^m b^{m'}\times q''$ and $N'=a^n b^{n'}\times q'''$, in which expressions a, b, q'', q''' are prime to each other, and m, m' are greater respectively than n, n', it may in like manner be shown that $a^n b^{n'}$ is the greatest common measure of the numbers N, N′: also that if m is greater than n, but m' less than n', then $a^n b^{m'}$ is the greatest common measure of N, N′.

If another simple factor, c, raised to the powers $c^{m''}$, $c^{n''}$, of which n'' is less than m'', is contained in N, N′ respectively, it follows, for the reasons already given, that $a^n b^{n'} c^{n''}$ is the greatest common measure of the numbers N, N′.

This reasoning is as applicable to all the simple factors common to two given numbers as it is to one, or two, or three such factors. Whence the greatest common measure of two numbers is equal to the product of all the prime factors common to the two numbers raised each to the lower of the two powers with which it is affected in either of the given numbers.

138. If a^m is written $a\times a\times a\times a$ . . . to m factors,
a^n . . . $a\times a\times a$ . . . . . to n factors,
and the same is done with $b^{m'}$, $b^{n'}$, $c^{m''}$, $c^{n''}$, &c. &c.,
so that all the repetitions of all the simple factors common to N, N′ are put in evidence, the conclusion just obtained may be more simply expressed thus: The greatest common measure of two numbers is equal to the product of all the simple factors common to the two numbers.

139. In the same manner it may be proved that the greatest common measure of three numbers is composed of the product of all the prime factors common to the three numbers; that of four numbers to the product of all the prime factors common to the four numbers, &c. &c.

140. If two numbers, N, N′, are multiplied together, the product $N\times N'$ is a multiple of both the numbers; and if N, N′ are decomposed into their simple factors, the product of all the factors being equal to $N\times N'$ is also a multiple of both the numbers.

When the simple factors into which N is decomposed are all different from those into which N′ is decomposed, it is evident that the product $N\times N'$ is the least number which can be divided by both the numbers N, N′ (Art. 115), or that it is the least common multiple of these numbers.

But when N, N′ have one or more common prime factors, the least common multiple of N and N′ is less than the product $N\times N'$.

Let N be resolved into the product of the prime factors a, b, and N′ be resolved into the product of the prime factors a, c.

Then, since N divides any multiple of ab, it divides $ab\times c$,
and since N′ divides any multiple of ac, it divides $ac\times b$.

Whence abc or acb is a common multiple of N and N′. This multiple of N, N′ is less than $N\times N'$ or $aabc$. Also, since abc contains only the prime factors which enter into N, N′, and since, to be capable of division by N, N′, the number abc must contain every prime factor found in N, N′, it follows that abc is the least common multiple of N, N′.

Next, let N contain the prime factors aab or a^2b,
and N′ . . . . ac:

Then, since N divides any multiple of a^2b, it divides $a^2b \times c$,
and since N′ divides any multiple of ac, it divides $ac \times ab$.

Whence $a^2b \times c = ac \times ab = a^2bc$ is a common multiple of N, N′; and since it contains every prime factor found in N, N′, and no more, it is the least common multiple of N, N′.

Since the product of the prime factors abc cannot be divided by a^2, or the product a^nbc by a^m, when m is greater than n, it follows that the highest power of each prime factor common to two numbers must be a factor of the least common multiple of these numbers.

Again, since the product of the prime factors a^2b cannot be divided by the prime factor c, or the product of ac by b (Art. 114), it is also evident that all the prime factors which are not common to N, N′ must enter into the least common multiple of N, N′.

That which has been proved in the case of a, a^2, a^m, a^n, being equally true of a second common prime factor, b, b^2, $b^{m'}$, $b^{n'}$, a third common prime factor, c, c^2, $c^{m''}$, $c^{n''}$, &c., it follows, that to form the least common multiple of two numbers it is necessary to decompose the numbers into their prime factors, and to multiply together all the prime factors not common to the two numbers and the highest powers of the common prime factors. The product thus obtained is the multiple required.

141. For the same reasons, if N, N′, N″. . . . are decomposed into prime factors, the least common multiple of N, N′, N″. . . . . is equal to a product of which all the prime factors not common to the numbers raised to their respective powers, and the prime factors common to the numbers raised to the highest powers which they obtain in any of the numbers N, N′, N″. . . . . are the several factors.

142. Instead of denoting the repetitions of any simple factor by an exponent, let every factor be put in evidence, thus,

Let $N = aabbce$ and $N' = aaabcc$,
or, $N = aabc \times be$ and $N' = aabc \times ac$.

Then, since N divides $aabc \times be$, it also divides $aabc \times be \times ac$;
And since N′ divides $aabc \times ac$, it divides $aabc \times ac \times be$.

Whence $aabc \times be \times ac$, or $aabc \times ac \times be$, or $aabc \times abce$, is a common multiple of N and N′.

Again, to render $aabc \times be$ or N capable of division by $aabc \times ac$ or N′, it is necessary to multiply $aabc \times be$ by ac; and to render $aabc \times ac$ capable of division by $aabc \times be$, it is necessary to multiply $aabc \times be$ by ac. Whence it follows, that $aabc \times abce$ is the least number which is exactly divisible by $aabc \times be$ and $aabc \times ac$; that is, $aabc \times abce$ is the least common multiple of the numbers N, N′.

The factors of the least common multiple of N, N′ are,

1st. The simple factors which are common to N and N′, taken once only.
2d. The simple factors contained in N, and not in N′.
3d. The simple factors contained in N′, and not in N.

Whence, if two numbers are decomposed into simple factors, and if all the simple factors which are common to both the numbers are cancelled in one of them, the continual product of the remaining simple factors is the least common multiple of the two numbers.

The product of all the simple factors common to N, N′, is the least common multiple of N, N′: let this be m.

$$\text{Then, } \frac{N}{m} = \frac{aabbce}{aabc} = be : \text{ let } be = q.$$

$$\frac{N'}{m} = \frac{aaabcc}{aabc} = ac : \text{ let } ac = q'.$$

Consequently $aabc \times be \times ac = mqq'$; and hence, if two given numbers are divided by their greatest common measure, the product of the greatest common measure and the two quotients is the least common multiple of the given numbers.*

143. When the quotient of an example in division is exact, that is, when $D = d \times q$, all the prime factors of d must be found in D (Art. 117).

Whence, if D, d are decomposed into prime factors, and those which are contained in d are cancelled in D, the product of the remaining prime factors of D must be equal to q.

Therefore, when $D = d \times q$, the operation of division may be performed by decomposing the dividend and divisor into their prime factors.

SECTION VI.

FRACTIONS.

144. In the division of one whole number by another, if the dividend is not a multiple of the divisor, the process ends in a remainder less than the divisor (Art. 82).

The remainder in such cases is written over, and the divisor under, the same horizontal line, to indicate that the former ought to be divided by the latter.

This expression of the quotient of an unexecuted division is termed a Fraction.

A fraction forms part of the quotient of every inexact division; and when the dividend is less than the divisor, it is the entire quotient.

Any dividend and divisor may be expressed fractionally, and any whole number may be represented in the fractional form by writing 1 in the place of the divisor.

145. Resuming the consideration of $\frac{3}{8}$, the fractional expression which results from the division of 43 by 8 (Art. 81), the 3 is a whole which it is required to divide into 8 equal parts, and 1 of these parts is the quotient. This part, or quotient, must contain unity as often as the dividend contains the divisor (Art. 79). But in all cases like the present the dividend is less than the divisor; the quotient is therefore less than unity. Consequently, some expedient must be devised in order to find a number which shall have to unity that relation which the dividend has to the divisor, or which shall form the same part of unity as the dividend forms of the divisor.

Let the divisor 8 be considered as one whole or unit composed of as many equal parts as it contains simple units, namely, of 8 equal parts.

Then	1	of these equal parts is	one eighth of the whole,	written,	$\frac{1}{8}$.
-	2	-	are two eighths	-	$\frac{2}{8}$.
-	3	-	three eighths	-	$\frac{3}{8}$.
-	4	-	four eighths	-	$\frac{4}{8}$.
-	5	-	five eighths	-	$\frac{5}{8}$.
-	6	-	six eighths	-	$\frac{6}{8}$.
-	7	-	seven eighths	-	$\frac{7}{8}$.
-	8	-	eight eighths	-	$\frac{8}{8} = 1$.

Now, the dividend, 3, is composed of three simple units or parts, each equal to the parts into which the divisor, taken as a whole, has been decomposed; and three parts of the divisor form three eighths of that unit of which the divisor is a whole. Whence the quotient of 3 divided by 8 is equal to $\frac{3}{8}$ of the quotient of 8 divided by 8; that is, it is equal to three eighths of one unit of the quotient, or three eighths of unity.

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* The least common multiple of three or more given numbers may be obtained in this manner: by finding, first, the least common multiple of two of the numbers, then the least common multiple of that least common multiple and a third number, &c. &c.

146. Generally, let $\frac{m}{n}$ be the expression of the quotient of any unexecuted division: if $m=n$, the quotient is unity; if m is greater that n, but not a multiple of it, the process of division may be employed till a remainder less than n is obtained. Whence it may always be assumed that m is less than n.

Let n be considered as a whole, composed of as many equal parts as it contains simple units, thus,

1 simple unit is one nth part of n taken as a whole: this is written $\frac{1}{n}$

2 simple units are two nth parts of n taken as a whole: this is written $\frac{2}{n}$

3 - - three nth - - - - - $\frac{3}{n}$

. . . .

$n-1$ simple units are $n-1$ nth parts of n taken as a whole, written $\frac{n-1}{n}$

n simple units are n nth parts of the whole n . . . . . . $\frac{n}{n}$, or 1.

Whence the quotient of $1\div n$ is equal to one nth part of one simple unit of the quotient resulting from the division of any number by n.

The quotient of $2\div n$ is equal to two nth parts of one simple unit of the same quotient.

The quotient of $(n-1)\div n$ is equal to $(n-1)$ nth parts of one simple unit of the same quotient. But m is some number in the series 1, 2, 3, ... $n-1$; therefore the quotient of $m\div n$ is equal to m, nth parts of one simple unit of the quotient resulting from the division of any number greater than n by n. In other words, the fractional expression $\frac{m}{n}$ is the same part of one simple unit of the quotient that the dividend m is of the divisor n.

Whence, considering n, the divisor of a fraction, as a unit or whole composed of as many equal parts as it contains simple units, and the dividend to consist of as many of these equal parts as it contains simple units, the relation of the fraction to unity is expressed by reckoning 1, 2, 3 . . . . m equal parts of the dividend as the 1 nth, 2 nth, 3 nth, m nth parts of unity.

Thus, in the fraction $\frac{3}{7}$ (which is read three sevenths) unity is conceived to be divided into seven equal parts, and the expression $\frac{3}{7}$ contains three of these parts.

The primitive signification, however, of the expression $\frac{3}{7}$ is, that the whole expressed by 3 is to be divided by the whole expressed by 7.

Whence it appears that three times the seventh part of unity or three sevenths, and three divided by seven or the seventh part of three, are identical expressions.

From Articles 79 and the present it follows, that when the greater of two numbers is divided by the less, the quotient contains unity as often as the dividend contains the divisor; and that when the less number is divided by the greater, the quotient contains a part of unity as often as the dividend contains the same part of the divisor.

147. In every fractional expression the divisor or lower number is called the Denominator, and the dividend, or upper number, the Numerator.

The denominator indicates the number of equal parts into which that whole, of which the fraction expresses some part, is divided; and the numerator, the number of these equal parts contained in the fraction.

The numerator and denominator of a fraction are called its Terms.

The name of a fractional expression is formed of the name of the number in the numerator followed by the name of the number in the denominator with the termination th or ths.

Thus $\frac{1}{5}$ is read one fifth; $\frac{3}{8}$, three eighths; $\frac{13}{27}$, thirteen twenty-sevenths; $\frac{145}{376}$, one hundred and forty-five three-hundred-and-seventy-sixths, &c., &c.

Exceptions; $\frac{1}{2}$ (which is read one-half),
- - $\frac{1}{3}$, $\frac{2}{3}$, . . (which are read one third, two thirds . .),
and all numbers, whose denominators, being greater than 20, end in one of the figures 1, 2, 3.

$\frac{1}{21}$, $\frac{1}{22}$, $\frac{1}{23}$, $\frac{2}{21}$, $\frac{2}{22}$, $\frac{2}{23}$, &c. may be read one twenty-first, one twenty-second, one twenty-third, two twenty-firsts, two twenty-seconds, two twenty-thirds, &c.

148. From the mutual relations of the numerator and denominator of any fractional expression are deduced the following consequences:

a. If the numerator of a given fraction is multiplied by any number, the denominator remaining unchanged, the result is the same number of times as great as the given fraction; and if the numerator is divided by any number, the denominator remaining unchanged, the given fraction is the same number of times as great as the result.

To multiply the numerator of a fraction by 2, 3, 4, . . . *n* is to take 2, 3, 4, . . . *n* times the number of parts expressed by the numerator. Now, the value of each of the parts of the numerator depends on the number of equal parts into which the denominator is divided; and this number is, by the hypothesis, unchanged; whence, since the value of the parts of the numerator is not changed, and the number of these parts is 2, 3, 4, . . . *n* times greater than before, the new fraction must be 2, 3, 4, . . . *n* times as great as the other.

Again, to divide the numerator of a fraction by 2, 3, 4, . . . *n*, is to take half, a third, a fourth, . . . one *n*th of the number of equal parts composing that numerator; and the values of these equal parts are unchanged so long as the denominator is not changed.

To multiply, therefore, or divide a fraction by any number, it is sufficient to multiply or divide the numerator by that number.

Taking, for the sake of illustration, the fraction $\frac{6}{25}$, if the numerator is multiplied by 2, 3, 4, successively, the products are 12, 18, 24; and $\frac{12}{25}$, $\frac{18}{25}$, $\frac{24}{25}$, are, respectively, 2, 3, 4 times as great (or contain $\frac{1}{25}$ 2, 3, 4 times as often) as $\frac{6}{25}$.

Again, if the numerator 6 is divided by 2 and by 3, the quotients are 3, 2, respectively; and $\frac{6}{25}$ are 2 and 3 times as great as $\frac{3}{25}$, $\frac{2}{25}$ respectively.

b. If the denominator of a given fraction is multiplied by any number, the numerator remaining unchanged, the given fraction is the same number of times as great as the result; and if the denominator is divided by any number, the numerator remaining unchanged, the result is the same number of times as great as the given fraction.

The multiplication of the denominator of a fraction by 2, 3, 4 . . . *n*, produces the effect of dividing the unit expressed by the denominator into 2, 3, 4, . . . *n* times as many parts. Each part is therefore 2, 3, 4, . . . *n* times less. The numerator containing, by hypothesis, the same number of parts as before, it follows that each of the parts is thus rendered 2, 3, 4 . . . *n* times less, and, by consequence, the fraction also 2, 3, 4, . . . *n* times less than before.

On the contrary, the division of the denominator of a fraction by 2, 3, 4, . . . *n* produces the effect of dividing the unit expressed by the denominator into half, one third, one fourth, ... one *n*th of the number of equal parts composing that denominator. Each part is thus rendered 2, 3, 4, . . . *n* times greater. The numerator containing the same number of these as of the parts which were 2, 3, 4, . . . *n* times smaller, it follows that the value of the fraction is 2, 3, 4, . . . *n* times greater than before the division of the denominator by 2, 3, 4, . . . *n*.

To divide, therefore, or multiply a fraction by any number, it is sufficient to multiply or divide the denominator of the fraction by that number.

149. From the preceding Article it follows that the multiplication of a fraction by any number is effected by either multiplying the numerator or dividing the denominator of the fraction by that number; and, conversely, that the division of a fraction by any number is effected by either dividing the numerator or multiplying the denominator of the fraction by that number.

Since the multiplication of the numerator is, in effect, multiplication of the fraction, and the multiplication of the denominator division of the fraction; and since the multiplication and division of any quantity by the same number cannot alter the absolute value of the quantity so multiplied and divided; it follows that if the terms of a fraction are both multiplied by the same number, the absolute value of the fraction remains unchanged.

In like manner, as division of the numerator amounts to division of the fraction, and division of the denominator to multiplication of the fraction; it follows that, if the terms of a fraction are both divided by the same number, the absolute value of the fraction remains unchanged.

Whence the fractions $\frac{4}{6}$, $\frac{6}{9}$, $\frac{8}{12}$, $\frac{10}{15}$, are all equivalent to (or have the same absolute value as) the fraction $\frac{2}{3}$; for they result from the multiplication of the terms of the fraction $\frac{2}{3}$ by the numbers 2, 3, 4, 5 respectively.

And the fraction $\frac{36}{48}$ is equivalent to $\frac{18}{24}$, or to $\frac{12}{16}$, or to $\frac{9}{12}$, or to $\frac{6}{8}$, or to $\frac{3}{4}$, since these fractions result from the division of the terms of the fraction $\frac{36}{48}$ by the numbers 2, 3, 4, 6, 9, respectively.

150. In fractions, as in whole numbers, the elementary operations are, addition, subtraction, multiplication, division.

To render these operations in some cases more easy, in others at all practicable, certain changes and reductions of fractional expressions are necessary.

For example, it appears (Art. 149) that the same absolute value is expressed by the fractions $\frac{36}{48}$ and $\frac{3}{4}$. Now, as it is more easy to calculate with small numbers than with such as are large, it is important to possess the means of reducing fractions, the terms of which are large numbers, to equivalent fractions whose terms are smaller numbers.

When the terms of a fraction are not prime to each other, this reduction can be always made.

For, the numerator and denominator of a fraction, being numbers not prime to each other, must be both divisible by some common measure. If both are divided by it the terms are rendered smaller numbers, but the absolute value of the fraction is not changed (Art. 149).

151. A fraction which cannot be reduced to an equivalent fraction whose numerator and denominator are smaller numbers, is said to be irreducible, or to be in its lowest terms.

Hence the terms of an irreducible fraction are prime to each other; for if not, they have a common measure; if divided by this common measure, the numerator and denominator are rendered smaller numbers, and the fraction is reduced to lower terms, which is contrary to the hypothesis.

Conversely, if the terms of a fraction are prime to each other, the fraction is irreducible.

Let *a*, *b* represent numbers which are prime to each other, and let *a* be the numerator, *b* the denominator, of a fraction. Then $\frac{a}{b}$ is the expression of the fraction.

Let $\frac{c}{d}$ be another fraction equal to $\frac{a}{b}$; if $\frac{c}{d}$ and $\frac{a}{b}$ are both multiplied by d, the products are equal. Now $\frac{c}{d}\times d=\frac{cd}{d}=c$ (Art. 148 a), and $\frac{a}{b}\times d=\frac{ad}{b}$. Whence $c=\frac{ad}{b}$.

Now c is a whole number; therefore $\frac{ad}{b}$ is also a whole number; and by hypothesis b is prime with a; therefore b divides d (Art. 111). Let d be divided by b, and let the quotient be q, so that $d=b\times q$; and in the expression $c=\frac{ad}{b}$, let bq be substituted for its equal d; then $c=\frac{abq}{b}=aq$. Now $d=bq$. Therefore $\frac{c}{d}=\frac{aq}{bq}$.

Whence, if a fraction, $\frac{c}{d}$, is equal to another fraction, $\frac{a}{b}$, whose terms are prime to each other, the terms c, d of the fraction $\frac{c}{d}$ are the same multiples of a, b, the terms of the fraction $\frac{a}{b}$.

The fraction $\frac{a}{b}$ cannot, consequently, be equal to any other fractions whose terms are smaller numbers than a, b; in other words, the fraction $\frac{a}{b}$, whose terms are prime to each other, is irreducible.

When the terms of a fraction have been divided by their greatest common measure, the result obtained is an irreducible fraction, for the terms are prime to each other.

152. Two irreducible fractions cannot be equivalent unless their numerators are the same number, and their denominators also the same number.

For the only fractions which can be equivalent to an irreducible fraction are those whose terms are equimultiples of the terms of the irreducible fraction (Art. 151); and such equivalent fractions being reduced to their lowest terms, the common result is that irreducible fraction of the terms of which their terms are equimultiples.

153. In conclusion, then, it appears that any fraction which is not irreducible may be reduced to lower terms, by dividing its terms by any number which is a measure of them both; and that to reduce a fraction to the lowest terms, it is necessary to divide its terms by their greatest common measure.

154. Examples of the reduction of fractions to the lowest terms:

1st. Let it be required to reduce the fraction $\frac{1836}{2052}$ to the lowest terms.

To effect the reduction it is necessary to divide 1836 and 2052 by their greatest common measure. This is found as follows (see Art. 108),

```
1836)2052(1
     1836
     ----
      216)1836(8
          1728
          ----
           108)216(2
               216
               ---
                 0
```

Whence 108 is the greatest common measure of 1836 and 2052.

Now $1836 \div 108 = 17$, and $2052 \div 108 = 19$.

Therefore $\frac{1836}{2052} = \frac{17}{19}$, which is the result required.

The division of the terms of the given fraction and the result may be indicated thus,

$$\frac{1836}{2052} \div \frac{108}{108} = \frac{17}{19}$$

Exercises :

	Fraction to be reduced.	G. C. M.	Fraction in lowest terms.
2d. ex.	$\frac{1965}{2620}$	655	$\frac{3}{4}$
3d.	$\frac{1856}{1920}$	64	$\frac{29}{30}$
4th.	$\frac{741}{767}$	13	$\frac{57}{59}$
5th.	$\frac{23111}{33759}$	121	$\frac{191}{279}$
6th.	$\frac{1001}{8393}$	77	$\frac{13}{109}$
7th.	$\frac{221154}{221400}$	246	$\frac{899}{900}$
8th.	$\frac{9681}{7376}$	461	$\frac{21}{16}$
9th.	$\frac{168597}{170300}$	1703	$\frac{99}{100}$
10th.	$\frac{585}{45360}$	45	$\frac{13}{1008}$
11th.	$\frac{6622}{7700}$	154	$\frac{43}{50}$
12th.	$\frac{9504}{10692}$	1188	$\frac{8}{9}$

154′. The reduction of a literal or algebraic fraction to an equivalent irreducible fraction, that is, to a fraction expressed in the lowest terms, is in like manner made by division of the terms of the fraction by their greatest common measure (Art. 153). This is equal to the product of all the prime factors common to the terms of the fraction (Art. 138).

The manner of resolving the terms into prime factors is explained in Art. 135.

The reduction of a literal fraction to the lowest terms, and the case of division (Art. 100 a) are identical operations.

Example. Reduce $\frac{a^5b^3cd}{a^4b^2c^2e}$ to the lowest terms.

The prime factors common to both terms are a, b, c.

The lower of the two powers to which these factors are raised respectively, are a^4, b^2, c.

Whence a^4b^2c is the greatest common measure of the terms (Art. 137).

Now $a^5b^3cd \div a^4b^2c = abd$,

and $a^4b^2c^2e \div a^4b^2c = ce$,

therefore $\frac{abd}{ce}$ is the fraction required.

Exercises in the reduction of literal fractions to the lowest terms;

1st. Reduce $\frac{ab^3c^2d}{b^3c\,d^2e}$ to the lowest terms?..............................*Ans.* $\frac{ac}{de}$.

2d. Reduce $\frac{a^7b^4c^2g}{a^6b^2df}$ to the lowest terms?............................*Ans.* $\frac{ab^2c^2g}{df}$.

3d. Reduce $\frac{a^{2m}b^nc^{n+2}}{a^mb^nc^n}$ to the lowest terms?..........................*Ans.* a^mc^2.

4th. Reduce $\frac{m^2n^3}{m^5n^8}$ to the lowest terms?.............................*Ans.* $\frac{1}{m^3n^5}$.

For the reduction to the lowest terms of fractions whose numerators and denominators are polynomial expressions, see Art. 27, Part II.

155. In the 81st Article it is required to divide 43 by 8. The result obtained, namely, $5\frac{3}{8}$, is composed of two parts, a whole number or integer, 5, and a fraction, $\frac{3}{8}$.

In this, as in every instance of inexact division, the fraction is equal to some part or parts of one simple unit of the integral part of the quotient;

and the whole quotient is composed of some collection of simple units, as 5, +some part or parts of one simple unit, as $\frac{3}{8}$.

The fraction is written after the figure expressing simple units, without the interposition of any symbol, although + might be written between them; thus $5\frac{3}{8}$ and $5+\frac{3}{8}$ are forms of expression which have the same meaning.

Any unexecuted division may be expressed by writing the dividend as the numerator, and the divisor as the denominator, of a fraction.

If the numerator is less than the denominator, the expression is termed a proper fraction; if greater, an improper fraction.

In the latter case when the division (being performed) is exact, the quotient is a whole number or integer; when inexact, it is named a mixed number.

The reduction of an improper fraction to a whole or mixed number, and of a mixed number to an improper fraction, are both necessary operations in fractional arithmetic.

An improper algebraic fraction, whose numerator is polynomial and the denominator monomial, is reduced to a mixed quantity by Art. 100 b; and if both terms are polynomial, by Art. 27. Part II.

156. From the preceding Article is deduced this rule for the reduction of an improper fraction to an equivalent whole or mixed number: Divide the numerator of the improper fraction by the denominator; if the division is exact, the quotient is the equivalent whole number; but if the division is inexact, the whole number of the quotient is the integral part, and the remainder and divisor, the numerator and denominator, respectively, of the fractional part of the equivalent mixed number.

157. To resolve the converse problem, that is, to reduce a mixed number back to the improper fraction from which it has been derived; let q denote the integral part, and $\frac{r}{d}$ the fractional part, of a mixed number; consequently $q+\frac{r}{d}$ is the expression of the mixed number

$$q=\frac{q}{1}, \text{ and } \frac{q}{1}\times\frac{d}{d}=\frac{qd}{d};$$

$$\text{therefore } q+\frac{r}{d}=\frac{qd}{d}+\frac{r}{d};$$

but the sum of qd times any quantity and r times the same quantity is equal to $qd+r$ times that quantity. Whence $\frac{qd}{d}+\frac{r}{d}=\frac{qd+r}{d}$, and $q+\frac{r}{d}=\frac{qd+r}{d}$.

From this result is derived the following rule for the reduction of a mixed number to an equivalent improper fraction: Multiply the integral part of the mixed number by the denominator of the fractional part, and to this product add the numerator of the fractional part; the result is the numerator of the improper fraction; and the denominator of the fractional part is the denominator of the improper fraction.

158. Examples in the reduction of improper fractions to equivalent whole or mixed numbers;

1st. Reduce $\frac{16462}{16}$ to an equivalent whole or mixed number?

$$\begin{array}{l} 16\ )\ 15466\ (\ 966\frac{5}{8} \\ \quad\ \ \underline{144} \\ \quad\ \ \ 106 \\ \quad\ \ \ \ \underline{96} \\ \quad\ \ \ 106 \\ \quad\ \ \ \ \underline{96} \\ \quad\ \ \ \ \frac{10}{16}=\frac{5}{8} \end{array}$$

Detail of the calculation; 15466÷16=966+10 rem.

966 is therefore the integral part of the mixed number required, and the remainder 10 being written over the divisor 16, $\frac{10}{16}=\frac{5}{8}$ is the fractional part.

Whence $\frac{15466}{16}=966\frac{5}{8}$, the mixed number required.

Reduce the following improper fractions to equivalent whole or mixed numbers;

2d. $\frac{374}{9}$?..............*Ans.* $41\frac{5}{9}$.
3d. $\frac{1729}{12}$?.........*Ans.* $144\frac{1}{12}$.
4th. $\frac{432}{8}$?..............*Ans.* 54.
5th. $\frac{2512}{23}$?.........*Ans.* $109\frac{5}{23}$.
6th. $\frac{27080}{72}$?.........*Ans.* $376\frac{1}{9}$.
7th. $\frac{110594}{539}$?.........*Ans.* $205\frac{9}{49}$.
8th. $\frac{20097}{81}$?............*Ans.* $248\frac{1}{9}$.
9th. $\frac{837964}{1728}$?.......*Ans.* $484\frac{403}{432}$.
10th. $\frac{79986}{795}$?.......*Ans.* $100\frac{162}{265}$.
11th. $\frac{374985}{83412}$?.......*Ans.* $4\frac{4593}{9268}$.

Reduce also the following algebraic fractions to mixed quantities;

1st. $\frac{a^2+ab+b^2}{a}$?.......*Ans.* $a+b+\frac{b^2}{a}$.

2d. $\frac{a^3x+a^2x^2+ax^3}{a^2x}$?.......*Ans.* $a+x+\frac{x^2}{a}$.

3d. $\frac{b^m+b^2+b}{b^2}$?.......*Ans.* $b^{m-2}+1+\frac{1}{b}$.

4th. $\frac{2a^4+4a^3b+6a^2b^2}{2a^3}$?.......*Ans.* $a+2b+\frac{3b^2}{a}$.

159. Examples in the reduction of mixed numbers to equivalent improper fractions.

1st. Reduce $46\frac{3}{11}$ to an equivalent improper fraction;

$$46=\frac{46}{1} \text{ and } \frac{46}{1}=\frac{46}{1}\times\frac{11}{11}=\frac{506}{11} \text{ (Art. 157).}$$

$$\therefore 46\tfrac{3}{11} \text{ or } 46+\frac{3}{11}=\frac{506}{11}+\frac{3}{11}.$$

Now the sum of 506 times any quantity and 3 times the same quantity is equal to 509 times that quantity;

$$\therefore \frac{506}{11}+\frac{3}{11}=\frac{509}{11}$$

or $46\frac{3}{11}=\frac{509}{11}$, the improper fraction required.

Reduce the following mixed numbers to equivalent improper fractions;

2d. $235\frac{5}{7}$?..........*Ans.* $\frac{1650}{7}$.
3d. $156\frac{2}{3}$?............*Ans.* $\frac{470}{3}$.
4th. $46\frac{11}{17}$?..........*Ans.* $\frac{793}{17}$.
5th. $113\frac{56}{57}$?.......*Ans.* $\frac{6497}{57}$.
6th. $142\frac{99}{100}$?......*Ans.* $\frac{14299}{100}$.
7th. $324\frac{14}{27}$?.............*Ans.* $\frac{8762}{27}$.
8th. $1089\frac{23}{64}$?..........*Ans.* $\frac{69719}{64}$.
9th. $197\frac{47}{568}$?.........*Ans.* $\frac{111943}{568}$.
10th. $127\frac{390}{3541}$?......*Ans.* $\frac{450097}{3541}$.
11th. $749\frac{333}{487}$?.......*Ans.* $\frac{365096}{487}$.

Reduce also the following mixed quantities to the form of improper fractions;

1st. $ab+\frac{c}{d}$?......*Ans.* $\frac{abd+c}{d}$.

2d. $px+qy+\frac{r}{s}$?......*Ans.* $\frac{psx+qsy+r}{s}$.

3d. $m+\frac{n}{v+x}$?......*Ans.* $\frac{mv+mx+n}{v+x}$.

4th. $3a^2+2b+\frac{5}{8c}$?......*Ans.* $\frac{24a^2c+16bc+5}{8c}$.

160. When the sum or difference of any given numbers is required, the numbers to be added or subtracted must be of the same kind, that is, composed of collections of the same simple unit.

If the given numbers are fractional, it is evident that they are not of the same kind (or are not referred to the same unit) unless they have the same denominator. An indispensable preparation, therefore, for either the addition or the subtraction of fractional quantities is the reduction of fractions having different denominators to equivalent fractions having the same number for denominator.

The principle (Art. 149) that the value of a fraction is not changed by the multiplication of its terms by the same number, supplies the means of making this reduction.

Let it, for example, be required to reduce the fractions $\frac{3}{5}$ and $\frac{6}{7}$ to other fractions of the same value, and having both the same number for their denominator.

If the terms of each fraction are multiplied by the denominator of the other fraction, the results,

$$\frac{3}{5}\times\frac{7}{7} \text{ or } \frac{21}{35},$$

$$\text{and } \frac{6}{7}\times\frac{5}{5} \text{ or } \frac{30}{35},$$

are equal respectively to $\frac{3}{5}$ and $\frac{6}{7}$ (Art. 149); and the denominators are of necessity equal, since $5\times7=7\times5$ (Art. 56).

When the fractions to be reduced are more in number than two, the process is in principle the same.

If it is required to reduce the fractions $\frac{2}{3}$, $\frac{5}{8}$, $\frac{7}{12}$, to equivalent fractions having the same denominator, let the terms of each fraction be multiplied by the denominators of the other fractions; then

$$\frac{2}{3}=\frac{2}{3}\times\frac{8}{8}\times\frac{12}{12}=\frac{192}{288} \text{ (Art. 149).}$$

$$\frac{5}{8}=\frac{5}{8}\times\frac{3}{3}\times\frac{12}{12}=\frac{180}{288} \quad \text{,,}$$

$$\frac{7}{12}=\frac{7}{12}\times\frac{3}{3}\times\frac{8}{8}=\frac{168}{288} \quad \text{,,}$$

The reduced are consequently equal to the given fractions, and they have the same number for their denominator; for the product of the three numbers, 3, 8, 12, is the same, whatever be the order in which they are multiplied together (Art. 68).

This process, derived from Articles 149 and 71, may be extended so as to include any number of fractions.

Whence, to reduce any number of fractions having different denominators to equivalent fractions having the same (or a common) denominator, multiply the terms of each of the fractions by the denominators, or by the product of the denominators, of all the other fractions; the results are equivalent fractions having the same denominator.

161. Examples: Reduce to the same denominator,

1st. $\frac{1}{3}$ and $\frac{2}{5}$?……………………………………*Ans.* $\frac{5}{15}$, $\frac{6}{15}$.

2d. $\frac{3}{4}$, $\frac{4}{5}$, $\frac{2}{7}$?……………………………*Ans.* $\frac{105}{140}$, $\frac{112}{140}$, $\frac{40}{140}$.

3d. $\frac{3}{5}$, $\frac{1}{2}$, $\frac{1}{3}$?………………………………*Ans.* $\frac{18}{30}$, $\frac{15}{30}$, $\frac{10}{30}$.

4th. $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{8}$?………………………………*Ans.* $\frac{24}{48}$, $\frac{32}{48}$, $\frac{30}{48}$.

5th. $\frac{2}{5}$, $\frac{2}{7}$, $\frac{5}{9}$, $\frac{7}{10}$?…………………*Ans.* $\frac{1260}{3150}$, $\frac{900}{3150}$, $\frac{1750}{3150}$, $\frac{2205}{3150}$.

6th. $\frac{4}{7}$, $\frac{5}{8}$, $3\frac{3}{4}$?……………………………*Ans.* $\frac{128}{224}$, $\frac{140}{224}$, $\frac{840}{224}$.

7th. $\frac{7}{8}$, $\frac{3}{22}$, $3\frac{1}{2}$?…………………………*Ans.* $\frac{308}{352}$, $\frac{48}{352}$, $\frac{1232}{352}$.

8th. $\frac{1}{6}$, $\frac{3}{8}$, $\frac{5}{11}$, $\frac{6}{7}$?…………………*Ans.* $\frac{616}{3696}$, $\frac{1386}{3696}$, $\frac{1680}{3696}$, $\frac{3168}{3696}$.

9th. $\frac{1}{4}$, $\frac{7}{8}$, $\frac{4}{9}$, $\frac{1}{14}$?…………………*Ans.* $\frac{1008}{4032}$, $\frac{3528}{4032}$, $\frac{1792}{4032}$, $\frac{288}{4032}$.

10th. $\frac{4}{15}$, $\frac{1}{12}$, $\frac{7}{8}$, $\frac{3}{16}$?………*Ans.* $\frac{6144}{23040}$, $\frac{1920}{23040}$, $\frac{20160}{23040}$, $\frac{4320}{23040}$.

A mixed number, as $3\frac{3}{4}$ in example 6, and $3\frac{1}{8}$ in example 7, is brought to an improper fraction (Art. 159).

162. If the number of fractions to be reduced is considerable, or the denominators are large numbers, it abridges labour to form the product of all the denominators, to divide this product by every denominator, and to multiply the terms of each fraction by the corresponding quotient.

Let it be required to reduce the fractions $\frac{3}{8}$, $\frac{7}{11}$, $\frac{10}{13}$, $\frac{23}{25}$, $\frac{29}{43}$, to equivalent fractions having a common denominator.

The product of all the denominators $=8\times11\times13\times25\times43=1229800$. This number, being a multiple of all the denominators, is divisible by each of them.

Divide this multiple by the denominator of the first fraction, and multiply the terms of the fraction by the quotient. Next, divide it by the denominator of the second fraction, and multiply its terms by the quotient, &c. &c. In this manner the reduction of the given fractions is accomplished as follows:

$$
\begin{array}{l|l}
1229800\div 8=153725 & \frac{3}{8}=\frac{461175}{1229800}. \\
1229800\div 11=111800 & \frac{7}{11}=\frac{782600}{1229800}. \\
1229800\div 13=\ \ 94600 & \frac{10}{13}=\frac{946000}{1229800}. \\
1229800\div 25=\ \ 49192 & \frac{23}{25}=\frac{1131416}{1229800}. \\
1229800\div 43=\ \ 28600 & \frac{29}{43}=\frac{829400}{1229800}.
\end{array}
$$

It is evident that the terms of each fraction are, in the preceding calculation, multiplied by the product of all the other denominators; and that the common denominator to which all the fractions are reduced is the product of the denominators of all the given fractions.

In this manner let the following examples be reduced to the same denominator:

1st. $\frac{7}{16}$, $\frac{8}{27}$, $5\frac{10}{11}$?..*Ans.* $\frac{2079}{4752}$, $\frac{1408}{4752}$, $\frac{28080}{4752}$.

2d. $\frac{2}{5}$, $\frac{1}{6}$, $\frac{3}{16}$, $\frac{11}{24}$?..............................*Ans.* $\frac{4608}{11520}$, $\frac{1920}{11520}$, $\frac{2160}{11520}$, $\frac{5280}{11520}$.

3d. $\frac{5}{12}$, $\frac{7}{16}$, $\frac{13}{18}$, $\frac{17}{20}$?..............................*Ans.* $\frac{28800}{69120}$, $\frac{30240}{69120}$, $\frac{49920}{69120}$, $\frac{58752}{69120}$.

4th. $\frac{9}{14}$, $\frac{17}{21}$, $\frac{8}{35}$, $\frac{23}{41}$?...................*Ans.* $\frac{271215}{421890}$, $\frac{341530}{421890}$, $\frac{96432}{421890}$, $\frac{236670}{421890}$.

163. If the denominators of any given fractions are prime to each other, the least number which can be divided by all these denominators,—in other words, their least common multiple,—is their continual product; but if the denominators are not all prime to each other, they have common multiples which are less than their continual product.

When the product of the denominators is a large number, the reduction of several fractions to a common denominator by the method of Articles 160 or 162 is laborious. If any common multiple less than the continual product of the given denominators is employed, the calculation is abridged, and the terms of the fractions having the same denominator are expressed by smaller numbers; and by taking the least common multiple of all the denominators the calculation is still more abridged, and the terms of the fractions expressed by the least numbers possible.

The least common multiple of the denominators of any given fractions is found by Article 141.

Let it be required to reduce the fractions $\frac{13}{60}$, $\frac{17}{28}$, $\frac{23}{240}$, $\frac{173}{225}$, $\frac{319}{490}$, $\frac{523}{720}$, to equivalent fractions which shall have the least common multiple of all their denominators for common denominator.

Decomposing the denominators into their prime factors (Art. 135), it is found that

$$
\begin{aligned}
60&=2^2\times3\times5.\\
28&=2^2\times7.\\
240&=2^4\times3\times5.\\
225&=3^2\times5^2.\\
490&=2\ \times5\times7^2.\\
720&=2^4\times3^2\times5.
\end{aligned}
$$

2, 3, 5, 7, are consequently the prime factors contained in the given denominators, and the highest powers to which they are raised are 2^4, 3^2, 5^2, 7^2.

Whence (Art. 141) $2^4 \times 3^2 \times 5^2 \times 7^2 = 176400$ is the least common multiple of the denominators.

Proceeding with this least common multiple as with the product of all the denominators in Article 162,

$$\begin{array}{r|l}
176400 \div 60 = 2940 & \frac{13}{60} = \frac{38220}{176400}. \\
176400 \div 28 = 6300 & \frac{17}{28} = \frac{107100}{176400}. \\
176400 \div 240 = 735 & \frac{23}{240} = \frac{16905}{176400}. \\
176400 \div 225 = 784 & \frac{173}{225} = \frac{135632}{176400}. \\
176400 \div 490 = 360 & \frac{319}{490} = \frac{114840}{176400}. \\
176400 \div 720 = 245 & \frac{523}{720} = \frac{128135}{176400}.
\end{array}$$

Since 176400 is a multiple of all the denominators, it can be divided by each of them; and since each of the reduced denominators is obtained by dividing 176400 by a number, as 60, 28, . . . and again multiplying the quotient by that very number, it is evident that the least common multiple of all the denominators is the common denominator of all the fractions.

Also, since the least common multiple of the denominators is the least number which is exactly divisible by all the denominators (Art. 140, 141), it follows that the fractions given in this example are reduced to their least common denominator.

The common denominator of the fractions $\frac{13}{60}$, $\frac{17}{28}$, $\frac{23}{240}$, $\frac{173}{225}$, $\frac{319}{490}$, $\frac{523}{720}$ is $60 \times 28 \times 240 \times 225 \times 490 \times 720 = 32006016000000$ (Art. 162). The least common denominator of the same fractions is only 176400. The advantage to be derived from the use of the latter when several fractions are given, and the denominators are expressed by large numbers, is sufficiently apparent from this example.

164. The following method of decomposing the denominators into prime factors, and from these forming the least common multiple of the denominators, may be adopted in practice:

Place the denominators in a line, divide by the least prime number by which two or more of the denominators are exactly divisible, and write the quotients and undivided denominators in a second line directly under the corresponding numbers in the first line. Repeat the division by this prime number as often as any two of the quotients can be divided by it, and after each division write the undivided numbers and quotients as before.

Divide in the same manner by the next prime number by which two or more of the undivided denominators or quotients are divisible, and write the undivided numbers and quotients in their proper places.

Divide similarly by a third prime number, a fourth, &c., until all the quotients, or undivided denominators and quotients, are prime to each other.

Then multiply together the numbers which are prime to each other (the numbers, namely, in the last line), and multiply this product by the product of all the prime divisors. The last product is the least common multiple required.

The least common denominator of the fractions given in the example of Article 163 is found by this method, as follows:

$$\begin{array}{r|rrrrrr}
2) & 60. & 28. & 240. & 225. & 490. & 720 \\ \hline
2) & 30. & 14. & 120. & 225. & 245. & 360 \\ \hline
2) & 15. & 7. & 60. & 225. & 245. & 180 \\ \hline
2) & 15. & 7. & 30. & 225. & 245. & 90 \\ \hline
3) & 15. & 7. & 15. & 225. & 245. & 45 \\ \hline
3) & 5. & 7. & 5. & 75. & 245. & 15 \\ \hline
5) & 5. & 7. & 5. & 25. & 245. & 5 \\ \hline
7) & 1. & 7. & 1. & 5. & 49. & 1 \\ \hline
 & 1. & 1. & 1. & 5. & 7. & 1 \\ \hline
\end{array}$$

$$5\times 7=35. \quad 2\times 2\times 2\times 2\times 3\times 3\times 5\times 7=5040,$$
$$\text{and } 5040\times 35=176400.$$

The prime factors found by this process are, in the present instance,
2 raised to the fourth power,
3 raised to the second,
5 raised to the second,
7 raised to the second; results which correspond with those already found by the method of Article 135.

This coincidence is not to be taken as a proof that the results will always correspond.

That they must always correspond is, however, easily proved; for the divisors comprehend all the prime factors common to the numbers, each being raised to the highest power which it attains in any of the numbers; and the numbers left in the last horizontal line (and which are prime to each other) contain all the prime factors not common to the numbers.

Now, the product of all the prime factors not common to any given numbers, and of the highest powers of the common prime factors, is the least common multiple of these numbers (Art. 141). The results must consequently agree.

165. Whole numbers may be expressed fractionally (Art. 144), and reduced to the same denominator with other fractions, by any of the preceding methods. Also, mixed numbers may be brought to improper fractions (Art. 157), and reduced in the same manner.

166. Of fractions having the same denominator, that is the greatest which has the greatest numerator, for the equal parts in all the numerators are of the same value; that, consequently, is the greatest which contains the greatest number of parts.

Fractions may, in like manner, be reduced to a common numerator, namely, by multiplying the terms of each fraction by the product of the numerators of all the others. Of fractions so reduced, that is the greatest which has the least denominator, for in this fraction the equal parts into which unity is divided are the greatest, and the same number of these equal parts is taken in all the fractions.

167. The methods to be followed in the calculation of the least common multiple of any given numbers, and in the reduction of fractions to the same denominator, being scattered through several articles, it may be useful to condense these methods into general rules.

The rule for finding the least common multiple of any given numbers is,

1st. Decompose the numbers into prime factors (Art. 135, 164).

2d. Multiply together all the prime factors which are found in one number, but not in any of the others; and multiply this product by the highest powers of the prime factors which are common to two or more of the given numbers; the last product is the multiple required.

Examples. Required the least common multiples of,

1st. 12, 16, 24, 32?......................................*Ans.* 96.
2d. 28, 56, 74?..*Ans.* 2072.
3d. 9, 14, 16, 21?......................................*Ans.* 1008.
4th. 6, 8, 10, 12, 14, 16, 18?..........................*Ans.* 5040.
5th. 18, 54, 48, 108, 96?...............................*Ans.* 864.
6th. 60, 72, 84, 96, 108?...............................*Ans.* 30240.
7th. 78, 84, 168, 234?..................................*Ans.* 6552.
8th. 308, 132, 154, 196?................................*Ans.* 6468.
9th. 26, 147, 364, 585?.................................*Ans.* 114660.
10th. 35, 45, 56, 65, 75?...............................*Ans.* 163800.

168. To reduce any given fractional expressions to a common denominator:

Rule. Reduce mixed numbers to improper fractions (Art. 157), whole numbers to a fractional form (Art. 144), and all the fractions to their lowest terms (Art. 153).

Then, 1st, multiply the terms of each of the given fractions by the denominators of all the other fractions.

Or, 2d, form the product of the denominators of all the given fractions, divide this product by the denominator of the first fraction, and multiply the terms of this fraction by the quotient. Next, divide the product of the denominators by the denominator of the second of the given fractions, and multiply its terms by the quotient, and in the same manner proceed with all the given fractions.

Or, 3d, find the least common multiple of the denominators of the given fractions, divide this least common multiple by the denominator of the first fraction, and multiply the terms of this fraction by the quotient; proceed in the same manner with a second, a third, ... of the given fractions; the results by each of the three methods of reduction are fractions equivalent to the fractions proposed, and having the same denominator.

The common denominator found by the first and second methods is the product of the denominators of all the given fractions. By the third method, it is the least common multiple of all the denominators.

Examples. Reduce to equivalent fractions having the same denominator,

1st. $\frac{3}{4}$, $\frac{4}{5}$, $\frac{2}{7}$?..................................*Ans.* $\frac{105}{140}$, $\frac{112}{140}$, $\frac{40}{140}$.
2d. $\frac{1}{2}$, $\frac{1}{6}$, $\frac{1}{7}$?......................................*Ans.* $\frac{42}{84}$, $\frac{14}{84}$, $\frac{12}{84}$.
3d. $\frac{1}{3}$, $\frac{3}{4}$, $\frac{5}{6}$?......................................*Ans.* $\frac{24}{72}$, $\frac{54}{72}$, $\frac{60}{72}$.
4th. $\frac{7}{8}$, $\frac{3}{22}$, $3\frac{1}{2}$?..............................*Ans.* $\frac{308}{352}$, $\frac{48}{352}$, $\frac{1232}{352}$.
5th. $\frac{4}{7}$, $\frac{5}{8}$, $3\frac{3}{4}$?................................*Ans.* $\frac{128}{224}$, $\frac{140}{224}$, $\frac{840}{224}$.
6th. $\frac{2}{5}$, $\frac{2}{7}$, $\frac{5}{9}$, $\frac{7}{10}$?.......................*Ans.* $\frac{1260}{3150}$, $\frac{900}{3150}$, $\frac{1750}{3150}$, $\frac{2205}{3150}$.
7th. $\frac{2}{5}$, $\frac{1}{6}$, $\frac{3}{16}$, $\frac{1}{24}$?..............*Ans.* $\frac{4608}{11520}$, $\frac{1920}{11520}$, $\frac{2160}{11520}$, $\frac{5280}{11520}$.
8th. $\frac{4}{15}$, $\frac{1}{12}$, $\frac{7}{8}$, $\frac{3}{16}$?..............*Ans.* $\frac{6144}{23040}$, $\frac{1920}{23040}$, $\frac{20160}{23040}$, $\frac{4320}{23040}$.
9th. $\frac{2}{3}$, $5\frac{1}{2}$, $\frac{3}{4}$, 3?..............................*Ans.* $\frac{16}{24}$, $\frac{132}{24}$, $\frac{18}{24}$, $\frac{72}{24}$.
10th. $\frac{1}{4}$, $\frac{7}{8}$, $\frac{4}{9}$, $\frac{1}{14}$?.......................*Ans.* $\frac{1008}{4032}$, $\frac{3528}{4032}$, $\frac{1792}{4032}$, $\frac{288}{4032}$.

Reduce to equivalent fractions having the least common denominator,

1st. $\frac{3}{4}$, $\frac{5}{12}$?..*Ans.* $\frac{9}{12}$, $\frac{5}{12}$.
2d. $\frac{9}{14}$, $\frac{11}{21}$?..*Ans.* $\frac{27}{42}$, $\frac{22}{42}$.

3d. $\frac{21}{22}, \frac{17}{33}$?........ *Ans.* $\frac{63}{66}, \frac{34}{66}$.

4th. $\frac{5}{52}, \frac{3}{13}$?........ *Ans.* $\frac{5}{52}, \frac{12}{52}$.

5th. $\frac{17}{18}, \frac{42}{45}$?........ *Ans.* $\frac{85}{90}, \frac{84}{90}$.

6th. $\frac{1}{4}, \frac{5}{6}, \frac{11}{12}$?........ *Ans.* $\frac{3}{12}, \frac{10}{12}, \frac{11}{12}$.

7th. $\frac{2}{3}, \frac{5}{6}, \frac{3}{8}$?........ *Ans.* $\frac{16}{24}, \frac{20}{24}, \frac{9}{24}$.

8th. $\frac{11}{16}, 4\frac{1}{9}, \frac{31}{36}$?........ *Ans.* $\frac{99}{144}, \frac{592}{144}, \frac{124}{144}$.

9th. $\frac{7}{8}, \frac{5}{12}, \frac{3}{16}, \frac{11}{24}$?........ *Ans.* $\frac{42}{48}, \frac{20}{48}, \frac{9}{48}, \frac{22}{48}$.

10th. $\frac{5}{8}, \frac{5}{12}, \frac{9}{16}, \frac{5}{18}$?........ *Ans.* $\frac{90}{144}, \frac{60}{144}, \frac{81}{144}, \frac{40}{144}$.

11th. $\frac{3}{11}, \frac{2}{7}, \frac{1}{22}, \frac{3}{14}$?........ *Ans.* $\frac{42}{154}, \frac{44}{154}, \frac{7}{154}, \frac{33}{154}$.

12th. $\frac{37}{42}, \frac{10}{21}, \frac{7}{12}, \frac{6}{7}$?........ *Ans.* $\frac{74}{84}, \frac{40}{84}, \frac{49}{84}, \frac{72}{84}$.

13th. $\frac{7}{13}, \frac{5}{9}, \frac{35}{39}, \frac{19}{24}$?........ *Ans.* $\frac{504}{936}, \frac{520}{936}, \frac{840}{936}, \frac{741}{936}$.

14th. $\frac{7}{12}, \frac{5}{144}, \frac{13}{1728}, \frac{19}{20736}$?... *Ans.* $\frac{12096}{20736}, \frac{720}{20736}, \frac{156}{20736}, \frac{19}{20736}$.

15th. $\frac{13}{20}, \frac{17}{48}, \frac{113}{280}, \frac{527}{960}, \frac{1211}{1800}, \frac{3613}{5040}, \frac{5237}{6860}$?........ *Ans.* $\frac{3210480}{4939200}$,
$\frac{1749300}{4939200}, \frac{1993320}{4939200}, \frac{2711415}{4939200}, \frac{3322984}{4939200}, \frac{3540740}{4939200}, \frac{3770640}{4939200}$.

168 a. The rules of Article 168, for the reduction of fractional expressions having different denominators to equivalent fractions having the same denominator, are deduced from investigations which are independent of particular number or quantity. These rules may therefore be employed for the reduction of algebraic fractions to the same denominator.

The fractions $\frac{a}{b}, \frac{c}{d}$, reduced by Article 168 to the same denominator, are respectively $\frac{ad}{bd}, \frac{bc}{bd}$.

In effect, it follows from Article 149, that $\frac{a}{b}=\frac{a}{b}\times\frac{d}{d}=\frac{ad}{bd}$;

and that $\frac{c}{d}=\frac{c}{d}\times\frac{b}{b}=\frac{bc}{bd}$.

If the fractions $\frac{a}{bc}, \frac{d}{ab^2}, \frac{b}{a^2c}$ are given, and it is required to reduce them to the least common denominator, it is necessary to find the least common multiple of bc, ab^2, a^2c; this (Art. 141) is a^2b^2c.

Next (Art. 168), $a^2b^2c \div bc = a^2b$, and $\frac{a}{bc}\times\frac{a^2b}{a^2b}=\frac{a^3b}{a^2b^2c}$.

$a^2b^2c \div ab^2 = ac$, and $\frac{d}{ab^2}\times\frac{ac}{ac}=\frac{acd}{a^2b^2c}$.

$a^2b^2c \div a^2c = b^2$, and $\frac{b}{a^2c}\times\frac{b^2}{b^2}=\frac{b^3}{a^2b^2c}$.

Reduce to the same denominator,

1st. $\frac{n}{d}, \frac{n'}{d'}$?........ *Ans.* $\frac{nd'}{dd'}, \frac{n'd}{dd'}$.

2d. $\frac{n}{d}, \frac{n'}{d'}, \frac{n''}{d''}$?........ *Ans.* $\frac{nd'd''}{dd'd''}, \frac{n'dd''}{dd'd''}, \frac{n''dd'}{dd'd''}$.

3d. $\frac{m}{a+b}, \frac{n}{b+c}$?........ *Ans.* $\frac{bm+cm}{ab+ac+b^2+bc}, \frac{an+bn}{ab+ac+b^2+bc}$.

4th. $\frac{a+b}{m+n}, \frac{a'+b'}{m'+n'}$?... *Ans.* $\frac{am'+an'+bm'+bn'}{mm'+mn'+m'n+n'n}, \frac{a'm+a'n+b'm+b'n}{mm'+mn'+m'n+n'n}$.

Reduce to the least common denominator,

1st. $\frac{n}{ad^2}, \frac{n'}{c^2d}$?........ *Ans.* $\frac{c^2n}{ac^2d^2}, \frac{adn'}{ac^2d^2}$.

2d. $\frac{a}{mn}, \frac{b}{m^2}, \frac{c}{mn^2}$?........ *Ans.* $\frac{amn}{m^2n^2}, \frac{bn^2}{m^2n^2}, \frac{cm}{m^2n^2}$.

3d. $\frac{ace}{b^2d^4}, \frac{abd}{c^2d^2e^4}, \frac{ce}{a^3b^3d}$?................*Ans.* $\frac{a^4bc^3e^5}{a^3b^3c^2d^4e^4}, \frac{a^4b^4d^3}{a^3b^3c^2d^4e^4}, \frac{c^3d^3e^5}{a^3b^3c^2d^4e^4}$.

4th. $\frac{a+b}{a^2c^3}, \frac{c+d}{c^2d^4}$?....................................*Ans.* $\frac{ad^4+bd^4}{a^2c^3d^4}, \frac{a^2c^2+a^2cd}{a^2c^3d^4}$.

ADDITION OF FRACTIONS.

169. In addition of fractions, as of whole numbers, the object is to find a single number which shall have the value of all the numbers which are to be added together.

If the given fractions have the same denominator, the numerators are added together, and their sum is written over the common denominator. This fractional expression is the result required.

For example, the sum of the fractions $\frac{3}{13}, \frac{5}{13}, \frac{4}{13}$,
is equal to 3 repetitions of the 13th part of unity,
+ 5 repetitions of the 13th part of unity,
+ 4 repetitions of the 13th part of unity.

But 3+5+4 repetitions of any quantity are equal to 12 repetitions of that quantity; therefore $\frac{3}{13}+\frac{5}{13}+\frac{4}{13}=\frac{12}{13}$, the sum of the given fractions.

In like manner the sum of the fractions $\frac{3}{16}, \frac{7}{16}, \frac{5}{16}, \frac{9}{16}$ is equal to $\frac{3+7+5+9}{16}=\frac{24}{16}$.

$$\text{But, } \frac{24}{16}=1\frac{8}{16} \text{ (Art. 156),}$$
$$\text{and } \frac{8}{16}=\frac{1}{2} \text{ (Art. 153).}$$
$$\therefore 1\frac{8}{16}=1\frac{1}{2},$$
$$\text{and } \frac{3}{16}+\frac{7}{16}+\frac{5}{16}+\frac{9}{16}=1\frac{1}{2}.$$

170. If the given fractions have different denominators, they are reduced to equivalent fractions having the same denominator by any of the methods of Article 168. This reduction made, the numerators are added together, and their sum placed over the common denominator, as in the last Article.

Example 1. Find the sum of the fractions $\frac{3}{8}, \frac{5}{12}, \frac{7}{15}, \frac{9}{20}$?

$$\begin{array}{l|l|l}
2)\underline{8.\ 12.\ 15.\ 20} & 120\div 8=15 & \frac{3}{8}=\frac{45}{120}. \\
2)\underline{4.\ \ 6.\ 15.\ 10} & 120\div 12=10 & \frac{5}{12}=\frac{50}{120}. \\
3)\underline{2.\ \ 3.\ 15.\ \ 5} & 120\div 15=\ 8 & \frac{7}{15}=\frac{56}{120}. \\
5)\underline{2.\ \ 1.\ \ 5.\ \ 5} & 120\div 20=\ 6 & \frac{9}{20}=\frac{54}{120}. \\
\quad\underline{2.\ \ 1.\ \ 1.\ \ 1} & & \frac{205}{120}=1\frac{85}{120}=1\frac{17}{24}.
\end{array}$$

$$2\times 2\times 3\times 5\times 2=120.$$

Detail of the preceding calculation:

1st. The least common multiple of the denominators is found. This is 120.

2d. The fractions are reduced to their least commom denominator. After reduction the numerators of the fractions are 45, 50, 56, 54.

3d. The numerators are added together, and their sum is placed over the common denominator, 45+50+56+54=205. This sum being written over the common denominator 120, $\frac{205}{120}$ is found to be the sum of the given fractions.

4th. The improper fraction, $\frac{205}{120}$, is reduced to the equivalent mixed number, $1\frac{85}{120}$ (Art. 156).

5th. The fractional part of the mixed number is reduced to the lowest terms (Art. 153).

Making this reduction $\frac{85}{120}=\frac{17}{24}$.

Whence $1\frac{17}{24}$ is the sum of the given fractions.

Example 2. Required the sum of the fractional expressions $3\frac{4}{5}$, $\frac{7}{12}$, $\frac{13}{15}$, $\frac{19}{24}$, $\frac{43}{50}$?

The mixed number $3\frac{4}{5}=\frac{19}{5}$ (Art. 157).

$$\begin{array}{l|l|l}
2\underline{)5.\ 12.\ 15.\ 24.\ 50} & 600\div 5=120 & \frac{19}{5}=\frac{2280}{600}. \\
2\underline{)5.\ \ 6.\ 15.\ 12.\ 25} & 600\div 12=\ 50 & \frac{7}{12}=\frac{350}{600}. \\
3\underline{)5.\ \ 3.\ 15.\ \ 6.\ 25} & 600\div 15=\ 40 & \frac{13}{15}=\frac{520}{600}. \\
5\underline{)5.\ \ 1.\ \ 5.\ \ 2.\ 25} & 600\div 24=\ 25 & \frac{19}{24}=\frac{475}{600}. \\
\ \ \underline{1.\ \ 1.\ \ 1.\ \ 2.\ \ 5} & 600\div 50=\ 12 & \frac{43}{50}=\underline{\frac{516}{600}}. \\
2\times2\times3\times5\times2\times5=600. & & \frac{4141}{600}=6\frac{541}{600}.
\end{array}$$

In this example one of the addends, $3\frac{4}{5}$, is a mixed number: it is brought to an improper fraction. This preparation made, the reduction of the given fractions to a common denominator, the addition of the numerators, and the reduction of the result to a mixed number, are all effected precisely as in the preceding example.

Example 3. Required the sum of $\frac{5}{8}$, $\frac{3}{10}$, $4\frac{1}{5}$, $\frac{9}{16}$, and 2?

$4\frac{1}{5}=\frac{21}{5}$, and $2=\frac{2}{1}$ (Art. 157 & 144).

The fractional expressions whose sum is required are therefore,

$$\frac{5}{8},\ \frac{3}{10},\ \frac{21}{5},\ \frac{9}{16},\ \frac{2}{1}.$$

$$\begin{array}{l|r|l}
\text{Calculation of l. c. m. of the denominators.} & \multicolumn{2}{l}{\text{Reduction to the same denominator and addition of the given expressions.}} \\
2\underline{)8.\ 10.\ \ 5.\ 16.\ \ 1} & 80\div 8=10 & \frac{5}{8}=\frac{50}{80}. \\
2\underline{)4.\ \ 5.\ \ 5.\ \ 8.\ \ 1} & 80\div 10=\ 8 & \frac{3}{10}=\frac{24}{80}. \\
2\underline{)2.\ \ 5.\ \ 5.\ \ 4.\ \ 1} & 80\div 5=16 & \frac{21}{5}=\frac{336}{80}. \\
5\underline{)1.\ \ 5.\ \ 5.\ \ 2.\ \ 1} & 80\div 16=\ 5 & \frac{9}{16}=\frac{45}{80}. \\
\ \ \underline{1.\ \ 1.\ \ 1.\ \ 2.\ \ 1} & 80\div 1=80 & \frac{2}{1}=\underline{\frac{160}{80}}. \\
2\times2\times2\times5\times2=80. & \text{The sum of the numerators} & -\ =\underline{615} \\
 & \text{The common denominator} & -\ =\ 80
\end{array}$$

The sum of $\frac{5}{8}$, $\frac{3}{10}$, $4\frac{1}{5}$, $\frac{9}{16}$, and 2 is therefore equal to $\frac{615}{80}$;

and $\frac{615}{80}=7\frac{55}{80}=7\frac{11}{16}$. Whence $7\frac{11}{16}$ is the sum of the given numbers.

171. From Articles 169 and 170 it follows,

That the addition of fractions having the same denominator is made by adding together the numerators of the fractions, and placing the sum over the common denominator;

That the addition of fractions having different denominators is made by reducing them to equivalent fractions having a common denominator, adding together the numerators and placing their sum over the common denominator;

And that the addition of quantities consisting of fractions having different denominators, mixed numbers, and whole numbers is made by reducing all the quantities to the fractional form, reducing these fractional expressions to a common denominator, adding together the numerators, and placing their sum over the common denominator.

Whence the general rule for the addition of any given quantities consisting of fractions, fractions and mixed numbers, or fractions, mixed numbers, and whole numbers, is,

Prepare the given numbers for addition by reducing mixed numbers to improper fractions, whole numbers to the form of fractions, and all the fractions to a common denominator. Then, add the numerators together, and place their sum over the common denominator. The result is the sum of the given quantities.

If this result is expressed by an improper fraction, it is reduced to a mixed number by Article 156, and the fractional part of the mixed number to the lowest terms by Article 153.

172. Exercises in the addition of fractions:

1st. $\frac{3}{17}+\frac{5}{17}+\frac{8}{17}$?……………… *Ans.* $\frac{16}{17}$.
2d. $\frac{4}{21}+\frac{13}{21}+\frac{11}{21}$?……………… *Ans.* $1\frac{1}{3}$.
3d. $\frac{2}{3}+\frac{3}{4}+\frac{5}{8}$?……………… *Ans.* $2\frac{1}{24}$.
4th. $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}$?……………… *Ans.* $\frac{248}{315}$.
5th. $\frac{3}{8}+7\frac{1}{2}+\frac{4}{15}$?……………… *Ans.* $8\frac{117}{120}$.
6th. $3\frac{1}{8}+4\frac{2}{5}+\frac{7}{40}+1$?……………… *Ans.* $8\frac{7}{10}$.
7th. $\frac{1}{2}+\frac{4}{7}+\frac{9}{22}+5\frac{3}{5}$?……………… *Ans.* $7\frac{31}{385}$.
8th. $\frac{5}{6}+\frac{7}{8}+\frac{9}{10}+\frac{11}{12}$?……………… *Ans.* $3\frac{21}{40}$.
9th. $\frac{5}{7}+\frac{3}{11}+\frac{2}{13}+\frac{17}{19}$?……………… *Ans.* $2\frac{6777}{19019}$.
10th. $\frac{7}{19}+9\frac{3}{8}+5\frac{1}{4}+\frac{18}{19}$?……………… *Ans.* $15\frac{143}{152}$.
11th. $\frac{5}{18}+1\frac{11}{12}+7+\frac{5}{84}$?……………… *Ans.* $9\frac{16}{63}$.
12th. $3\frac{1}{2}+5\frac{1}{4}+4\frac{1}{7}+3\frac{1}{5}$?……………… *Ans.* $16\frac{13}{140}$.
13th. $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}$?……………… *Ans.* $1\frac{201}{280}$.
14th. $\frac{3}{8}+\frac{1}{2}+7+\frac{8}{25}+3$?……………… *Ans.* $11\frac{39}{200}$.
15th. $1+\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{5}{6}+\frac{7}{8}$?……………… *Ans.* $6\frac{79}{280}$.
16th. $172\frac{3}{16}+6\frac{11}{40}+\frac{3}{80}$?……………… *Ans.* $178\frac{1}{2}$.
17th. $26\frac{10}{21}+37\frac{14}{35}+5$?……………… *Ans.* $68\frac{92}{105}$.
18th. $8\frac{2}{13}+4\frac{1}{39}+5\frac{14}{45}+14\frac{2}{25}$?……………… *Ans.* $31\frac{1669}{2925}$.
19th. $1804\frac{1}{2}+5246\frac{5}{18}+2618\frac{35}{36}$?……………… *Ans.* $9669\frac{3}{4}$.
20th. $3854\frac{1}{13}+1756\frac{23}{39}$?……………… *Ans.* $5610\frac{2}{3}$.

173. The sum of any numbers consisting of integers and fractions, integers and mixed numbers, mixed numbers and fractions, or of all the three forms of expression, is found by the rule of Article 171. But when the integers, or the integral parts of the mixed numbers, are expressed by several figures the calculation by the general rule is very laborious. Examples 19 and 20 may suffice to prove this assertion.

Hence it is of importance to discover a more concise process for the addition of such numbers.

The sum of an integer and a fraction can always be indicated by interposing the symbol + between the integer and fraction. Thus the sum of the integer 8 and the fraction $\frac{3}{5}$ is indicated by the expression $8+\frac{3}{5}$.

The whole number being composed of eight repetitions of one simple unit, and the fraction of three fifths of one simple unit, the expression $8+\frac{3}{5}$ is equal to eight and three fifth repetitions of one simple unit. This is written $8\frac{3}{5}$ (Art. 155), and, generally, the sum of any integer and fraction, which are both related to the same unit, is expressed by writing the fraction to the right of that figure which expresses the simple units of the integer.

The sum of 8 and $\frac{3}{5}$, found by the general rule, is - - -
$$\left\{\begin{array}{l} 8=\frac{8}{1}=\frac{8}{1}\times\frac{5}{5}=\frac{40}{5} \\ \frac{3}{5}=\frac{3}{5}\times\frac{1}{1}=\underline{\ \ \frac{3}{5}} \\ \qquad\qquad\qquad\ \ \frac{43}{5}=8\frac{3}{5}. \end{array}\right.$$

$$\therefore 8+\tfrac{3}{5}=8\tfrac{3}{5},\text{ as before.}$$

The addition, however, of a fraction to a whole number of the same kind may be considered a matter of notation merely, and the employment of the rule unnecessary.

174. Next, let it be required to add another fraction, $\frac{7}{8}$ for example, to the mixed number $8\frac{3}{5}$, without reducing the latter to an improper fraction.

$$8\tfrac{3}{5}+\tfrac{7}{8}=8+\tfrac{3}{5}+\tfrac{7}{8};$$

Or, since $\frac{3}{5}$ and $\frac{7}{8}$, reduced to the same denominator, are $\frac{24}{40}$, $\frac{35}{40}$, $8\frac{3}{5}+\frac{7}{8}=8+\frac{24}{40}+\frac{35}{40}$.

$$\text{But } \tfrac{24}{40}+\tfrac{35}{40}=\tfrac{59}{40}=1\tfrac{19}{40}=1+\tfrac{19}{40}.$$

$$\therefore 8\tfrac{3}{5}+\tfrac{7}{8}=8+1+\tfrac{19}{40}=9+\tfrac{19}{40}=9\tfrac{19}{40}.$$

This result is obtained by, first, adding together the two fractions, and, second, adding the sum of the fractions to the integral part of the mixed number. The process is independent of the values of the addenda $8\frac{3}{5}$ and $\frac{7}{8}$, or of any particular numbers; whence to find the sum of a mixed number and a fraction,

Rule. Add the fractions together, and add their sum to the integral part of the mixed number.

175. Lastly, let it be required to find the sum of two mixed numbers (for example, $27\frac{5}{12}$ and $35\frac{13}{16}$) without reducing the mixed numbers to improper fractions.

$$27\tfrac{5}{12}=27+\tfrac{5}{12} \text{ and } 35\tfrac{13}{16}=35+\tfrac{13}{16}.$$

$$\therefore 27\tfrac{5}{12}+35\tfrac{13}{16}=27+\tfrac{5}{12}+35+\tfrac{13}{16},$$

$$\text{or } 27\tfrac{5}{12}+35\tfrac{13}{16}=27+35+\tfrac{5}{12}+\tfrac{13}{16} \text{ (Art. 14).}$$

Now $27+35=62=$sum of the integral parts,

and $\frac{5}{12}+\frac{13}{16}=\frac{20}{48}+\frac{39}{48}=\frac{59}{48}=1\frac{11}{48}$, or $1+\frac{11}{48}=$sum of the fractional parts;

$$\therefore 27\tfrac{5}{12}+35\tfrac{13}{16}=62+1+\tfrac{11}{48}, \text{ and } 62+1+\tfrac{11}{48}=63+\tfrac{11}{48}=63\tfrac{11}{48}.$$

Whence, in conclusion, $27\frac{5}{12}+35\frac{13}{16}=63\frac{11}{48}$.

In this case also the process of addition is independent of the particular values of the mixed numbers which are added together.

The sum of any two mixed numbers is, consequently, obtained by adding the integral parts of the given mixed numbers into one sum, the fractional parts into another, and adding together the two results.

176. In like manner it is made evident if the mixed numbers, whose sum is required, are three, four, ... in number, or if some of the addenda are integers, others mixed numbers, and others fractions; that in any case the integral numbers or parts may be added into one sum, the fractions or fractional parts into another, and the two results into one, which is the sum of the given numbers.

Whence, to find the sum of any given whole numbers, mixed numbers, and fractions,

Rule. Combine the integers or integral parts of the given quantities into one number, the fractions or fractional parts into another, and add together the two results.

177. Exercises in the addition of mixed numbers and fractions.

1st. Required the sum of $16\frac{3}{4}$, $10\frac{5}{12}$, and $19\frac{7}{8}$?

Calculation. $16\frac{3}{4}=16+\frac{3}{4}$, $10\frac{5}{12}=10+\frac{5}{12}$, $19\frac{7}{8}=19+\frac{7}{8}$,

$$\therefore 16\tfrac{3}{4}+10\tfrac{5}{12}+19\tfrac{7}{8}=16+\tfrac{3}{4}+10+\tfrac{5}{12}+19+\tfrac{7}{8} \text{ (Art. 173),}$$

$$\text{or } 16\tfrac{3}{4}+10\tfrac{5}{12}+19\tfrac{7}{8}=16+10+19+\tfrac{3}{4}+\tfrac{5}{12}+\tfrac{7}{8} \text{ (Art. 14).}$$

The sum of the integral parts is $16+10+19=45$.

The sum of the fractional parts is found by Article 171, thus:

Least common multiple of 4. 12. 8.	Reduction to least common denominator, and addition of the fractions.
$2)\underline{4.\ 12.\ \ 8}$	$24\div 4=6 \;\big\vert\; \frac{3}{4}=\frac{18}{24}$
$2)\underline{2.\ \ 6.\ \ 4}$	$24\div 12=2 \;\big\vert\; \frac{5}{12}=\frac{10}{24}$
$\underline{1.\ \ 3.\ \ 2}$	$24\div 8=3 \;\big\vert\; \frac{7}{8}=\underline{\frac{21}{24}}$
$2\times2\times3\times2=24$	$\frac{49}{24}=2\frac{1}{24}.$

The sum of the integral parts is 45.

The sum of the fractional parts is $2+\frac{1}{24}$.

$$\therefore 16\tfrac{3}{4}+10\tfrac{5}{12}+19\tfrac{7}{8}=45+2+\tfrac{1}{24}=47+\tfrac{1}{24}=47\tfrac{1}{24}.$$

2d. $176\frac{3}{11}+492\frac{7}{8}+245\frac{5}{44}$?............................ *Ans.* $914\frac{23}{88}$.
3d. $18\frac{14}{45}+9\frac{7}{8}+364\frac{5}{12}+\frac{25}{54}$?........................ *Ans.* $393\frac{71}{1080}$.
4th. $1284\frac{29}{36}+28645\frac{3}{11}+\frac{5}{132}$?...................... *Ans.* $29930\frac{23}{198}$.
5th. $7\frac{3}{10}+18\frac{5}{8}+\frac{7}{12}$?.. *Ans.* $26\frac{61}{120}$.
6th. $1827\frac{5}{12}+4004\frac{7}{52}+27\frac{9}{13}$?........................ *Ans.* $5859\frac{19}{78}$.
7th. $37461\frac{56}{119}+4938\frac{19}{68}$?................................ *Ans.* $42399\frac{3}{4}$.
8th. $3\frac{1}{2}+4\frac{1}{3}+5\frac{1}{5}+6\frac{1}{6}+7\frac{1}{7}$?................................ *Ans.* $26\frac{12}{35}$.
9th. $135\frac{3}{4}+291\frac{5}{18}+2618\frac{34}{35}$?...................... *Ans.* $3045\frac{1259}{1260}$.
10th. $74\frac{3}{8}+1\frac{1}{12}+171\frac{19}{36}$?.................................. *Ans.* $246\frac{71}{72}$.
11th. $274\frac{3}{8}+186\frac{1}{5}+36\frac{5}{16}+27\frac{63}{64}$?...................... *Ans.* $526\frac{279}{320}$.
12th. $19\frac{1}{8}+210\frac{1}{4}+79\frac{1}{16}$?.................................. *Ans.* $308\frac{7}{16}$.
13th. $7\frac{1}{2}+\frac{13}{16}+120\frac{9}{32}+\frac{5}{8}$?................................ *Ans.* $129\frac{7}{32}$.
14th. $12\frac{1}{3}+18\frac{5}{6}+27\frac{5}{9}+10\frac{11}{12}$?............................ *Ans.* $69\frac{23}{36}$.
15th. $\frac{7}{8}+1\frac{1}{2}+25\frac{5}{16}+\frac{3}{4}+147$?.......................... *Ans.* $175\frac{7}{16}$.
16th. $62\frac{1}{8}+16\frac{3}{5}+\frac{9}{20}+26\frac{1}{4}$?.............................. *Ans.* $105\frac{17}{40}$.
17th. $17\frac{6}{10}+35\frac{1}{5}+24\frac{9}{16}+478\frac{9}{100}+38\frac{11}{40}$?........... *Ans.* $593\frac{291}{400}$.
18th. $26\frac{1}{15}+\frac{8}{15}+\frac{2}{3}$?.. *Ans.* 34.
19th. $54\frac{7}{8}+26+9\frac{5}{18}+128\frac{12}{13}$?.......................... *Ans.* $219\frac{71}{936}$.
20th. $819\frac{1}{2}+2043\frac{5}{16}+461\frac{17}{36}$?........................ *Ans.* $3324\frac{41}{144}$.

177′. The sum of any given algebraic fractional expressions is found in the same manner as the sum of fractions which are expressed by numbers; for example, the fractional expressions $\frac{n}{d}$, $\frac{n'}{d}$ are equal, the first to n repetitions of the $\frac{1}{d}$ part of unity; and the second, to n' repetitions of the $\frac{1}{d}$ part of unity; their sum is, consequently, equal to $n+n'$ repetitions of the $\frac{1}{d}$ part of unity, that is, to $\frac{n+n'}{d}$.

If the sum of the fractions $\frac{n}{d}$, $\frac{n'}{d'}$ is required, the addition may be indicated by the interposition of the symbol + between them; thus, $\frac{n}{d}+\frac{n'}{d'}$; but to obtain the expression of the sum as in the preceding example, it becomes necessary to reduce the fractions $\frac{n}{d}$, $\frac{n'}{d'}$ to the same denominator (Art. 160).

The given fractions reduced to the same denominators are equal,

$$\frac{n}{d} \text{ to } \frac{n}{d}\times\frac{d'}{d'} \text{ or } \frac{nd'}{dd'}, \text{ and } \frac{n'}{d'} \text{ to } \frac{n'}{d'}\times\frac{d}{d} \text{ or } \frac{n'd}{dd'}.$$

The sum of these fractions is $\frac{nd'+n'd}{dd'}$,

therefore $\frac{n}{d}+\frac{n'}{d'}=\frac{nd'+n'd}{dd'}$.

Required the sum of the following fractional expressions:

1st. $\frac{a}{b}$, $\frac{m}{n}$, and $\frac{p}{q}$?.................................*Ans.* $\frac{anq+mbq+pbn}{bnq}$.
2d. $\frac{n}{d}$, $\frac{n'}{d^2}$, and $\frac{n''}{d^3}$?... *Ans.* $\frac{nd^2+n'd+n''}{d^3}$.

3d. $\frac{m+n}{mn}$, $\frac{m+n}{m^2n^2}$?....... *Ans.* $\frac{m(mn+1)+n(mn+1)}{m^2n^2}$, or $\frac{(m+n)\ (mn+1)}{m^2n^{\ 2}}$.

4th. $\frac{a+b}{c^2d^3}$ and $\frac{c+1}{c^3d^2}$?.................................... *Ans.* $\frac{ac+bc+cd+d}{c^3d^3}$.

5th. $a+\frac{n}{d}$ and $\frac{5a}{d^2}$?.. *Ans.* $\frac{ad^2+nd+5a}{d^2}$.

6th. $\frac{m^2n}{b^2cd}$ and $\frac{m^3n^5}{a^2bd^3}$?.................................. *Ans.* $\frac{a^2d^2m^2n+bcm^3n^5}{a^2b^2cd^3}$.

SUBTRACTION OF FRACTIONS.

178. In subtraction of fractions, two fractions or two fractional expressions are given, and it is required to find the excess of the greater over the less.

If the given fractions have the same denominator, that is the greater which has the greater numerator (Art. 166).

Of the two fractions $\frac{11}{12}$ and $\frac{5}{12}$, for example, $\frac{11}{12}$ is the greater. To find its excess over $\frac{5}{12}$:

The fraction $\frac{11}{12}$ is equal to 11 repetitions of $\frac{1}{12}$ of unity;

And the fraction $\frac{5}{12}$ is equal to 5 repetitions of $\frac{1}{12}$ of unity:

Therefore $\frac{11}{12}$ and $\frac{5}{12}$ are equal, respectively, to 11 and to 5 repetitions of the same quantity.

But if from 11 repetitions of a quantity 5 are taken away, 6 repetitions of that quantity are left. Whence, from 11 repetitions of the quantity $\frac{1}{12}$, taking away 5, the remainder is 6 repetitions of $\frac{1}{12}$ or $\frac{6}{12}$ (Art. 168 a), or $\frac{1}{2}$; that is, $\frac{11}{12}-\frac{5}{12}=\frac{6}{12}=\frac{1}{2}$.

This result is obtained by subtracting 5, the numerator of the subtrahend, from 11, the numerator of the minuend, and placing the remainder 6 over the common denominator.

The reasoning employed in this particular case being applicable to any fractions having a common denominator, it follows that the difference of any two fractions having a common denominator is obtained by subtracting the less numerator from the greater, and writing the remainder over the common denominator.

179. If the given fractions have different denominators, they can be reduced to equivalent fractions having a common denominator by the rule of Article 168. Their difference is then found in the manner already explained.

Example. From $\frac{13}{14}$ subtract $\frac{5}{8}$.

$$\begin{array}{lr} 2\underline{)14.\ \ 8} & 56\div14=4)\ \frac{13}{14}=\frac{52}{56} \\ \quad\ 7.\ \ 4 & \\ & 56\div8=7)\ \ \frac{5}{8}=\underline{\frac{35}{56}} \\ 2\times7\times4=56 & \frac{17}{56} \end{array}$$

Detail: the least common multiple of the denominators is 56.

The fractions, after reduction to the least common denominator, are $\frac{52}{56}$, $\frac{35}{56}$.

The difference of 52 and 35, the numerators, is 17.

Therefore $\frac{17}{56}$ is the difference of the fractions $\frac{13}{14}$ and $\frac{5}{8}$.

180. The minuend and subtrahend may be, the one a whole or mixed number, and the other a fraction, or both may be mixed numbers.

The whole or mixed numbers in such cases are reducible to fractional expressions by Articles 157 and 144, and the fractional expressions to a common denominator by Article 168. After these reductions the subtraction is made as in Article 178.

Example 1. From 5 subtract $\frac{3}{8}$:

Since $5=\frac{5}{1}$, the fractional expressions whose difference is required are $\frac{5}{1}$ and $\frac{3}{8}$.

$$\frac{5}{1}=\frac{5}{1}\times\frac{8}{8}=\frac{40}{8}$$
$$\frac{3}{8}=\frac{3}{8}\times\frac{1}{1}=\frac{3}{8}$$
$$\frac{37}{8}=4\tfrac{5}{8}.$$

Explanation: $5=\frac{40}{8}$; $\frac{3}{8}=\frac{3}{8}$.

$$\frac{40}{8}-\frac{3}{8}=\frac{40-3}{8}=\frac{37}{8}=4\tfrac{5}{8}\ \therefore 5-\tfrac{3}{8}=4\tfrac{5}{8}.$$

Example 2. From $2\frac{1}{4}$ subtract $\frac{4}{5}$.

Since $2\frac{1}{4}=\frac{9}{4}$, the fractional expressions whose difference is required are $\frac{9}{4}$ and $\frac{4}{5}$.

$$\frac{9}{4}=\frac{9}{4}\times\frac{5}{5}=\frac{45}{20}$$
$$\frac{4}{5}=\frac{4}{5}\times\frac{4}{4}=\frac{16}{20}$$
$$\frac{29}{20}=1\tfrac{9}{20}.$$

Explanation: $2\frac{1}{4}=\frac{45}{20}$; $\frac{4}{5}=\frac{16}{20}$.

$$\frac{45}{20}-\frac{16}{20}=\frac{29}{20}=1\tfrac{9}{20}\ \therefore 2\tfrac{1}{4}-\tfrac{4}{5}=1\tfrac{9}{20}.$$

Example 3. From $8\frac{1}{6}$ take $5\frac{4}{9}$.

$$8\tfrac{1}{6}=\frac{49}{6}\text{ and }5\tfrac{4}{9}=\frac{49}{9}.$$

$$\begin{array}{r|l} 3 & 6.\quad 9 \\ \hline & 2.\quad 3 \end{array}\qquad \begin{array}{l|l} 18\div6=3 & \frac{49}{6}=\frac{147}{18}. \\ 18\div9=2 & \frac{49}{9}=\frac{98}{18}. \end{array}$$

$$3\times2\times3=18.\qquad\qquad \frac{49}{18}=2\tfrac{13}{18}.$$

Explanation: $8\frac{1}{6}=\frac{147}{18}$; $5\frac{4}{9}=\frac{98}{18}$.

$$\frac{147}{18}-\frac{98}{18}=\frac{147-98}{18}=\frac{49}{18}=2\tfrac{13}{18}\ \therefore 8\tfrac{1}{6}-5\tfrac{4}{9}=2\tfrac{13}{18}.$$

181. From Articles 178, 179, and 180 is deduced the following general rule for subtraction of fractions:

Prepare the given numbers for subtraction by reducing, if necessary, mixed numbers to improper fractions, whole numbers to the form of fractions, and fractional expressions having different denominators to equivalent fractions having a common denominator; then subtract the less numerator from the greater, and place the remainder over the common denominator for the result required.

182. Exercises in subtraction of fractions:

1st. $\frac{7}{8}-\frac{3}{8}$?..*Ans.* $\frac{1}{2}$.
2d. $\frac{7}{12}-\frac{5}{12}$?..*Ans.* $\frac{1}{6}$.
3d. $\frac{5}{7}-\frac{4}{9}$?..*Ans.* $\frac{17}{63}$.
4th. $2\frac{1}{11}-\frac{10}{11}$?..*Ans.* $1\frac{2}{11}$.
5th. $10-4\frac{3}{8}$?..*Ans.* $5\frac{5}{8}$.
6th. $\frac{5}{8}-\frac{3}{10}$?..*Ans.* $\frac{13}{40}$.
7th. $\frac{4876}{4875}-1$?..*Ans.* $\frac{1}{4875}$.
8th. $\frac{11}{12}-\frac{5}{7}$?..*Ans.* $\frac{17}{84}$.
9th. $\frac{21}{20}-\frac{20}{21}$?..*Ans.* $\frac{41}{420}$.
10th. $2\frac{1}{4}-\frac{5}{12}$?..*Ans.* $1\frac{5}{6}$.
11th. $12\frac{5}{9}-9\frac{4}{7}$?..*Ans.* $2\frac{62}{63}$.
12th. $9\frac{2}{7}-5\frac{1}{2}$?..*Ans.* $3\frac{11}{14}$.
13th. $\frac{19}{20}-\frac{13}{17}$?..*Ans.* $\frac{63}{340}$.
14th. $13\frac{3}{4}-5\frac{11}{13}$?..*Ans.* $7\frac{47}{52}$.
15th. $7\frac{13}{18}-4$?..*Ans.* $3\frac{13}{18}$.
16th. $\frac{127}{129}-\frac{117}{119}$?..*Ans.* $\frac{20}{15351}$.
17th. $2-\frac{145}{144}$?..*Ans.* $\frac{143}{144}$.

18th. $\frac{190}{189}-\frac{189}{190}$?...................................*Ans.* $\frac{379}{35910}$.

19th. $200\frac{7}{12}-129\frac{3}{4}$?...................................*Ans.* $70\frac{5}{6}$.

20th. $1165\frac{4}{27}-795\frac{19}{81}$?...................................*Ans.* $369\frac{74}{81}$.

183. The subtraction of a mixed number or a fraction from a whole or mixed number can always be effected by the rule of Article 181. But when the whole numbers or the integral parts of the mixed numbers are expressed by several figures, the reductions are tedious. In these cases the subtraction may be more easily made by other methods, for the more convenient explanation of which,

1st. Let it be required to subtract $\frac{5}{11}$ from 1 :

Since unity is equal to any number divided by itself,

$$1=\tfrac{11}{11},$$

$$\therefore 1-\tfrac{5}{11}=\tfrac{11}{11}-\tfrac{5}{11}=\tfrac{6}{11}=\text{remainder required.}$$

In practice, this subtraction is readily made by subtracting the numerator of the fraction from the denominator, and placing the difference over the denominator for the remainder required.

2d. Let it be required to subtract $\frac{13}{17}$ from 9.

1 must be borrowed from 9, which is thus reduced to 8.

$$\begin{array}{ll}\text{Then} \quad - \quad - & 1-\tfrac{13}{17}=\tfrac{4}{17}, \\ \text{and therefore} \quad - & 9-\tfrac{13}{17}=8\tfrac{4}{17}.\end{array}$$

The subtraction of a fraction from a whole number can therefore be made (without reduction of the latter) by subtracting the fraction from 1, and annexing the remainder to the minuend diminished by unity.

3d. Let it be required to subtract $15\frac{3}{11}$ from 32.

1 must be borrowed from 32, which is thus reduced to 31.

$$\begin{array}{ll}\text{Then} \quad - \quad - & 1-\tfrac{3}{11}=\tfrac{8}{11}, \\ \text{and} \quad - \quad - & 31-15=16, \\ \text{therefore} \quad - & 32-15\tfrac{3}{11}=16\tfrac{8}{11}.\end{array}$$

Whence a mixed number can be subtracted from a whole number (without the reduction of either to an improper fraction) by subtracting the fractional part of the subtrahend from unity, and the integral part from the minuend diminished by unity.

4th. Let it be required to subtract 7 from $15\frac{3}{8}$.

$15\frac{3}{8}$ may be divided into the parts $8\frac{3}{8}$ and 7.

$$\therefore 15\tfrac{3}{8}-7=8\tfrac{3}{8}+7-7=8\tfrac{3}{8},$$

$$\text{or } 15\tfrac{3}{8}-7=8\tfrac{3}{8}.$$

To subtract, therefore, a whole number from a mixed number, subtract the whole number from the integral part of the mixed number; the remainder is the difference of the given numbers.

5th. Let it be required to subtract $\frac{5}{9}$ from $1\frac{1}{6}$.

$\frac{5}{9}$ and $\frac{1}{6}$, reduced to the same denominator, are $\frac{10}{18}$ and $\frac{3}{18}$.

$$\begin{array}{rr}\therefore 1\tfrac{1}{6}=1\tfrac{3}{18}= & \tfrac{21}{18} \\ \tfrac{5}{9}= & \underline{\tfrac{10}{18}} \\ & \tfrac{11}{18} \text{ remainder.}\end{array}$$

Since $\frac{10}{18}$ is greater than $\frac{3}{18}$, it becomes necessary to reduce $1\frac{3}{18}$ to an improper fraction. This, in practice, is accomplished by adding the numerator of the fraction to the denominator, and placing the sum over the denominator.

$$\begin{array}{ll}\text{Thus} & 3+18=21 \text{ and } 1\tfrac{3}{18}=\tfrac{21}{18}. \\ \text{Then} & \tfrac{21}{18}-\tfrac{10}{18}=\tfrac{11}{18}, \text{ which is the excess of } 1\tfrac{1}{6} \text{ over } \tfrac{5}{9}.\end{array}$$

6th. Let it be required to subtract $\frac{5}{6}$ from $8\frac{1}{4}$.

The fractions, reduced to a common denominator, are $\frac{10}{12}$ and $\frac{3}{12}$.

$$\begin{array}{rr}\text{Whence } 8\tfrac{1}{4}= & 8\tfrac{3}{12} \\ \tfrac{5}{6}= & \underline{\tfrac{10}{12}} \\ & 7\tfrac{5}{12} \text{ remainder.}\end{array}$$

Since $\frac{10}{12}$ cannot be taken from $\frac{3}{12}$, 1 must be borrowed from 8; the 8 is ntly reduced to 7.

The fraction $\frac{10}{12}$ is then subtracted, as in example 5, from $1\frac{3}{12}$.

$$1\tfrac{3}{12}=\tfrac{15}{12}\text{; and }\tfrac{15}{12}-\tfrac{10}{12}=\tfrac{5}{12}.$$
$$\therefore 8\tfrac{3}{12}-\tfrac{10}{12}=7\tfrac{5}{12},$$
$$\text{or } 8\tfrac{1}{4}-\tfrac{5}{6}=7\tfrac{5}{12}.$$

7th. Let it be required from $69\frac{5}{6}$ to subtract $46\frac{3}{8}$.

The fractions $\frac{5}{6}$ and $\frac{3}{8}$, reduced to the same denominator, are $\frac{20}{24}$, $\frac{9}{24}$.

$$\begin{aligned}\therefore\ 69\tfrac{5}{6}&=69\tfrac{20}{24}\\ \text{and } 46\tfrac{3}{8}&=\underline{46\tfrac{9}{24}}\\ &\quad\underline{23\tfrac{11}{24}}\text{ remainder.}\end{aligned}$$

In this example the fractional and integral parts of the subtrahend are both less than the corresponding parts of the minuend. The only reduction required is, therefore, that of the fractional parts to the same denominator.

8th. Let it be required to subtract $29\frac{3}{4}$ from $54\frac{2}{7}$.

$$\begin{aligned}\tfrac{2}{7}=\tfrac{2}{7}\times\tfrac{4}{4}=\tfrac{8}{28}\ \therefore 54\tfrac{2}{7}&=54\tfrac{8}{28}\\ \tfrac{3}{4}=\tfrac{3}{4}\times\tfrac{7}{7}=\tfrac{21}{28}\ \therefore 29\tfrac{3}{4}&=\underline{29\tfrac{21}{28}}\\ &\quad\underline{24\tfrac{15}{28}}\text{ remainder.}\end{aligned}$$

The fractions $\frac{2}{7}$ and $\frac{3}{4}$ being reduced to the same denominator, and the subtrahend written under the minuend, it is found that $\frac{21}{28}$, the fractional part of the former, is greater than $\frac{8}{28}$, the fractional part of the latter.

To render the subtraction possible 1 is borrowed from the integral part of the minuend, and $\frac{21}{28}$ is subtracted from $1\frac{8}{28}$, as in Example 5.

$$1\tfrac{8}{28}=\tfrac{36}{28}\text{; and }\tfrac{36}{28}-\tfrac{21}{28}=\tfrac{15}{28}.$$

Next, for the 1 borrowed, take 1 from 54, or add 1 to 29; in either case the difference $53-29$, or $54-30,=24$.

$$\text{Whence } 54\tfrac{8}{28}-29\tfrac{21}{28}=24\tfrac{15}{28},$$
$$\text{or } 54\tfrac{2}{7}-29\tfrac{3}{4}\ =24\tfrac{15}{28}.$$

184. From the examples and illustrations given in the last Article are deduced the following rules for the subtraction of fractions or mixed numbers from whole or mixed numbers without reducing the mixed or whole numbers to improper fractions:

a. To subtract a proper fraction from unity: subtract the numerator of the given fraction from the denominator; the remainder placed over the denominator expresses the excess of unity over the proper fraction.

b. To subtract a proper fraction from any whole number: subtract the fraction from unity, and annex the fractional remainder to the whole number diminished by unity; the mixed number, thus obtained, is the difference between the given whole number and proper fraction.

c. To subtract a mixed number from a whole number: subtract the fractional part of the mixed number from unity; carry 1 to the integral part of the mixed number; subtract the sum from the whole number; the mixed number composed of the integral and fractional remainders is the excess of the whole over the mixed number.

d. To subtract a whole number from a mixed number: subtract the whole number from the integral part of the mixed number, and to the remainder annex the fractional part of the latter; the mixed number thus formed is the excess of the mixed over the whole number.

e. To subtract the greater of two fractions having a common denominator from the less preceded by 1: add together the numerator and denominator of the less fraction; subtract the numerator of the greater fraction from the sum, and place the remainder over the common denominator for the difference required.

f. To find the difference of two mixed numbers: reduce the fractional parts to a common denominator, and place the less mixed number under the greater.

Then, if the fractional part of the subtrahend is less than that of the minuend, the difference of the fractions is the fractional part; and the difference of the whole numbers the integral part of the mixed number which expresses the difference of the given quantities.

But if the fractional part of the subtrahend is greater than that of the minuend, subtract the former from the latter augmented by unity: the remainder is the fractional part of the difference; carry 1, for that borrowed, to the integral part of the subtrahend, and subtract the sum from the integral part of the minuend; the remainder is the integral part of the difference. The fractional part being annexed to this, the result is the difference of the given mixed numbers.

If the fractional parts are equal, the remainder is a whole number; and if the integral parts are equal, the remainder is a proper fraction.

185. Additional exercises in the subtraction of fractions:

1st. $1-\frac{9}{17}$?............................*Ans.* $\frac{8}{17}$.
2d. $100-\frac{13}{20}$?............................*Ans.* $99\frac{7}{20}$.
3d. $95-31\frac{29}{45}$?............................*Ans.* $63\frac{16}{45}$.
4th. $79\frac{8}{11}-56$?............................*Ans.* $23\frac{8}{11}$.
5th. $1\frac{8}{45}-\frac{43}{45}$?............................*Ans.* $\frac{2}{9}$.
6th. $72\frac{3}{16}-56\frac{11}{40}$?............................*Ans.* $15\frac{73}{80}$.
7th. $2050\frac{4}{27}-172\frac{5}{9}$?............................*Ans.* $1877\frac{16}{27}$.
8th. $200-99\frac{11}{23}$?............................*Ans.* $100\frac{12}{23}$.
9th. $375-\frac{14}{17}$?............................*Ans.* $374\frac{3}{17}$.
10th. $7854\frac{1}{4}-5236\frac{4}{17}$?............................*Ans.* $2618\frac{1}{68}$.
11th. $1745\frac{3}{8}-998\frac{3}{4}$?............................*Ans.* $746\frac{5}{8}$.
12th. $31415\frac{11}{12}-4134\frac{23}{24}$?............................*Ans.* $27280\frac{23}{24}$.
13th. $74\frac{26}{133}-38\frac{18}{57}$?............................*Ans.* $35\frac{117}{133}$.
14th. $2-\frac{7855}{7854}$?............................*Ans.* $\frac{7853}{7854}$.
15th. $3786\frac{9}{56}-1735\frac{15}{16}$?............................*Ans.* $2050\frac{25}{112}$.
16th. $8374\frac{7}{15}-3598\frac{99}{100}$?............................*Ans.* $4775\frac{143}{300}$.
17th. $3768\frac{33}{57}-2998\frac{67}{114}$?............................*Ans.* $769\frac{113}{114}$.
18th. $1746\frac{3}{16}-986\frac{29}{64}$?............................*Ans.* $759\frac{47}{64}$.
19th. $157\frac{2}{9}-54\frac{4}{7}$?............................*Ans.* $102\frac{41}{63}$.
20th. $5831\frac{2}{13}-1827\frac{11}{52}$?............................*Ans.* $4003\frac{49}{52}$.

185′. The reasoning employed in Articles 178 and 179 is general, and therefore applicable to literal fractions.

The fractions $\frac{n}{d}$, $\frac{n'}{d}$ being equal, respectively, to n repetitions of the $\frac{1}{d}$ part of unity, and to n' repetitions of the $\frac{1}{d}$ part of unity, their difference is equal to $n-n'$ repetitions of the common quantity expressed by $\frac{1}{d}$, that is, to $\frac{n-n'}{d}$.

If the difference of the fractions $\frac{n}{d}, \frac{n'}{d'}$, which have different denominators, is required, it becomes necessary to reduce $\frac{n}{d}, \frac{n'}{d'}$ to the same denominator (Art. 160).

$$\text{Now } \frac{n}{d}=\frac{n}{d}\times\frac{d'}{d'}=\frac{nd'}{dd'} \text{ and } \frac{n'}{d'}=\frac{n'}{d'}\times\frac{d}{d}=\frac{n'd}{dd'},$$

$$\text{And } \frac{nd'}{dd'}-\frac{n'd}{dd'}=\frac{nd'-n'd}{dd'}.$$

$$\text{Therefore } \frac{n}{d}-\frac{n'}{d'}=\frac{nd'-n'd}{dd'}.$$

Required the difference of the following fractional expressions:

1st. $\frac{mn^2}{d^3}$ and $\frac{n^3}{d^4}$?..*Ans.* $\frac{mn^2d-n^3}{d^4}$.

2d. $\frac{a}{b^2c}$ and $\frac{a}{b^3c^5}$?..*Ans.* $\frac{a(bc^4-1)}{b^3c^5}$.

3d. $\frac{m+n}{b^3c^2}$ and $\frac{m^2+n^3}{ab^2c^4}$?......................*Ans.* $\frac{ac^2m+ac^2n-bm^2-bn^3}{ab^3c^4}$.

4th. $m+\frac{n}{d}$ and $\frac{b}{c}$?..*Ans.* $\frac{cdm+cn-bd}{cd}$.

MULTIPLICATION OF FRACTIONS.

186. In the multiplication of whole numbers two integral factors, the multiplicand and multiplier, are given, and it is required to find a third number which shall contain the multiplicand as often as the multiplier contains unity.

In multiplication of fractions the integral are replaced by fractional factors, and it is required to find a product which shall contain some part or parts of the multiplicand as often as the multiplier contains the same part or parts of unity.

Combining these cases, the problem of multiplication may be enunciated more generally thus: given two numbers to find a third which shall have to the first the same relation which the second has to unity.

It may be required in the multiplication of fractional quantities,

1st. To multiply a fraction or a mixed number by a whole number.

2d. To multiply a whole number by a fraction or a mixed number.

3d. To multiply a mixed number or a fraction by a mixed number or a fraction.

187. Let it be required to multiply $\frac{3}{11}$ by 7, $\frac{5}{12}$ by 4, and $\frac{5}{21}$ by 3.

$$\text{By Art. 148}\quad \frac{3}{11}\times7=\frac{3\times7}{11}=\frac{21}{11}=1\frac{10}{11}$$

$$-\qquad \frac{5}{12}\times4=\frac{5}{12\div4}=\frac{5}{3}=1\frac{2}{3}$$

$$-\qquad \frac{5}{21}\times3=\frac{5\times3}{21}=\frac{15}{21}=\frac{5}{7};$$

$$-\quad\text{or } \frac{5}{21}\times3=\frac{5}{21\div3}=\frac{5}{7}.$$

Let it also be required to multiply $5\frac{2}{3}$ by 8 and $4\frac{7}{16}$ by 4.

$$5\frac{2}{3}=\frac{17}{3} \text{ and } 4\frac{7}{16}=\frac{71}{16} \quad (\text{Art. 157}).$$

$$\therefore\ 5\frac{2}{3}\times8=\frac{17}{3}\times8=\frac{17\times8}{3}=\frac{136}{3}=45\frac{1}{3},$$

$$\text{and } 4\frac{7}{16}\times4=\frac{71}{16}\times4=\frac{71}{16\div4}=\frac{71}{4}=17\frac{3}{4}.$$

This case has been fully considered in Article 148.

The conclusion is, that to multiply a fraction or a mixed number by a whole number, it is necessary to bring the mixed number to an improper fraction, and then to multiply the numerator of the fractional multiplicand by the integral multiplier, or to divide the denominator of the fractional multiplicand by the integral multiplier; the result of either process is the product required.

188. Let it be required to multiply 8 by $\frac{3}{5}$, and 14 by $5\frac{1}{4}$.

1st. Since the multiplier $\frac{3}{5}$ is equal to 3 times the 5th part of unity, the product of 8 by $\frac{3}{5}$ must be equal to three times the 5th part of 8 (Art. 186).

Now the 5th part of 8 is $\frac{8}{5}$, and by Article 187

$$3 \text{ times } \frac{8}{5} \text{ or } \frac{8}{5}\times 3=\frac{8\times 3}{5}=\frac{24}{5}=4\frac{4}{5}.$$

$$\therefore\ 8\times\frac{3}{5}=\frac{24}{5}=4\frac{4}{5}.$$

2d. The mixed number $5\frac{1}{4}$ being reduced to the equivalent improper fraction $\frac{21}{4}$, the product of 14 by $5\frac{1}{4}$ is equal to the product of 14 by $\frac{21}{4}$. The multiplier $\frac{21}{4}$ is equal to 21 times the 4th part of unity; therefore $14\times\frac{21}{4}$ is equal to 21 times the 4th part of 14.

But the 4th part of $14=\frac{14}{4}$,

$$\text{and 21 times } \frac{14}{4}=\frac{14}{4}\times 21=\frac{14\times 21}{4}=\frac{294}{4}=73\frac{2}{4}=73\frac{1}{2}.$$

$$\text{Whence } 14=5\frac{1}{4}=73\frac{1}{2}.$$

Consequently, to find the product of a whole number by a proper or by an improper fraction (the latter being substituted for a mixed number), multiply the whole number by the numerator of the fractional multiplier and divide the product by its denominator; the result is the product required.

189. Let it be required to multiply, 1st. $\frac{2}{3}$ by $\frac{5}{7}$; 2d. $2\frac{1}{3}$ by $\frac{4}{5}$; 3d. $\frac{8}{9}$ by $6\frac{1}{3}$; 4th. $2\frac{1}{3}$ by $5\frac{3}{7}$.

1st. It is required to multiply $\frac{2}{3}$ by $\frac{5}{7}$.

Since the multiplier is equal to 5 times the 7th part of unity, the product of $\frac{2}{3}$ by $\frac{5}{7}$ must be equal to 5 times the 7th part of $\frac{2}{3}$.

$$\text{Now the 7th part of } \frac{2}{3}=\frac{2}{3\times 7},$$

$$\text{and 5 times } \frac{2}{3\times 7}=\frac{2\times 5}{3\times 7}.$$

$$\therefore\ \frac{2}{3}\times\frac{5}{7}=\frac{2\times 5}{3\times 7}=\frac{10}{21}.$$

The product of the fractional multiplicand $\frac{2}{3}$ by the fractional multiplier $\frac{5}{7}$ is, hence, expressed by a fraction whose numerator is the product of the numerators of the given factors, and its denominator the product of the denominators of the given factors.

2d. It is required to multiply $2\frac{1}{3}$ by $\frac{4}{5}$.

$$2\frac{1}{3}=\frac{7}{3}\quad \therefore\ 2\frac{1}{3}\times\frac{4}{5}=\frac{7}{3}\times\frac{4}{5}.$$

The multiplier being equal to 4 times the 5th part of unity, the product $\frac{7}{3}\times\frac{4}{5}$ must be equal to 4 times the 5th part of $\frac{7}{3}$.

$$\text{The 5th of } \frac{7}{3}=\frac{7}{3\times 5},$$

$$\text{and 4 times the 5th part of } \frac{7}{3\times 5}=\frac{7\times 4}{3\times 5}.$$

$$\text{Therefore } 2\frac{1}{3}\times\frac{4}{5}\text{ or }\frac{7}{3}\times\frac{4}{5}=\frac{7\times 4}{3\times 5}=\frac{28}{15}=1\frac{13}{15}.$$

The product of a mixed number by a proper fraction is hence obtained by reducing the mixed number to an improper fraction, and forming the product of the two fractional factors as in the 1st example of this Article.

3d. It is required to multiply $\frac{8}{9}$ by $6\frac{1}{3}$.

$$6\frac{1}{3}=\frac{19}{3}\ \therefore\ \frac{8}{9}\times 6\frac{1}{3}=\frac{8}{9}\times\frac{19}{3}.$$

The multiplier being equal to 19 times the 3d part of unity, the product $\frac{8}{9}\times\frac{19}{3}$ must be equal to 19 times the 3d part of $\frac{8}{9}$.

$$\text{The 3d part of } \frac{8}{9}=\frac{8}{9\times 3},$$

$$\text{and 19 times the 3d part of } \frac{8}{9\times 3}=\frac{8\times 19}{9\times 3}.$$

$$\therefore\ \frac{8}{9}\times 6\frac{1}{3}=\frac{8}{9}\times\frac{19}{3}=\frac{8\times 19}{9\times 3}=\frac{152}{27}=5\frac{17}{27}.$$

Wherefore the product of a proper fraction by a mixed number is obtained by reducing the mixed number to an improper fraction, and forming the product of the two fractional factors again, as in the 1st Example of this Article.

4th. It is required to multiply $2\frac{1}{5}$ by $5\frac{2}{7}$.

$$2\tfrac{1}{5}=\tfrac{11}{5}, \text{ and } 5\tfrac{2}{7}=\tfrac{37}{7}.$$

$$\therefore 2\tfrac{1}{5}\times 5\tfrac{2}{7}=\tfrac{11}{5}\times\tfrac{37}{7}.$$

The multiplier contains the 7th part of unity 37 times, therefore the product of $\frac{11}{5}$ by $\frac{37}{7}$ must contain the 7th part of $\frac{11}{5}$ 37 times.

The 7th part of $\frac{11}{5}=\frac{11}{5\times 7}$,

and 37 times $\frac{11}{5\times 7}=\frac{11\times 37}{5\times 7}=\frac{407}{35}=11\frac{22}{35}$,

therefore $2\frac{1}{5}\times 5\frac{2}{7}=\frac{11}{5}\times\frac{37}{7}=\frac{11\times 37}{5\times 7}=\frac{407}{35}=11\frac{22}{35}$.

Whence, also, the product of two mixed numbers is obtained by reducing the mixed numbers to improper fractions, and forming the product of the two fractional factors as in the 1st Example of this Article.

190. The first and second cases of Article 187 may be rendered identical with the first and second cases of Article 189, by giving the integral factor unity as a denominator; and the first and second of Article 188 with the first and third of Article 189, by the same means.

Whence, employing the phrase, *a fractional expression*, to describe either a proper fraction, or a whole number reduced to the fractional form, or a mixed number reduced to an improper fraction, the product of any two fractional expressions is equal to a fraction whose numerator is the product of the numerators of the given fractional factors, and its denominator the product of their denominators.

This conclusion is inferred from results obtained by the application of general reasoning to particular examples. As the process for one example of a case, and for all the examples falling under that case, is the same, and all the cases which can occur have been considered, it follows that the conclusion is general.

191. The general rule for the multiplication of any fractional expressions may be obtained more directly by means of general symbols.

Let a represent the numerator, and b the denominator, of the multiplicand; c the numerator, and d the denominator, of the multiplier; a, b, c, d denoting any whole numbers whatever; then $\frac{a}{b}$ represents the fractional multiplicand, $\frac{c}{d}$ the fractional multiplier, and the product of $\frac{a}{b}$ by $\frac{c}{d}$ is indicated by the expression $\frac{a}{b}\times\frac{c}{d}$.

If the fractional expression $\frac{a}{b}$ is actually multiplied by $\frac{c}{d}$, the product must have to $\frac{a}{b}$ the same relation which $\frac{c}{d}$ has to unity.

Now, the relation of $\frac{c}{d}$ to unity is obtained by dividing unity into a number of equal parts expressed by the numerical value of d, and taking of these equal parts a number expressed by the numerical value of c. One of these parts is the dth part of unity, and the number of such parts is c. Whence the multiplier is equal to c times the dth part of unity. The product of $\frac{a}{b}$ by $\frac{c}{d}$ must consequently be equal to c times the dth part of $\frac{a}{b}$. But the dth part of $\frac{a}{b}$ is $\frac{a}{bd}$ (Art. 148 b), and c times $\frac{a}{bd}$ is $\frac{ac}{bd}$ (Art. 148 a).

Wherefore the product of the fractional expressions $\frac{a}{b}$, $\frac{c}{d}$ is $\frac{ac}{bd}$.

The numerator of this product is the product of the numerators of the fractional factors, and its denominator the product of the denominators of the fractional factors. Whence, generally, the product of two fractional factors is equal to a fractional expression, whose numerator is the product of the numerators of the given factors, and its denominator the product of their denominators.

192. Since $ac=ca$, $bd=db$ (Art. 56), and therefore $\frac{ac}{bd}=\frac{ca}{db}$, it follows that either of the fractions may be made the multiplicand, and the other the multiplier.

Also, since either b or d may be equal to 1, it follows that the product of a whole number by a fractional expression is equal to the product of the fractional expression by the whole number.

If any of the numerators and denominators are not prime numbers, they may be resolved into simple factors: thus if $a=m^2n^3$, $b=m'^3n'$, $c=m'n'^2$, $d=m^3n$, the product $\frac{ac}{bd}=\frac{m^2n^3m'n'^2}{m'^3n'm^3n}$.

$$\text{But } \frac{m^2n^3m'n'^2}{m'^3n'm^3n}=\frac{n^2n'}{m'^2m} \text{ (Art. 153).}$$

$$\therefore \frac{ac}{bd}=\frac{n^2n'}{m'^2m}.$$

Whence, if the numerator and denominator of the product of two fractional expressions are resolved into simple factors, and the simple factors common to both terms are omitted, the value of the product is not altered, and the fraction expressing the product is reduced to its lowest terms.

As an illustration, let it be required to multiply $\frac{9}{16}$ by $\frac{8}{15}$.

$$9=3^2,\ 16=2^4,\ 8=2^3,\ 15=3\times5.$$

$$\therefore \frac{9}{16}\times\frac{8}{15}=\frac{9\times8}{16\times15}=\frac{3^2\times2^3}{2^4\times3\times5}=\frac{3}{2\times5}=\frac{3}{10},$$

$$\text{or } \frac{9}{16}\times\frac{8}{15}=\frac{3}{10}.$$

193. Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$ represent three fractional expressions, and let it be required to find the continual product of $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$.

The continual product of three integral factors is obtained by multiplying together two of the factors, and then multiplying their product by the third factor (Art. 67); that is (Art. 186), a first product is found which has to one of the factors, taken as multiplicand, that relation which another of the factors, taken as multiplier, has to unity. Then, a second product is found which has to the first product, taken as multiplicand, that relation which the remaining factor, taken as multiplier, has to unity. The second product is that of the three factors.

Applying these principles to the case of three fractional factors, the last product must be equal to the product of $\frac{a}{b}\times\frac{c}{d}$, considered as multiplicand, by $\frac{e}{f}$, taken as multiplier.

$$\text{Now } \frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd} \text{ (Art. 191),}$$

$$\text{and } \frac{ac}{bd}\times\frac{e}{f}=\frac{ace}{bdf};$$

whence the product of three fractional expressions is obtained by multiplying together the numerators of the fractional factors for the numerator of the product, and the denominators for its denominator. It is evident that the factors may be multiplied together in any order (Art. 71), and the simple factors common to the numerators and denominators omitted in forming the product (Art. 153).

The product of four fractional factors is formed from that of three as the product of three is from that of two, and similarly for a greater number of factors than four.

Let it be required to find the continual product of the three fractions, $\frac{5}{7}$, $\frac{9}{10}$, $\frac{14}{15}$:

$$\frac{5}{7}\times\frac{9}{10}\times\frac{14}{15}=\frac{5\times9\times14}{7\times10\times15}=\frac{630}{1050}.$$

The greatest common measure of 630 and 1050 is 210 (Art. 108).

$$\therefore \frac{630}{1050}=\frac{3}{5}, \text{ or } \frac{5}{7}\times\frac{9}{10}\times\frac{14}{15}=\frac{3}{5}.$$

$$\text{Otherwise } \frac{9}{10}=\frac{3^2}{2\times5}, \quad \frac{14}{15}=\frac{2\times7}{3\times5}.$$

$$\therefore \frac{5}{7}\times\frac{9}{10}\times\frac{14}{15}=\frac{5}{7}\times\frac{3^2}{2\times5}\times\frac{2\times7}{3\times5}=\frac{5\times3^2\times2\times7}{7\times2\times5\times3\times5}=\frac{3}{5}.$$

In this example the calculation is presented in two forms, differing slightly the one from the other.

1st. The product of the numerators, 5, 9, 14, is 630; and that of the denominators, 7, 10, 15, is 1050. The product of the given fractions is therefore $\frac{630}{1050}$.

To reduce this result to the lowest terms, it is necessary to find the greatest common measure of 630 and 1050. This is 210.

Then $630\div210=3$, and $1050\div210=5$.

$$\text{Whence } \frac{5\times9\times14}{7\times10\times15}=\frac{630}{1050}=\frac{3}{5}.$$

2d. The terms of the fractions are decomposed into simple factors; viz. 9 into 3^2, 10 into 2×5, 14 into 2×7, and 15 into 3×5.

The product, expressed in terms of these simple factors, is $\frac{5\times3\times3\times2\times7}{7\times2\times5\times3\times5}$. Omitting the simple factors common to the numerator and denominator of the product (which are 5, 3, 2, 7), the result obtained is $\frac{3}{5}$, as before.

Let it be required to find the product of the three factors $\frac{4}{9}$, $4\frac{1}{5}$, and 3:

$$\frac{4}{9}=\frac{2^2}{3^2}, \quad 4\frac{1}{5}=\frac{21}{5}=\frac{3\times7}{5}, \quad 3=\frac{3}{1}.$$

$$\therefore \frac{4}{9}\times4\frac{1}{5}\times3=\frac{2^2\times3\times7\times3}{3^2\times5\times1}=\frac{2^2\times7}{5}=\frac{28}{5}=5\frac{3}{5}.$$

$5\frac{3}{5}$ is therefore the product required.

Let it be required to find the product of the factors $\frac{7}{10}$, $3\frac{4}{7}$, $\frac{11}{12}$, and 14:

$$\frac{7}{10}=\frac{7}{2\times5}, \quad 3\frac{4}{7}=\frac{25}{7}=\frac{5^2}{7}, \quad \frac{11}{12}=\frac{11}{2^2\times3}, \quad 14=\frac{14}{1}=\frac{2\times7}{1}.$$

$$\therefore \frac{7}{10}\times3\frac{4}{7}\times\frac{11}{12}\times\frac{14}{1}=\frac{7\times5^2\times11\times2\times7}{2\times5\times7\times2^2\times3\times1}=\frac{5\times11\times7}{2^2\times3}=\frac{385}{12}=32\frac{1}{12}.$$

194. To find the product of any factors consisting of fractions, fractions and whole numbers, fractions and mixed numbers, whole numbers and mixed numbers, or mixed numbers:

General Rule. Reduce mixed numbers to improper fractions, and whole numbers to the form of fractions; then multiply together the numerators of all the fractional expressions for the numerator of the product, and the denominators of all the fractional expressions for the denominator of the product.

If the terms of the fractional expressions are decomposed into prime factors, and all the prime factors which are common to both terms of the product are omitted, the product will be obtained in the lowest terms; or, if the factors are multiplied together without decomposition, and the products are divided by their greatest common measure, the result will be the product of the given fractions also, expressed in the lowest terms.

195. Exercises in the multiplication of fractions:

1st. $\frac{3}{4}\times\frac{5}{7}$? .... *Ans.* $\frac{15}{28}$.
2d. $\frac{4}{17}\times\frac{3}{13}$? .... *Ans.* $\frac{12}{221}$.
3d. $\frac{3}{8}\times\frac{6}{7}$? .... *Ans.* $\frac{9}{28}$.
4th. $\frac{4}{5}\times 13$? .... *Ans.* $10\frac{2}{5}$.
5th. $153\times\frac{7}{9}$? .... *Ans.* 119.
6th. $6\frac{1}{7}\times\frac{4}{5}$? .... *Ans.* $4\frac{32}{35}$.
7th. $\frac{5}{9}\times 5\frac{1}{4}$? .... *Ans.* $2\frac{11}{12}$.
8th. $8\frac{2}{5}\times 12\frac{13}{14}$? .... *Ans.* $108\frac{3}{5}$.
9th. $19\frac{5}{6}\times 45\frac{3}{8}$? .... *Ans.* $899\frac{15}{16}$.
10th. $\frac{2}{3}\times\frac{9}{10}\times\frac{10}{11}$? .... *Ans.* $\frac{6}{11}$.
11th. $\frac{3}{5}\times\frac{7}{8}\times\frac{16}{21}$? .... *Ans.* $\frac{2}{5}$.
12th. $2\frac{1}{3}\times\frac{1}{2}\times\frac{4}{9}$? .... *Ans.* $\frac{14}{27}$.
13th. $9\frac{3}{4}\times\frac{8}{33}\times 7\frac{1}{3}$? .... *Ans.* $17\frac{1}{3}$.
14th. $12\times 10\frac{3}{4}\times\frac{7}{24}$? .... *Ans.* $37\frac{5}{8}$.
15th. $2\frac{3}{4}\times 3\frac{1}{5}\times 6\frac{2}{7}$? .... *Ans.* $55\frac{11}{35}$.
16th. $\frac{1}{2}\times\frac{7}{13}\times\frac{3}{5}\times\frac{4}{11}$? .... *Ans.* $\frac{42}{715}$.
17th. $\frac{6}{17}\times\frac{85}{114}\times\frac{33}{34}\times 1\frac{6}{11}$? .... *Ans.* $\frac{15}{38}$.
18th. $\frac{5}{21}\times 3\frac{1}{7}\times\frac{14}{15}\times\frac{3}{11}$? .... *Ans.* $\frac{4}{21}$.
19th. $\frac{2}{3}\times 1\frac{1}{4}\times\frac{3}{8}\times 17$? .... *Ans.* $5\frac{5}{16}$.
20th. $3\frac{1}{5}\times 5\frac{5}{8}\times 3\frac{5}{9}$? .... *Ans.* 64.
21st. $10\frac{1}{2}\times\frac{3}{4}\times 11\times 2\frac{1}{7}$? .... *Ans.* $185\frac{5}{8}$.
22d. $3\times\frac{2}{3}\times 5\frac{1}{2}\times\frac{3}{4}\times\frac{19}{22}\times 6$? .... *Ans.* $42\frac{3}{4}$.
23d. $3206\frac{11}{12}\times 709$? .... *Ans.* $2273703\frac{11}{12}$.
24th. $\frac{10}{11}\times 3316\frac{2}{15}$? .... *Ans.* $3014\frac{2}{3}$.
25th. $246\frac{7}{8}\times 351\frac{8}{9}$? .... *Ans.* $86872\frac{41}{72}$.

196. When the integral parts of the factors of any product are large numbers, the process of multiplication by the rule of Article 194 is laborious. The labour could in such cases be abridged by multiplying the factors together without reducing them to improper fractions.

In multiplying a mixed number or a fraction by an integer the multiplicand is repeated as often as the multiplier contains simple units; and in multiplying by a fraction, a part of the multiplicand, determined by the denominator of the multiplier, is repeated as often as the numerator of the multiplier contains simple units. The product of any mixed number or fraction by an integral or fractional multiplier is, consequently, equal to some number of repetitions of either the whole multiplicand or a part of the multiplicand. The sum of these repetitions of the multiplicand, or of some part of it, can be found by addition. Now, in addition of mixed numbers the integers and fractions are added together separately, and the results added into one sum (Art. 176).

Whence, if a multiplicand is decomposed into two parts, the one an integer, the other a fraction, and if the repetitions of the integer or of some part of it are combined into one result, and the repetitions of the fraction or of some part of it are combined into another, and the two results are added together, their sum is equal to that of the same number of repetitions of the undecomposed multiplicand or of some part of it.

The sum of any number of repetitions of a quantity being found by multiplication, it follows that if the integral and fractional parts of a multiplicand are separately multiplied by the multiplier, and the partial products are added together, their sum is equal to the product of the multiplicand and multiplier.

Consequently the product of a mixed number by a whole number or a fraction is obtained, without reduction of the factors to improper fractions, by multiplying the integral and fractional parts of the multiplicand separately by the multiplier, and taking the sum of the partial products for that required.

Examples of the application of this method:

1st. Let it be required to multiply 8547 by $\frac{3}{4}$.

The product of a whole number by a fraction is obtained by multiplying the whole number by the numerator of the fraction, and dividing the product by the denominator (Art. 188).

Now $8547\times3=25641$, and $25641\div4=6410\frac{1}{4}$.

$$\therefore 8547\times\tfrac{3}{4}=6410\tfrac{1}{4}.$$

The calculation may be made as follows:

$$\begin{array}{r} 8547 \\ \underline{\quad 3} \\ 4)\underline{25641} \\ \underline{6410\frac{1}{4}} \end{array}$$

Since the product of a fraction by an integer is equal to the product of the integer by the fraction (Art. 191), it follows that, the fraction being made the multiplicand and the integer the multiplier, the product is still formed in the same manner.

2d. Let it be required to multiply $6835\frac{3}{5}$ by 7.

$$\begin{aligned} 6835\tfrac{3}{5}&=6835+\tfrac{3}{5}, \\ \text{and } 6835\tfrac{3}{5}\times7&=6835\times7+\tfrac{3}{5}\times7. \\ 6835\times7&=47845. \\ \tfrac{3}{5}\times7&=\qquad\ \ 4\tfrac{1}{5} \\ \hline \therefore 6835\tfrac{3}{5}\times7&=47849\tfrac{1}{5}. \end{aligned}$$

3d. Let it be required to multiply $2857\frac{3}{8}$ by $\frac{5}{6}$.

The product of $2857\frac{3}{8}$ by $\frac{5}{6}$ is equal to 5 repetitions of the 6th part of 2857 plus 5 repetitions of the 6th part of $\frac{3}{8}$.

$$\text{The 6th part of } 2857=\frac{2857}{6}=476\frac{1}{6}.$$

$$\text{5 repetitions of } 476\frac{1}{6}=\left\{\begin{array}{ll} 476\times5, \text{ or} & 2380 \\ +\frac{1}{6}\times5, \text{ or} & \frac{5}{6} \end{array}\right\}=2380\frac{5}{6}.$$

$$\text{The 6th part of } \frac{3}{8}=\frac{3}{8\times6}=\frac{3}{48}=\frac{1}{16},$$

$$\text{and 5 repetitions of } \frac{1}{16}=\frac{5}{16}.$$

$$\begin{array}{lrr} \therefore 2857\ \times\frac{5}{6}= & 2380\frac{5}{6}= & 2380\frac{40}{48}, \\ \quad\ \ \frac{3}{8}\times\frac{5}{6}= & \frac{5}{16}= & \frac{15}{48}, \\ \hline \text{and } 2857\frac{3}{8}\times\frac{5}{6}= & & 2381\frac{7}{48}. \end{array}$$

4th. Let it be required to multiply $246\frac{3}{5}$ by $86\frac{7}{10}$.

The product must be equal to 86 repetitions of $246\frac{3}{5}$ plus 7 repetitions of the 10th part of $246\frac{3}{5}$.

The product of $246\frac{3}{5}$ by 86 is found as in Example 2 of this Article, and the product of $246\frac{3}{5}$ by $\frac{7}{10}$ is found as in Example 3, thus,

$$\begin{array}{rl|l} 246\frac{3}{5}\times86=\Bigg\{ & \begin{array}{lr} 246\times86= & 21156 \\ \frac{3}{5}\times86= & 51\frac{3}{5} \end{array} & \begin{array}{l} \\ \frac{3}{5}\ =\frac{30}{50} \end{array} \\ 246\frac{3}{5}\times\frac{7}{10}=\Bigg\{ & \begin{array}{lr} 246\times\frac{7}{10}= & 172\frac{1}{5} \\ \frac{3}{5}\times\frac{7}{10}= & \frac{21}{50} \end{array} & \begin{array}{l} \frac{1}{5}\ =\frac{10}{50} \\ \frac{21}{50}=\frac{21}{50} \end{array} \\ \hline \therefore 246\frac{3}{5}\times86\frac{7}{10}= & 21380\frac{11}{50} & \quad 1\frac{11}{50}. \end{array}$$

From the principles and examples of this Article it appears that the product of a mixed number by a whole number or a fraction is composed of two partial products; the one being the product of the integral part of the mixed number by the multiplier; and the other, the product of the fractional part of the mixed number by the multiplier.

When the multiplier also is a mixed number, the product of the multiplicand by the integral part of the multiplier is composed of two partial products, and the product of the multiplicand by the fractional part of the multiplier is also composed of two partial products.

To obtain the product of two mixed numbers without reducing them to improper fractions, it is, hence, necessary to form four partial products. These are,

1st. The product of the integral parts of the factors.

2d. The product of the fractional part of the multiplicand by the integral part of the multiplier.

3d. The product of the integral part of the multiplicand by the fractional part of the multiplier.

4th. The product of the fractional parts of the two factors.

197. Let a, a' denote the integral parts, and b, b' the fractional parts, of two mixed numbers.

Then $a+b$, $a'+b'$ denote the mixed numbers.

Multiplying the first of these expressions by the second (Art. 76) $(a+b)\ (a'+b')=aa'+a'b+ab'+bb'$.

This formula expresses, generally, the conclusion of Article 196.

To find the product of a large mixed number by a whole number, a fraction, or another mixed number,

Rule. Consider each mixed number to be decomposed into an integral part and a fractional part; then find the product of each of the parts into which the multiplicand is decomposed by each of the parts into which the multiplier is decomposed, and add together the partial products. The result is the product of the given factors.

198. Additional exercises in the multiplication of fractions:

1st. $100 \times 99\frac{8}{9}$? .... *Ans.* $9988\frac{8}{9}$.
2d. $23571 \times \frac{3}{4}$? .... *Ans.* $17678\frac{1}{4}$.
3d. $4509\frac{3}{7} \times \frac{5}{9}$? .... *Ans.* $2505\frac{5}{21}$.
4th. $44963\frac{9}{10} \times \frac{5}{6}$? .... *Ans.* $37469\frac{11}{12}$.
5th. $4517\frac{3}{4} \times 9$? .... *Ans.* $40659\frac{3}{4}$.
6th. $763\frac{1}{3} \times 100\frac{9}{10}$? .... *Ans.* $77020\frac{1}{3}$.
7th. $71643 \times 509\frac{1}{9}$? .... *Ans.* $36474247\frac{1}{3}$.
8th. $135000\frac{3}{8} \times 72\frac{3}{4}$? .... *Ans.* $9730152\frac{9}{32}$.
9th. $261795 \times 5\frac{1}{2}$? .... *Ans.* $1439872\frac{1}{2}$.
10th. $5760\frac{2}{3} \times 27\frac{1}{5}$? .... *Ans.* $156690\frac{2}{15}$.
11th. $3159\frac{2}{7} \times 150\frac{1}{4}$? .... *Ans.* $474682\frac{19}{28}$.
12th. $72918\frac{4}{11} \times 65\frac{7}{9}$? .... *Ans.* $4796407\frac{91}{99}$.
13th. $32486\frac{4}{27} \times 36\frac{3}{8}$? .... *Ans.* $1181683\frac{23}{36}$.
14th. $64120\frac{2}{3} \times 107\frac{3}{4}$? .... *Ans.* $6908973\frac{1}{10}$.
15th. $7970\frac{19}{26} \times 879\frac{78}{133}$? .... *Ans.* $7010946\frac{461}{494}$.
16th. $5249\frac{1}{3} \times 67\frac{1}{3}$? .... *Ans.* $353455\frac{1}{9}$.
17th. $7942\frac{14}{15} \times 246\frac{3}{20}$? .... *Ans.* $1955153\frac{1}{25}$.
18th. $745\frac{1}{2} \times 100\frac{1}{4}$? .... *Ans.* $74736\frac{3}{8}$.
19th. $87468\frac{1}{5} \times 700\frac{1}{2}$? .... *Ans.* $61267971\frac{3}{5}$.
20th. $1782\frac{3}{8} \times 225\frac{2}{5}$? .... *Ans.* $401747\frac{13}{40}$.
21st. $1610\frac{4}{9} \times 819\frac{5}{7}$? .... *Ans.* $1320104\frac{20}{63}$.

22d. $5009\frac{1}{3} \times \frac{2}{7} \times 3\frac{1}{4}$?……………………………… *Ans.* $4651\frac{11}{21}$.

23d. $48\frac{1}{3} \times 37\frac{4}{5} \times 11\frac{19}{25}$?……………………………… *Ans.* $21485\frac{13}{25}$.

24th. $3000\frac{1}{7} \times \frac{1}{3} \times 11\frac{4}{7}$?……………………………… *Ans.* $11571\frac{48}{49}$.

25th. $\frac{n}{d} \times \frac{n'}{d'}$ (see Art. 191, 192, and 193) ?……………………… *Ans.* $\frac{nn'}{dd'}$.

26th. $\frac{ac}{d^2} \times \frac{bd}{c^3}$?……………………………………………… *Ans.* $\frac{ab}{c^2d}$.

27th. $\frac{abc}{d^2e^2} \times \frac{bd}{c^3e}$?……………………………………………… *Ans.* $\frac{ab^2}{c^2de^3}$.

28th. $\frac{n}{d} \times \frac{n'}{d'} \times \frac{n''}{d''}$?……………………………………………… *Ans.* $\frac{nn'n''}{dd'd''}$.

29th. $\frac{n}{d} \times \frac{n}{d} \times \frac{n}{d}$?……………………………………………… *Ans.* $\frac{n^3}{d^3}$.

30th. $\frac{a+b}{c+d} \times \frac{a+b}{c+d}$?……………………………………… *Ans.* $\frac{a^2+2ab+b^2}{c^2+2cd+d^2}$.

DIVISION OF FRACTIONS.

199. Division of fractions differs from division of integers in this respect, that the dividend or the divisor, or both the dividend and the divisor, are fractional expressions instead of whole numbers.

The result which is to be found, namely, the quotient, being a number whose product by the divisor shall be equal to the dividend (Art. 79), it follows from Article 186 that the dividend must contain some part or parts of the quotient as often as the divisor contains the same part or parts of unity; or, conversely, that the divisor must have to unity the same numerical relation as the dividend has to the quotient.

In division of fractions, it may be required,

1st. To divide a fractional expression by a whole number.
2d. To divide a whole number by a fractional expression.
3d. To divide a fractional expression by a fractional expression.

200. Let it be required to divide $\frac{8}{9}$ by 4, $\frac{3}{5}$ by 8, and $5\frac{2}{3}$ by 9.

1st. To divide $\frac{8}{9}$ by 4:

Since unity is the 4th part of the divisor, the quotient is the 4th part of $\frac{8}{9}$.

Now, by Article 148 the 4th part of $\frac{8}{9}$ is obtained by either dividing the numerator 8 by 4, or multiplying the denominator 9 by 4. As 8 is a multiple of 4, the first method is to be preferred on account of its giving the result in the lowest terms.

Whence $\frac{8}{9} \div 4 = \frac{8 \div 4}{9} = \frac{2}{9}$, the quotient required.

If the denominator is multiplied by 4,

$\frac{8}{9} \div 4 = \frac{8}{9 \times 4} = \frac{2 \times 4}{9 \times 4} = \frac{2}{9}$, the same result as before.

2d. To divide $\frac{3}{5}$ by 8:

Since unity is the 8th part of the divisor, the quotient is the 8th part of the dividend.

3, the numerator of the dividend, is not a multiple of 8; therefore, in order to divide the fraction $\frac{3}{5}$ by 8, the denominator 5 must be multiplied by 8.

Whence $\frac{3}{5} \div 8 = \frac{3}{5 \times 8} = \frac{3}{40}$, the quotient required.

3d. To divide $5\frac{2}{3}$ by 9:

The dividend in this example is a mixed number.

Expressed as an improper fraction, $5\frac{2}{3} = \frac{17}{3}$.

Consequently $5\frac{2}{3} \div 9 = \frac{17}{3} \div 9$.

Unity being the 9th part of the divisor, the quotient must be the 9th part of $\frac{17}{3}$.

17, the numerator of the dividend, is not a multiple of 9; the denominator of the dividend must therefore be multiplied by 9.

Whence $5\frac{2}{3}\div 9=\frac{17}{3}\div 9=\frac{17}{3\times 9}=\frac{17}{27}$, the quotient required.

From the investigations in this Article it appears, that when the dividend is a fractional expression and the divisor a whole number, the quotient is obtained by either dividing the numerator or multiplying the denominator of the dividend by the divisor.

201. Let it be required to divide 8 by $\frac{7}{9}$, and 15 by $3\frac{1}{2}$.

1st. To divide 8 by $\frac{7}{9}$:

Unity contains the 7th part of the divisor 9 times;

Therefore 8 must contain the 7th part of the quotient 9 times.

Now, the 7th part of 8 is $\frac{8}{7}$, the 9th of the quotient;

and 9 times $\frac{8}{7}=\frac{8\times 9}{7}=\frac{72}{7}=10\frac{2}{7}$, the quotient required.

2d. To divide 15 by $3\frac{1}{2}$:

$$3\frac{1}{2}=\frac{7}{2}\therefore 15\div 3\frac{1}{2}=15\div\frac{7}{2}.$$

Unity contains the 7th part of the divisor twice;

Therefore 15 contains the 7th part of the quotient, also, twice.

Now, the 7th part of 15 is $\frac{15}{7}$;

$\frac{15}{7}$ is, consequently, half the quotient;

and $\frac{15}{7}\times 2=\frac{15\times 2}{7}=\frac{30}{7}=4\frac{2}{7}$ is the quotient required.

Hence it appears, that to divide a whole number by a fractional expression, it is necessary to multiply the whole number by the denominator of the divisor, and to divide this product by its numerator.

Let it be required to divide 1 by $\frac{3}{5}$:

The division of 1 by $\frac{3}{5}$ is effected by multiplying 1 by 5, and dividing the product by 3:

Whence $1\div\frac{3}{5}=\frac{1\times 5}{3}=1\times\frac{5}{3}=\frac{5}{3}$.

The fraction obtained by dividing 1 by $\frac{3}{5}$, namely $\frac{5}{3}$, is termed the reciprocal of the fraction $\frac{3}{5}$; and, generally, a fraction, whose numerator and denominator are the denominator and numerator of another fraction, is termed the reciprocal of that other fraction.

The rule for the division of a whole number by a fraction may now be expressed thus: multiply the dividend by the reciprocal of the divisor; the result is the quotient required.

202. Let it be required to divide $\frac{3}{5}$ by $\frac{8}{11}$, and $12\frac{3}{4}$ by $6\frac{2}{3}$.

1st. To divide $\frac{3}{5}$ by $\frac{8}{11}$:

1 contains the 8th part of the divisor 11 times;

Therefore $\frac{3}{5}$ contains the 8th part of the quotient 11 times.

The 8th part of $\frac{3}{5}$ is $\frac{3}{5\times 8}$, which is the 11th part of the quotient,

and 11 times $\frac{3}{5\times 8}=\frac{3\times 11}{5\times 8}=\frac{33}{40}$.

Whence $\frac{33}{40}$ is the quotient required.

2d. To divide $12\frac{3}{4}$ by $6\frac{2}{3}$:

$$\therefore 12\frac{3}{4}=\frac{51}{4}, \text{ and } 6\frac{2}{3}=\frac{20}{3}\therefore 12\frac{3}{4}\div 6\frac{2}{3}=\frac{51}{4}\div\frac{20}{3}.$$

Now, it may be shown, as in the last example, that

$$\frac{51}{4}\div\frac{20}{3}=\frac{51}{4}\times\frac{3}{20}=\frac{51\times 3}{4\times 20}=\frac{153}{80}=1\frac{73}{80}.$$

Wherefore $12\frac{3}{4}\div 6\frac{2}{3}=1\frac{73}{80}$, the quotient required.

To obtain the quotient in the 1st and 2d examples of this Article, it has been found requisite to multiply the dividend by the denominator, and to divide it by the numerator, of the divisor.

The reasoning employed is general. Whence, to divide one fractional expression by another, multiply the dividend by the reciprocal of the divisor.

When any of the given quantities are mixed numbers, reduce them to improper fractions.

203. The general rule for the division of fractions may be investigated without reference to particular quantity, thus,

Let a represent the numerator and b the denominator of the dividend; c the numerator, and d the denominator, of the divisor:

Then $\frac{a}{b}$ expresses the fractional dividend, and $\frac{c}{d}$ the divisor.

Since 1 contains the cth part of the divisor d times, therefore the dividend $\frac{a}{b}$ contains the cth part of the quotient d times; but the cth part of $\frac{a}{b}$ is $\frac{a}{b\times c}$; whence $\frac{a}{b\times c}$ is the dth part of the quotient, and $\frac{a}{b\times c}\times d=\frac{a\times d}{b\times c}$ is the whole quotient.

Wherefore $\frac{a}{b}\div\frac{c}{d}=\frac{ad}{bc}=\frac{a}{b}\times\frac{d}{c}$; that is, the quotient is equal to the product of the dividend multiplied by the reciprocal of the divisor.

When $d=1$, the formula $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}$ is applicable to the case of Article 200; and when $b=1$, it coincides with that of Article 201.

204. Since the division of $\frac{a}{b}$ by c can be made by either dividing a by c, or multiplying b by c (Art. 148), the result $\frac{a}{b\times c}=\frac{a\div c}{b}$;

And, by the same Article, $\frac{a}{b\times c}$, or its equivalent $\frac{a\div c}{b}$, multiplied by d, gives $\frac{a\times d}{b\times c}=\frac{a\div c}{b\div d}$.

Whence, if the terms of the dividend are multiples of the terms of the divisor, the quotient can be obtained by dividing the terms of the dividend by the corresponding terms of the divisor.

205. From Articles 200, 201, 202, 203, 204, are deduced the following practical rules for the division of fractional quantities, it being premised that, to render the rules applicable, mixed numbers must first be reduced to improper fractions;

1st. To divide a fraction by a whole number; divide the numerator, or multiply the denominator, of the dividend by the divisor.

2d. To divide a whole number by a fraction; multiply the dividend by the denominator of the divisor, and divide this product by its numerator.

3d. To divide a fraction by a fraction; multiply the dividend by the reciprocal of the divisor, or, if the numerator of the dividend contains the numerator of the divisor, and the denominator of the dividend also contains the denominator of the divisor, divide the terms of the dividend by the corresponding terms of the divisor.

The result is in each case the quotient required.

206. Exercises in the division of fractions:

1st. $\frac{5}{7} \div 10$?.. *Ans.* $\frac{1}{14}$.
2d. $\frac{7}{9} \div 14$?.. *Ans.* $\frac{1}{18}$.
3d. $26\frac{2}{5} \div 6$?.. *Ans.* $4\frac{2}{5}$.
4th. $6 \div \frac{3}{11}$?.. *Ans.* 22.
5th. $20 \div \frac{8}{13}$?.. *Ans.* $32\frac{1}{2}$.
6th. $4 \div 37\frac{5}{9}$?.. *Ans.* $\frac{18}{169}$.
7th. $10 \div 12\frac{6}{7}$?.. *Ans.* $\frac{7}{9}$.
8th. $1 \div \frac{17}{23}$?.. *Ans.* $1\frac{6}{23}$.
9th. $1 \div 3\frac{1}{5}$?.. *Ans.* $\frac{5}{16}$.
10th. $3797\frac{3}{4} \div 11$?.. *Ans.* $345\frac{1}{4}$.
11th. $2402\frac{2}{5} \div 12$?.. *Ans.* $200\frac{1}{5}$.
12th. $4483 \div \frac{5}{6}$?. . ... *Ans.* $5379\frac{3}{5}$.
13th. $13884 \div 69\frac{1}{3}$?.. *Ans.* $200\frac{1}{4}$.
14th. $\frac{16}{21} \div \frac{8}{3}$?.. *Ans.* $\frac{2}{7}$.
15th. $\frac{77}{108} \div \frac{7}{12}$?.. *Ans.* $1\frac{2}{9}$.
16th. $\frac{52}{121} \div \frac{4}{11}$?.. *Ans.* $1\frac{2}{11}$.
17th. $\frac{13}{21} \div \frac{7}{3}$?.. *Ans.* $\frac{13}{49}$.
18th. $\frac{299}{595} \div \frac{13}{17}$?.. *Ans.* $\frac{23}{35}$.
19th. $\frac{35}{78} \div \frac{10}{13}$?.. *Ans.* $\frac{7}{12}$.
20th. $\frac{27}{45} \div \frac{3}{5}$?.. *Ans.* 1.
21st. $4\frac{32}{35} \div \frac{4}{5}$?.. *Ans.* $6\frac{1}{7}$.
22d. $2\frac{11}{12} \div \frac{5}{9}$?.. *Ans.* $5\frac{1}{4}$.
23d. $\frac{33}{40} \div 3\frac{3}{10}$?.. *Ans.* $\frac{1}{4}$.
24th. $\frac{14}{27} \div 2\frac{1}{3}$?.. *Ans.* $\frac{2}{9}$.
25th. $16\frac{1}{4} \div 2\frac{1}{2}$?.. *Ans.* $6\frac{1}{2}$.
26th. $4\frac{1}{5} \div 13\frac{1}{6}$?.. *Ans.* $\frac{126}{395}$.
27th. $7\frac{4}{5} \div 6\frac{1}{2}$?.. *Ans.* $1\frac{1}{5}$.
28th. $147\frac{5}{13} \div 1\frac{1}{26}$?.. *Ans.* $141\frac{25}{27}$.
29th. $98\frac{35}{104} \div 365\frac{1}{4}$?.. *Ans.* $\frac{7}{26}$.
30th. $4545\frac{43}{72} \div 1\frac{5}{6}$?.. *Ans.* $2479\frac{5}{12}$.
31st. $766\frac{1}{8} \div 1021\frac{1}{2}$?.. *Ans.* $\frac{3}{4}$.
32d. $6049\frac{1}{2} \div 3$?.. *Ans.* $2016\frac{1}{2}$.
33d. $40659\frac{3}{4} \div 9$?.. *Ans.* $4517\frac{3}{4}$.
34th. $1757\frac{113}{196} \div \frac{5}{28}$?.. *Ans.* $9842\frac{3}{7}$.
35th. $3087\frac{5}{6} \div 6\frac{3}{10}$?.. *Ans.* $490\frac{25}{189}$.
36th. $53786\frac{29}{32} \div 1\frac{3}{4}$?.. *Ans.* $30735\frac{3}{8}$.

207. When the dividend is a mixed number greater than the divisor, the division may be effected without the reduction of the former to an improper fraction; for, if the divisor is a whole number less than the dividend, it is contained in the integral part of the dividend a certain number of times, either exactly, or with a remainder less than the divisor.

If exactly, the quotient obtained by dividing the integral part of the dividend by the divisor, + the quotient obtained by dividing the fractional part of the dividend by the divisor, must be the quotient arising from the division of the whole dividend by the divisor, for the product of this quotient by the divisor is equal to the dividend (Art. 70 and 199).

If a certain number of times with a remainder less than the divisor, the whole dividend is decomposed into two parts, the first a multiple of the divisor, the second a mixed number less than the divisor. The result of the division of the first part by the divisor is a whole number; that of the second part (found by Art. 205) a proper fraction; and the sum of these partial quotients is, as in the other case, the quotient required.

When the divisor is a fractional expression, the dividend may be multiplied by the denominator of the divisor, as in Article 197, and this product divided by its numerator, in the manner already explained.

As illustrations of this Article, let it be required,

1st. To divide $748695\frac{3}{8}$ by 5.

The integral part of this dividend is a multiple of the divisor (Art. 122). Decomposing the dividend into the parts 748695 and $\frac{3}{8}$, it is found that

$$748695 \div 5 = 149739$$

$$\text{and} \quad \frac{3}{8} \div 5 = \frac{3}{40}$$

Collecting the partial quotients, $149739 + \frac{3}{40}$ or $149739\frac{3}{40}$ is the quotient required.

When the divisor is less than 10, the calculation may be made as in Article 86, thus,

$$\begin{array}{r} 5\,)\,\underline{748695\frac{3}{8}} \\ 149739\frac{3}{40}. \end{array}$$

2d. Let it be required to divide $4557\frac{3}{5}$ by 9;

By Article 127 it is found that 4557 is not a multiple of 9, also that the remainder from the division of 4557 by 9 is 3; therefore $4557\frac{3}{5}$ must be decomposed into the parts 4554 and $3\frac{3}{5}$.

$$4554 \div 9 = 506, \text{ and } 3\frac{3}{5} \div 9 = \frac{18}{5} \div 9 = \frac{2}{5},$$

consequently $4557\frac{3}{5} \div 9 = 506 + \frac{2}{5} = 506\frac{2}{5}$.

The process of division may be represented as follows:

$$\begin{array}{l} 9\,)\,\underline{4557\frac{3}{5}} \\ \quad 506 + 3\frac{3}{5} \text{ rem., } 3\frac{3}{5} = \frac{18}{5}, \text{ and } \frac{18}{5} \div 9 = \frac{2}{5}, \\ \therefore 506\frac{2}{5} \text{ is the quotient required.} \end{array}$$

3d. Let it be required to divide $4545\frac{43}{72}$ by $1\frac{5}{6}$.

Since the divisor $= 1\frac{5}{6} = \frac{11}{6}$, it is required to multiply the dividend by 6, and to divide this product by 11;

$$\begin{array}{l} \quad 4545\frac{43}{72} \\ \qquad\quad \underline{6} \\ 11\,)\,\underline{27273\frac{7}{12}} = \text{product of dividend by 6.} \\ \quad 2479 + 4\frac{7}{12} \text{ rem., } 4\frac{7}{12} = \frac{55}{12}, \text{ and } \frac{55}{12} \div 11 = \frac{5}{12}, \\ \therefore 2479\frac{5}{12} \text{ is the quotient required.} \end{array}$$

The calculation by this method is sometimes more concise than that by the general rule for the division of fractional quantities. On this account a rule and appropriate examples are subjoined.

Rule. To divide a large mixed number by a whole number;

1st. Divide the whole number of the dividend by the divisor: the result is the integral part of the quotient.

2d. Divide the remainder (which, if a mixed number, must be brought to an improper fraction) by the divisor: the result is the fractional part of the quotient.

3d. Annex the fractional to the integral part of the quotient: the mixed number thus formed is the quotient required.

To divide a large mixed number by a fraction or by a mixed number (which must be reduced to an improper fraction);

Multiply the dividend by the denominator of the divisor, and divide this product by its numerator.

208. Additional exercises in division of fractions:

1st. $437850\frac{6}{7} \div 9$?..........*Ans.* $48650\frac{2}{21}$.

2d. $33378\frac{3}{5} \div 7$?..........*Ans.* $4768\frac{13}{35}$.

3d. $2914\frac{1}{3} \div \frac{7}{9}$?..........*Ans.* 3747.

4th. $15627\frac{1}{3} \div \frac{4}{5}$?..........*Ans.* $19534\frac{1}{6}$.

5th. $48269\frac{2}{25} \div \frac{7}{12}$?..........*Ans.* $82746\frac{174}{175}$.

6th. $524\frac{17}{42} \div \frac{5}{6}$?..........*Ans.* $629\frac{2}{7}$.

7th. $4651\frac{11}{21} \div \frac{13}{14}$?..........*Ans.* $5009\frac{1}{3}$.

8th. $5956\frac{4}{11} \div 35$?..........*Ans.* $170\frac{2}{11}$.

9th. $216308\frac{22}{35} \div 20\frac{3}{5}$?..........*Ans.* $10500\frac{3}{7}$.

10th. $7396\frac{1}{2} \div 13\frac{1}{5}$?..........*Ans.* $560\frac{15}{44}$.

11th. $273772\frac{1}{2} \div 30\frac{1}{4}$?..........*Ans.* $9050\frac{40}{121}$.

12th. $156690\frac{2}{15} \div 27\frac{1}{5}$?..........*Ans.* $5760\frac{2}{3}$.

13th. $1320104\frac{20}{63} \div 819\frac{5}{7}$?..........*Ans.* $1610\frac{4}{9}$.

14th. $1351185\frac{1}{20} \div 300\frac{1}{4}$?..........*Ans.* $4500\frac{1}{5}$.

15th. $108118749\frac{6}{7} \div 441\frac{7}{12}$?..........*Ans.* $244843\frac{13877}{37093}$.

16th. $308856\frac{4}{11} \div 40\frac{15}{22}$?..........*Ans.* 7592.

17th. $9730152\frac{9}{32} \div 72\frac{3}{4}$?..........*Ans.* $135000\frac{3}{8}$.

18th. $474682\frac{19}{28} \div 150\frac{1}{4}$?..........*Ans.* $3159\frac{2}{7}$.

19th. $353455\frac{1}{9} \div 67\frac{1}{3}$?..........*Ans.* $5249\frac{1}{3}$.

20th. $85622845\frac{26}{27} \div 1000\frac{1}{9}$?..........*Ans.* $85613\frac{1}{3}$.

21st. $\frac{n}{d} \div \frac{n'}{d'}$ (see Art. 203 and 204) ?..........*Ans.* $\frac{nd'}{n'd}$

22d. $\frac{n}{d} \div d$?..........*Ans.* $\frac{n}{d^2}$.

23d. $\frac{n}{d} \div \frac{1}{d}$?..........*Ans.* n.

24th. $\frac{n}{d} \div \frac{n}{d}$?..........*Ans.* 1.

25th. $\frac{a^2c^2e^3}{b^3d^4f} \div \frac{a^3ce^4}{b^2df^3}$?..........*Ans.* $\frac{cf^2}{abd^3e}$.

26th. $\frac{b^2c^3d}{a^3e^4g} \div \frac{bc^2}{a^2e^3}$?..........*Ans.* $\frac{bcd}{aeg}$.

209. Remarks on multiplication and division of fractions.

In multiplication, when the multiplier is a proper fraction the product is less than the multiplicand; for, the multiplier being less than unity, the product, which is a like part of the multiplicand, is less than once the multiplicand.

In division, when the divisor is a proper fraction the quotient is greater than the dividend; for, the divisor being less than unity, its reciprocal is greater than unity, and the quotient is the product of the dividend by the reciprocal of the divisor.

210. The fractional expressions considered in the preceding rules are fractions of unity; $\frac{3}{5}$, for example, and $\frac{3}{5}$ of 1 being expressions of the same signification (Art. 146).

But a fractional part or fractional parts may, in like manner, be taken of any numbers either less or greater than unity, as $\frac{5}{7}$ of $\frac{3}{4}$, $\frac{8}{9}$ of 2, $\frac{9}{11}$ of $5\frac{3}{8}$.

To distinguish a fraction of unity from a fraction of any number less or greater than unity, it is usual to term the former a simple and the latter a complex or compound fraction.

When the sum or difference of any numbers, of which some are complex fractions, is to be found, it is necessary to reduce the complex to equivalent simple fractions, as without such reduction the complex cannot be brought to the same denominator with the simple fractions or with each other.

Let a given complex fraction be $\frac{5}{7}$ of $\frac{3}{4}$, and let it be required to reduce it to a simple fraction. The signification of an expression such as $\frac{3}{4}$, or $\frac{3}{4}$ of 1, is that 1 is divided into 4 equal parts, and that 3 of these 4th parts are taken to compose the fraction $\frac{3}{4}$. So in the complex fraction $\frac{5}{7}$ of $\frac{3}{4}$, it is to be understood that $\frac{3}{4}$ is divided into 7 equal parts, and that 5 of these 7th parts are taken to compose the fraction.

Now, if $\frac{3}{4}$ or $\frac{3}{4}$ of 1 is divided into 7 equal parts, one of these 7th parts is equal to $\frac{3}{4} \div 7$, or $\frac{3}{28}$ of 1, and 5 such parts are equal to 5 times $\frac{3}{28}$ of 1 or $\frac{15}{28}$ of 1, that is, to the simple fraction $\frac{15}{28}$.

Wherefore $\frac{5}{7}$ of $\frac{3}{4}=\frac{15}{28}=\frac{3\times5}{4\times7}$.

Similarly $\frac{8}{9}$ of $2=8$ times the 9th part of $2=\frac{2\times8}{9}=\frac{16}{9}=1\frac{7}{9}$;

$\frac{9}{11}$ of $5\frac{2}{3}=\frac{9}{11}$ of $\frac{17}{3}=9$ times the 11th part of $\frac{17}{3}=\frac{17\times9}{3\times11}$;

but $\frac{17\times9}{3\times11}=\frac{17\times3^2}{3\times11}=\frac{51}{11}=4\frac{7}{11}$ $\therefore$ $\frac{9}{11}$ of $5\frac{2}{3}=4\frac{7}{11}$.

In these instances the reduction of a complex to a simple fraction is made by multiplying together the numerators of the complex fraction for the numerator of the equivalent simple fraction, and the denominators for its denominator. The method being independent of the particular values of the given fractions, it follows that any complex fraction, consisting of a fraction of some number less or greater than unity, is reducible to a simple fraction by this process, which is identical with that for finding the product of two fractional factors.

Next, let it be required to reduce the complex fraction $\frac{3}{4}$ of $\frac{5}{9}$ of $\frac{2}{7}$ to a simple fraction.

In this example it is required to reduce $\frac{3}{4}$ of $\frac{5}{9}$ of $\frac{2}{7}$ to a fraction of unity.

Now $\frac{5}{9}$ of $\frac{2}{7}$ is equal to 5 times the 9th part of $\frac{2}{7}=\frac{2\times5}{7\times9}$.

Therefore $\frac{3}{4}$ of $\frac{5}{9}$ of $\frac{2}{7}=\frac{3}{4}$ of $\frac{2\times5}{7\times9}$.

Similarly $\frac{3}{4}$ of $\frac{2\times5}{7\times9}=3$ times the 4th part of $\frac{2\times5}{7\times9}=\frac{2\times5\times3}{7\times9\times4}$;

but $\frac{2\times5\times3}{7\times9\times4}=\frac{2\times5\times3}{7\times3^2\times2^2}=\frac{5}{42}$ $\therefore$ $\frac{3}{4}$ of $\frac{5}{9}$ of $\frac{2}{7}=\frac{5}{42}$.

That is, the complex fraction $\frac{3}{4}$ of $\frac{5}{9}$ of $\frac{2}{7}$ is equal to the simple fraction $\frac{5}{42}$.

From a consideration of the method followed in the reduction of these complex fractions to simple fractions, it appears that the process of reduction coincides with that for finding the product of all the fractional expressions composing the complex fraction.

Whence, to reduce any complex fraction to an equivalent simple fraction,

Rule. Multiply together the numerators of the complex fraction for the numerator of the simple fraction, and the denominators for its denominator.

211. Exercises in the reduction of complex to simple fractions:

1st. $\frac{2}{3}$ of $\frac{7}{8}$? .... *Ans.* $\frac{7}{12}$.
2d. $\frac{4}{5}$ of $1\frac{1}{2}$? .... *Ans.* $\frac{6}{5}$.
3d. $\frac{2}{3}$ of $\frac{6}{7}$ of $\frac{1}{2}$? .... *Ans.* $\frac{2}{7}$.
4th. $\frac{1}{7}$ of $\frac{9}{16}$ of $4\frac{1}{3}$? .... *Ans.* $\frac{39}{112}$.
5th. $\frac{4}{11}$ of $\frac{7}{12}$ of $3\frac{1}{2}$? .... *Ans.* $\frac{49}{66}$.
6th. $\frac{1}{5}$ of 4 of $\frac{3}{17}$ of $7\frac{1}{12}$? .... *Ans.* 1.
7th. $\frac{5}{12}$ of $\frac{42}{45}$ of $\frac{13}{63}$ of 9? .... *Ans.* $\frac{13}{18}$.

8th. $\frac{4}{11}$ of $\frac{5}{12}$ of $\frac{19}{23}$ of $3\frac{3}{5}$?........................... *Ans.* $\frac{114}{253}$.
9th. $\frac{1}{3}$ of $\frac{2}{5}$ of $\frac{3}{7}$ of $\frac{4}{9}$ of 5?.............................. *Ans.* $\frac{8}{63}$.
10th. $\frac{20}{21}$ of $\frac{35}{44}$ of $\frac{11}{15}$ of $9\frac{3}{4}$?........................ *Ans.* $\frac{65}{12}$ or $5\frac{5}{12}$.
11th. $\frac{1}{8}$ of $\frac{17}{19}$ of $2\frac{2}{3}$ of $2\frac{1}{9}$?............................... *Ans.* $\frac{17}{27}$.
12th. $\frac{4}{15}$ of $\frac{9}{21}$ of $\frac{5}{8}$ of 7?.................................... *Ans.* $\frac{1}{2}$.

To distinguish fractions of the form $\frac{n}{d}$ from fractions of the form $\frac{n}{10^m}$ the former are termed *Vulgar* or common, and the latter *Decimal*, fractions.

SECTION VII.

OF DECIMAL FRACTIONS.

213. If the figures of any number are taken consecutively from right to left, the first figure expresses units of the first order or simple units; the second, units of the second order, or tens; the third, units of the third order, or hundreds, &c. &c; the units expressed by any figure being ten times greater than those expressed by the figure which is immediately on its right.

Conversely, the units expressed by any figure which is on the right of another, are tenth parts of the units expressed by that other figure.

This principle may be extended so as to include figures written to the right of that digit which expresses the simple units of any number; thus, if 1 is written to the right of the digit expressing the simple units of a number, its relative value is one tenth part of unity, or, simply, one tenth; similarly the values of the figures 2, 3 ..... 9, written to the right of the digit expressing the simple units of a number are respectively two tenths, three tenths, .. nine tenths.

The relative value of 1 written to the right of the figure expressing tenths is one tenth part of one tenth, or one hundredth part of unity (Art. 210); and similarly the values of 2, 3 ...... 9, in the second place from the figure expressing simple units, are two hundredths, three hundredths, ... nine hundredths of unity.

In like manner the relative values of the figures 1, 2, 3 ..... 9, written to the right of the figure expressing hundredths, are one thousandth, two thousandths, three thousandths ..... nine thousandths of unity.

214. If the relative values of the figures equidistant on both sides from the digit expressing the simple units of a number are compared with each other,

Of the figures on the left of the place of simple units:	Of the figures on the right of the place of simple units:
The first expresses tens.	The first expresses tenths.
The second expresses hundreds.	The second expresses hundredths.
The third expresses thousands.	The third expresses thousandths.
The fourth expresses ten thousands.	The fourth expresses ten thousandths.
. . . .	. . . .

And, generally, whatever multiple of unity is expressed by a figure on the left of the digit which expresses the simple units of a number, the corresponding part of unity is expressed by the same figure, at an equal distance from that digit, on the right.

215. The fractional numbers expressed by figures written to the right of the digit which occupies the place of simple units, are termed decimal fractions, or, simply, decimals.

To distinguish the integral from the fractional part of a mixed number written in this manner, a mark, named the decimal point, is placed after that digit which expresses the simple units of the number; the mark used in this Treatise is a point (·).

Thus the figures 24·8 express the mixed number 24 units and 8 tenths; the figures 46·54, 46 units 5 tenths and 4 hundredths; and the figures 78·246, 78 units 2 tenths 4 hundredths and 6 thousandths.

The integral part of each of these mixed numbers is expressed as a multiple of unity, but every figure of the fractional part is taken by itself; the manner of expressing both may, however, be rendered uniform.

For, since 2 tenths are equal to 20 hundredths or to 200 thousandths,
and 4 hundredths to 40 thousandths,
it follows that 2 tenths 4 hundredths and 6 thousandths are equal to 200 thousandths + 40 thousandths + 6 thousandths, or to 246 thousandths; and that the number 78·246 may be read 78 units and 246 thousandths.

216. Each figure of a decimal fraction has its own denominator, which, according as the figure is the 1st, 2d, 3d, &c. from the decimal point, is 10, 100, 1000, &c.; hence it is evident that the preceding expedient to reduce the isolated names of the several figures composing a decimal fraction into one name of number amounts, in effect, to the reduction of the fractions expressed by these figures to equivalent fractions having a common denominator, and their combination into one fractional number having that denominator for its denominator.

This common denominator must always be the denominator of that figure which is most remote from the decimal point.

A decimal is, consequently, read as a fraction whose numerator is the number expressed by the figures of the given decimal, and its denominator that of the last figure of the given decimal.

217. The integral and decimal parts of an expression composed of a whole number and a decimal fraction may be both comprehended under one name of number.

Taking, for example, the number 57·496; the decimal part is 496 thousandths; the integral part 57 is equal to 57000 thousandths; and therefore the sum of both, or 57000 thousandths + 496 thousandths, is 57496 thousandths.

This process is analogous to that for the reduction of a mixed number to an improper fraction. In the case of a decimal fraction, however, neither multiplication of the integral part nor addition of the fractional part is necessary. It is sufficient to read the number as if there were no decimal point, and to name after it the denominator of the last decimal figure contained in the number.

218. The manner of expressing in words the value of a decimal fraction being explained, it is requisite next to show how the spoken name of a decimal fraction is expressed in figures.

Let it be required, for the sake of illustration, to express in figures the number thirty-seven units four hundred and thirty-five thousandths:

First, 37, the integral part of the number, must be written to the left of the decimal point.

Next, since 435 thousandths are equal to 400 thousandths + 30 thousandths + 5 thousandths, and that 400 thousandths are equal to 40 hundredths, or 4 tenths, and 30 thousandths are equal to 3 hundredths, the figure 4, which expresses tenths, must be written to the right of the decimal point, and then, successively, the figures, 3, expressing the hundredths, and 5, the thousandths of the number.

Whence the number expressed in figures is 37·435.

Let it be also required to express in figures the number, eight units and thirty-seven thousandths.

The figure 8, expressing the eight units, must be written to the left of the decimal point.

The decimal part, 37 thousandths, or 30 thousandths + 7 thousandths, or 3 hundredths + 7 thousandths, contains no figure expressing tenths; the figure 0 must, therefore, be written after the decimal point to occupy the place of the deficient tenths, then the 3, expressing the hundredths, and lastly the 7, expressing the thousandths, of the given number, which in figures is 8·037.

From the preceding examples and remarks it follows, that to translate into figures a decimal number expressed in words, it is necessary to write first the integral part of the number, then the decimal point, lastly in succession to the right of this point the figures which represent the tenths, hundredths, thousandths, &c. named in the words expressing the number, the place of any deficient tenth, hundredth, &c. being filled with a zero.

If the number which is to be expressed in figures is wholly a decimal fraction, a zero may be written to the left of the decimal point to occupy the place of the deficient whole number.

219. In the verbal expression of a mixed number composed of an integer and a decimal fraction, the integral may not be distinguished from the decimal part of the number. This is the converse of Article 217; from which it follows that in determining the position of the decimal point in such a case it is necessary to place it so that the last decimal figure may have for denominator that denominator which is given to the whole number, in such manner that if the word expressing the denominator is tenths there is one decimal figure, if hundredths, two decimal figures, &c. &c.

220. A vulgar fraction is expressed by means of two numbers, the numerator and the denominator, both of which are written at length. In decimal fractions the numerator only is expressed in figures; but the decimal point, by its position, indicates the magnitude of the denominator, which is equal to unity followed by as many zeros as there are figures on the right of the decimal point.

The numerator is the number expressed by the figures on the right of the decimal point.

The number 75·437, written as a vulgar fraction, is $75\frac{437}{1000}$ or $\frac{75437}{1000}$.

$$2{\cdot}00409=2\frac{409}{100000}=\frac{200409}{100000}.$$

$$0{\cdot}0002154=\frac{2154}{10000000}.$$

$$\text{Conversely, } 17\frac{2049}{10000} \text{ or } \frac{172049}{10000}=17{\cdot}2049.$$

$$5\frac{49}{1000} \text{ or } \frac{5049}{1000} = 5{\cdot}049.$$

$$\frac{7684}{10000} = 0{\cdot}7684.$$

$$\frac{98}{100000} = 0{\cdot}00098.$$

221. If a decimal is written as a vulgar fraction, and both the terms of the latter are multiplied by any power of 10, the value of the fraction is not altered.

Taking the number 5·736, which is equal to $\frac{5736}{1000}$,

$$\frac{5736}{1000}=\frac{57360}{10000}=\frac{573600}{100000}=\frac{5736000}{1000000}=\frac{57360000}{10000000},$$

$$\text{or } 5{\cdot}736=5{\cdot}7360=5{\cdot}73600=5{\cdot}736000=5{\cdot}7360000.$$

Consequently the value of a decimal is not changed by annexing to it any number of zeros; and hence, if several decimal fractions are composed, some of more and some of fewer decimal figures, by annexing zeros, the inequality may be removed without any change in the magnitude of the numbers.

By means of zeros thus annexed any given decimals are reduced to the same denominator.

Thus, the decimals 17·49, ·025, 58·3841, 126·8, are equal,

17·49 to 17·4900,
·025 to 0·0250,
58·3841 to 58·3841,
126·8 to 126·8000,

under which form they have the common denominator 10000.

All figures on the right of the decimal point, and at the same distance from it, have a common denominator.

222. In the case of decimal fractions, as well as of whole numbers, ten units of any order make one unit of the next higher order (Art. 213).

Thus, 10 thousandths (for example) make 1 hundredth, or $\frac{10}{1000}=\frac{1}{100}=\cdot01$,
10 hundredths make 1 tenth, or $\frac{10}{100}=\frac{1}{10}=\cdot1$,
10 tenths make 1 unit or whole, $\frac{10}{10}=1$;
precisely in the same manner as
10 units make 1 ten, or 10,
10 tens make 1 hundred, or 100,
10 hundred make 1 thousand, or 1000.

By this extension of the decimal scale to numbers less than unity, the reductions necessary for the combination of fractions with other fractions and with whole numbers are avoided.

Addition of Decimal Fractions.

223. In addition of decimal fractions certain numbers, altogether or partly expressed by decimal figures, are given; and it is required to find a single number which shall be equal to the whole of the given numbers taken together.

This sum must comprehend all the units and parts of unity contained in the given numbers. In forming it units of the same order alone, whether integral or decimal, can be combined together. Therefore the separate sums of all the units of each order contained in the given numbers must be found, and these sums collected into one result.

Since ten units of any order make one unit of the next higher order in the fractional as well as in the integral parts of the given numbers, and ten tenths make one whole (Art. 222), it follows that the combination into one number of the given quantities, whether fractional or integral, is made uniformly by the rule for the addition of whole numbers.

In arranging the given numbers for addition care must be taken to place in the same vertical column all the figures which have the same relative value, and no others. Thus, of the integral parts of the numbers, units must be written under units, tens under tens, hundreds under hundreds, &c.; and of the decimal parts, tenths must be written under tenths, hundredths under hundredths, &c.; and, by consequence, the decimal points also under each other.

224. When all the given quantities contain the same number of decimal figures, if the figures of the lowest order, or the last in each quantity, are placed under each other in a vertical column, and the other figures as if the numbers were integral, the corresponding decimal figures, the decimal points, and the same orders of units of the whole numbers will fall under each other.

If there are more decimal figures in some of the numbers than in others, the arrangement must be made to proceed from the decimal point towards the left for the whole numbers, according to the relative values of the figures; and towards the right for the decimals, also according to the relative values of the figures, in the manner described, Art. 223.

The unusual appearance of broken columns on the right may occasion some embarrassment. On this account it may be found convenient at first to reduce the decimals to a common denominator by Article 221, and to arrange the figures as directed in the preceding paragraph of this Article.

225. Example 1. Let it be required to find the sum of the decimals ·0795, ·3854, ·0072, and ·9588.

The number of decimal figures in these quantities being the same, the arrangement for addition is made by Article 224.

Arrangement of the numbers, and calculation:

Units.	Decimal point.	Tenths.	Hundredths.	Thousandths.	Ten thousandths.
0	·	0	7	9	5
0	·	3	8	5	4
0	·	0	0	7	2
0	·	9	5	8	8
1	·	4	3	0	9

The detail of the calculation, beginning with the right column (which in this example expresses ten thousandths of unity), is,

8+2+4+5=19 ten thousandths=10 ten thousandths+9 ten thousandths =1 thousandth+9 ten thousandths.

The 9 is written under the column of ten thousandths, and the 1 thousandth reserved for addition to the column of thousandths.

1+8+7+5+9=30 thousandths=3 hundredths+0 thousandths.

The 0 is written under the column of thousandths, and the 3 reserved for addition to the column of hundredths.

3+5+8+7=23 hundredths=20 hundredths+3 hundredths=2 tenths +3 hundredths.

The 3 is written under the column of hundredths, and the 2 reserved for addition to the column of tenths.

2+9+3=14 tenths=10 tenths+4 tenths=1 whole+4 tenths.

The 4 is written under the column of tenths, the decimal point to the left of the figure expressing tenths, and the 1 whole to the left of the decimal point, under the column of simple units. Consequently the sum of the given numbers is 1·4309.

Example 2. Let it be required to find the sum of the decimals 35·847, 9·3546, 125·09546, ·42, and 76·3865.

The common denominator of the decimals contained in this example is 100000. Bringing the decimals to this denominator, arranging the numbers by Article 224, adding them together as integers, and placing the decimal point to the left of the figure which expresses tenths, the sum is found to be 247·10356. A difference between this example and example 1 is, that most of the addenda are mixed numbers. In finding their sum, the 2 wholes, arising from the column of tenths, are carried to the column of simple units.

If the decimals are not brought to the same denominator, the addenda must be arranged by Article 223. The process of addition and the result are the same, in whichever of the two ways the given numbers are arranged.

Both forms of arrangement and the result of the addition are subjoined. A lengthened detail of the process (after that given in Example 1) seems unnecessary.

Hundreds.	Tens.	Units.	Decimal point.	Tenths.	Hundredths.	Thousandths.	Ten thousandths.	Hundred thousandths.
	3	5	·	8	4	7	0	0
		9	·	3	5	4	6	0
1	2	5	·	0	9	5	4	6
		0	·	4	2	0	0	0
	7	6	·	3	8	6	5	0
2	4	7	·	1	0	3	5	6

Hundreds.	Tens.	Units.	Decimal point.	Tenths.	Hundredths.	Thousandths.	Ten thousandths.	Hundred thousandths.
	3	5	·	8	4	7		
		9	·	3	5	4	6	
1	2	5	·	0	9	5	4	6
		0	·	4	2			
	7	6	·	3	8	6	5	
2	4	7	·	1	0	3	5	6

226. Rule for the addition of decimal fractions:

Arrange the numbers so that the figures of the same relative value, and, consequently, also the decimal points, may fall under each other in vertical columns.

Then add as in whole numbers, and place the decimal point of the sum in the vertical column of the other decimal points.

227. Exercises in the addition of decimal fractions:

1st. ·0962+·5847+·9754+·0029+·0506?............*Ans.* 1·7098.

2d. ·07+·003+·00034+·00562?........................*Ans.* ·07896.

3d. 17·039+13·7+·0075+·186+345·03?.........*Ans.* 375·9625.

4th. ·08+·46+·23+·22+·0048+·0055?..............*Ans.* 1·0003.

5th. 274·5+·075+4·0076+290·0486+45·07+394.09?
Ans. 1007·7912.

6th. 17·019+13·7+·7+·0075+$\frac{186}{1000}$+345·03?...*Ans.* 376·6425.

7th. ·08+·46+·23+·22+·0045+·0055?......................*Ans.* 1.

8th. 987·+·85+·000734+845·01+90$\frac{9}{10}$?.....*Ans.* 1923·760734.

9th. $\frac{4}{10}+\frac{8}{1000}+\frac{8}{100}+\frac{8}{1000000}$?........................*Ans.* ·488008.

10th. 235·9+17$\frac{99}{100}$+32$\frac{5}{1000}$+$\frac{4}{100}$?..................*Ans.* 285·935.

11th. 5$\frac{4}{10}$+79·25+·00074+29$\frac{247}{10000}$+394·09? *Ans.* 507·76544.

12th. 176+$\frac{176}{10}+\frac{176}{100}+\frac{176}{1000}+\frac{176}{10000}$?..............*Ans.* 195·5536.

SUBTRACTION OF DECIMAL FRACTIONS.

228. If the minuend and subtrahend contain the same number of decimals, the last figures of each have the same relative value. Whence, if the last figure of the latter is written under the last of the former, and the others in order, the figures, fractional as well as integral, which have the same relative values, and also the decimal points, must fall directly under each other. Then, since 10 units of any order make 1 unit of the next higher in the fractional as well as the integral part of a number expressed decimally, and 10 tenths also make 1, if the difference is taken as in whole numbers, and the decimal point of the remainder is placed in the same vertical column with the decimal points of the minuend and subtrahend, the result will be the difference of the given numbers.

229. If the minuend and subtrahend contain different numbers of decimal figures, the inequality may be removed by annexing the proper number of zeros to that of the two which has fewer*; or, having, as in addition (Art. 223), arranged the numbers so that the decimal points and those figures which have the same relative value shall fall in vertical columns, the subtraction may be made as before.

230. Let it be required from ·1354 to subtract ·0436.

Arrangement of the numbers, and calculation:

$$\begin{array}{r} \cdot1354 \\ \underline{\cdot0436} \\ \underline{\cdot0918} \end{array}$$

Detail: the denominator of the last figure is 10000: 4 ten thousandths—6 ten thousandths, the subtraction is not possible; 1 thousandth=10 ten thousandths is borrowed from the next figure on the left; then 10 ten thousandths+4 ten thousandths=14 ten thousandths; and 14 ten thousandths

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[*] Though not essentially necessary, it is convenient at first to reduce to the same denominator the decimal fractions whose difference is to be found. A beginner might suppose ·098, for example, greater than ·1; but if the ·1 is written ·100, such a mistake is less likely to be made.

—6 ten thousandths=8 ten thousandths. 8 is written under the column of ten thousandths, and 1 carried to the next figure for the 1 thousandth borrowed: 1 thousandth+3 thousandths=4 thousandths; and 5 thousandths—4 thousandths=1 thousandth.

The remainder, 1, is written under the column of thousandths, and the subtraction continued.

3 hundredths—4 hundredths; the subtraction is not possible: 1 tenth,=10 hundredths, is borrowed; 10 hundredths+3 hundredths=13 hundredths; and 13 hundredths—4 hundredths=9 hundredths. The 9 is written in its proper place, and 1 tenth carried for the 1 tenth or 10 hundredths borrowed.

1 tenth+0=1 tenth, which, taken from the 1 tenth of the minuend, gives 0 to be written in the place of tenths of the remainder.

The remainder is, therefore, ·0918 or 918 ten thousandths.

2d Example. Let it be required from 176·384 to take 98·75864.

The arrangement of the numbers, both with and without the reduction of the decimals to the same denominator, is exhibited below. The remainder, also, is given, but not the detail of the calculation, this seeming unnecessary.

$$\begin{array}{r} 176{\cdot}38400 \\ 98{\cdot}75864 \\ \hline 77{\cdot}62536 \\ \hline \end{array} \qquad \begin{array}{r} 176{\cdot}384 \\ 98{\cdot}75864 \\ \hline 77{\cdot}62536 \\ \hline \end{array}$$

231. Rule for the subtraction of decimal fractions:

Arrange the figures of the minuend and subtrahend, the latter under the former, according to their relative values. Then subtract as in whole numbers, and place the decimal point of the remainder in the vertical column containing the decimal points of the minuend and subtrahend.

232. Exercises in the subtraction of decimal fractions.

1st. Find the difference between 274·5 and 4·0076? *Ans.* 270·4924.

2d. - - - - 1·0009 and 2·009? *Ans.* 1·0081.

3d. - - - - 10· and ·001?...... *Ans.* 9·999.

4th. - - - - ·00032 and ·000085? *Ans.* ·000235.

5th. - - - - $472\frac{3}{10}$ and $\frac{3847}{100}$? *Ans.* 433·83.

6th. - - - - 1· and ·1?................ *Ans.* ·9.

7th. - - - - $\frac{37}{1000}$ and $\frac{37}{10}$?...... *Ans.* 3·663.

8th. - - - - 1982 and 19·82? *Ans.* 1962·18.

9th. - - - - ·009 and ·01?....... *Ans.* ·001.

10th. - - - - 79·84 and 794·8? *Ans.* 714·96.

11th. - - - - 43·2543 and ·29354? *Ans.* 42·96076.

12th. - - - - 319·006 and $78\frac{188}{10000}$? *Ans.* 240·9862.

233. Rules for the addition and subtraction of decimals may be easily deduced from the rules for the addition and subtraction of vulgar fractions.

For any given decimals may be reduced to a common denominator by Article 221; and, considered as fractions having a common denominator, their sum is equal to the sum, and their difference to the difference, of the numerators, placed over the common denominator.

In decimal fractions the value of the denominator is indicated by the position of the decimal point. This value being the same for the numbers which are added or subtracted, and for their sum or difference, the position of the decimal point in the sum or difference must be the same as it is in the numbers added or subtracted.

Whence, generally, to find the sum or difference of any decimal fractions, reduce the given decimals to a common denominator, add or subtract them as whole numbers, and point off as many decimal figures in the sum or difference as there are decimal figures in each of the numbers after reduction to the same denominator.

MULTIPLICATION OF DECIMAL FRACTIONS.

234. The decimal fractions whose product is required, can be expressed as vulgar fractions (Art. 220), and their product obtained by multiplying together the numerators of the factors for the numerator of the product, and the denominators for its denominator (Art. 194).

Now, the numerator of a decimal, expressed as a vulgar fraction, is the given number without the decimal point; that is, it is the given number considered as an integer; and the denominator is 1 with as many zeros annexed as there are decimal figures in the given number (Art. 217 and 220).

Whence the numerator of the product of two decimal factors is equal to the product of the given figures, considered as integers; and the denominator of their product is equal to the product of 1, followed by as many zeros as there are decimal figures in the multiplicand into 1 followed by as many zeros as there are decimal figures in the multiplier; that is, to 1 followed by as many zeros as there are decimal figures in both multiplicand and multiplier.

In decimal fractions this denominator is indicated by the decimal point, which must be placed so as to cut off, from the right of the numerator, as many figures for decimals as there are zeros in the denominator (Art. 220).

Consequently the product of two given decimal fractions is found by,

1st, multiplying together the given factors as whole numbers;

And, 2d, pointing off from this product as many figures for decimals as there are decimal figures in both the multiplicand and multiplier.

The product of three decimal fractions is obtained by forming the product of any two of the factors, and then multiplying this product by the third factor; and in the same manner is found the product of four or any greater number of decimal fractions.

235. To apply these deductions, let it be required to multiply 35·407 by 12·54:

$$35{\cdot}407=\frac{35407}{1000} \text{ and } 12{\cdot}54=\frac{1254}{100} \text{ (Art. 220)},$$

$$\therefore 35{\cdot}407 \times 12{\cdot}54=\frac{35407}{1000}\times\frac{1254}{100}.$$

$$\text{Now } \frac{35407}{1000}\times\frac{1254}{100}=\frac{35407\times1254}{1000\times100}=\frac{44400378}{100000}=444{\cdot}00378.$$

$$\therefore 35{\cdot}407\times12{\cdot}54=444{\cdot}00378.$$

Performed in this manner the calculation serves to illustrate the preceding observations on the multiplication of decimal, considered as vulgar, fractions.

In accordance, however, with the conclusion which has been deduced, 35407 ought to be multiplied by 1254, and 3+2 or 5 decimal figures to be pointed off from the product thus,

$$\begin{array}{r}
35{\cdot}407 \\
\underline{12{\cdot}54} \\
141628 \\
177035 \\
70814 \\
\underline{35407} \\
\underline{\underline{444{\cdot}00378}}
\end{array}$$

236. Multiply 0·03054 by 0·023.

$$\begin{array}{r} \cdot03054 \\ \cdot023 \\ \hline 9162 \\ 6108 \\ \hline 70242 \end{array}$$

The product of 3054 by 23 is 70242. To give this product its proper relative value: the multiplicand contains 5 decimal figures, and the multiplier 3; the product must consequently contain 8 decimal figures; and therefore the last figure, 2, is the 8th from the decimal point. But the product of the factors, taken as whole numbers, contains only 5 figures. In order, therefore, that the last figure may hold the 8th place from the decimal point, it is necessary to write three zeros to the left of the significant figures of the product, which thus becomes ·00070242.

The calculation by the factors, expressed fractionally, is as follows,

$$\cdot03054=\frac{3054}{100000};\quad \cdot023=\frac{23}{1000},$$

$$\text{and } \frac{3054}{100000}\times\frac{23}{1000}=\frac{3054\times23}{100000\times1000}=\frac{70242}{100000000}=\cdot00070242.$$

A conclusion to be drawn from this example (or any other in which the absolute product of the given factors contains fewer figures than the factors contain decimals) is, that when the number of figures in the product of two decimal fractions is less than the number of decimal figures in the multiplicand and multiplier, zeros must be prefixed to the product, till the number of decimal figures contained in it is made equal to the number contained in both factors.

237. Let any decimal, expressed as a vulgar fraction, be multiplied by 10; the multiplication can be effected by dividing the denominator of the multiplicand by 10 (Art. 149); the denominator of the product consequently contains 1 zero fewer than that of the multiplicand, and, therefore, the product 1 decimal figure fewer than the multiplicand.

Similarly, the multiplier being 100, the product contains two decimal figures fewer than the multiplicand; being 1000, the product contains 3 decimal figures fewer than the multiplicand. . . . . . .

Whence, to multiply a decimal by any power of 10, remove the decimal point of the multiplicand 1 figure to the right for each zero contained in the multiplier; the result is the product required.

Example 1. Multiply 0·00587 by 1000.

The multiplier contains 3 zeros; therefore to obtain the product, the decimal point of the multiplicand must be removed three figures to the right. The product consequently is 005·87, or, omitting the zeros, which on the left of a whole number have no value, 5·87.

Example 2. Multiply 4·76 by 10000.

The multiplier contains 4 zeros; therefore the product is obtained by removing the decimal point of the multiplicand 4 figures to the right. The latter containing but two decimal figures, this transference of the decimal point to the right can only be made by annexing 2 zeros to the multiplicand.

Therefore 47600 is the product required.

The multiplication of 4·76 by 10000 may be represented also thus,

$$4\cdot76=\frac{476}{100};\quad 10000=\frac{10000}{1};$$

$$\frac{476}{100}\times\frac{10000}{1}=\frac{476}{1}\times\frac{100}{1}=47600, \text{ the same result as before.}$$

238. **Rule for the multiplication of decimal fractions:**

Multiply the given factors together as whole numbers, and point off from the product as many figures for decimals as there are decimal figures in both the multiplicand and multiplier.

When the number of figures in the product is less than the number of decimal figures in the multiplicand and multiplier, zeros, equal in number to the deficient figures, must be written to the left of the number which expresses the absolute product of the factors.

To multiply a decimal by any power of 10, remove the decimal point of the multiplicand one figure to the right for each zero contained in the multiplier.

239. Exercises in the multiplication of decimal fractions:

1st. 3·795 × 75·437 ?................................*Ans.* 286·283415.
2d. 39·745 × ·067 ?................................*Ans.* 2·662915.
3d. 0·3734 × 0·7954 ?................................*Ans.* 0·29700236.
4th. 0·0475 × 1·3756 ?................................*Ans.* 0·065341.
5th. 0·57305 × 0·00754 ?................................*Ans.* 0·004320797.
6th. 0·001 × 375· ?................................*Ans.* 0·375.
7th. $200\frac{1}{10} \times 3\frac{1}{100}$?................................*Ans.* 602·301.
8th. 3·795 × ·7487 ?................................*Ans.* 2·8413165.
9th. ·00475 × ·13756 ?................................*Ans.* 0·00065341.
10th. $81\frac{31}{1000} \times \frac{834}{10000}$?................................*Ans.* 6·7579854.
11th. $56\frac{7}{100} \times \frac{13}{10000}$?................................*Ans.* 0·072891.
12th. 3·05 × 1·42 × 7·64 ?................................*Ans.* 33·08884.
13th. ·01819 × 100 ?................................*Ans.* 1·819.
14th. 3·415 × 1000 ?................................*Ans.* 3415.
15th. ·0076 × 100000 ?................................*Ans.* 760.
16th. 588·4 × 10000 ?................................*Ans.* 5884000.
17th. 4·79 × 54·6 × 39·625 ?................................*Ans.* 10363·28475.
18th. ·1234 × ·0001 ?................................*Ans.* 0·00001234.
19th. ·1 × ·1 ?................................*Ans.* 0·01.
20th. ·01 × ·01 ?................................*Ans.* 0·0001.
21st. ·0074 × 100 ?................................*Ans.* 0·74.
22d. ·0074 × 1000000 ?................................*Ans.* 7400.
23d. ·001 × 50 ?................................*Ans.* ·05.
24th. ·0001 × 1000 ?................................*Ans.* 0·1.
25th. ·0001 × ·0001 ?................................*Ans.* 0·00000001.
26th. ·076 × 6000 ?................................*Ans.* 456.
27th. 1·05 × 12000 ?................................*Ans.* 12600.
28th. 9 × ·9 × ·09 ?................................*Ans.* 0·729.

Division of Decimal Fractions.

240. If the dividend and divisor are brought to the same denominator, and then expressed as vulgar fractions, the denominators are equal; the quotient of the decimal is also equal to that of the vulgar fractions.

Now, in division of vulgar fractions, the quotient is obtained by multiplying the dividend by the reciprocal of the divisor; but, since the numerator of the reciprocal of the divisor is equal to the denominator of the dividend, these factors may be cancelled in the numerator and denominator of the quotient, which is thus reduced to a fractional expression, having for numerator the numerator of the dividend, and for denominator the numerator of the divisor.

These numbers are, respectively, the given dividend and divisor, reduced to the same denominator, and then considered as whole numbers.

Whence, to divide one decimal fraction by another, bring the decimals to the same denominator (or make the number of decimal figures in the dividend and divisor equal), and divide as in whole numbers.

241. Example. Divide 3·4703 by 0·027.

$$3·4703=\frac{34703}{10000}\,;\quad ·027=·0270=\frac{270}{10000}.$$

$$\frac{34703}{10000}+\frac{270}{10000}=\frac{34703}{10000}\times\frac{10000}{270}=\frac{34703}{270}\,;\ \text{and}\ \frac{34703}{270}=128\frac{143}{270}\ \text{(Art. 156.)}$$

The fractional part of this result being a vulgar, and not a decimal, fraction, it becomes necessary to investigate a means of converting the former into the latter form of expression.

Now $\frac{143}{270}$ is $\frac{143}{270}$ of one simple unit.

Since one simple unit is equal to 10 tenths, $\frac{143}{270}$ of one simple unit= $\frac{1430}{270}$ tenths=$5\frac{80}{270}$ tenths, or $\frac{143}{270}$ of one unit=5 tenths+$\frac{80}{270}$ of a tenth. Again, 1 tenth=10 hundredths, ∴ $\frac{80}{270}$ tenths=$\frac{800}{270}$ hundredths, and $\frac{800}{270}$ hundredths=$2\frac{260}{270}$ hundredths=2 hundredths+$\frac{260}{270}$ of a hundredth.

Similarly $\frac{260}{270}$ hundredths=$9\frac{170}{270}$ thousandths.

∴ $\frac{143}{270}$ of 1 simple unit=$5\frac{80}{270}$ tenths.
=5 tenths+$2\frac{260}{270}$ hundredths.
=5 tenths+2 hundredths+$9\frac{170}{270}$ thousandths.
=5 tenths+2 hundredths+9 thousandths+$6\frac{80}{270}$ ten thousandths+ &c. &c.

Whence $\frac{143}{270}$=0·5296+*,
And $\frac{34703}{270}$ =128·5296+.

From the preceding discussion it appears that to convert a proper vulgar into a decimal fraction, it is necessary to annex zeros to the numerator, and to divide by the denominator.

The number of decimal figures in the result is equal to the number of zeros annexed to the numerator of the vulgar fraction; for the quotient of the first division is tenths; that of the second, hundredths, &c. If any partial division give no significant figure to the quotient, a zero must be written to occupy the place, in order that their proper relative values may be preserved to the succeeding figures.

The reduction of $\frac{143}{270}$ to a decimal may be represented thus,

```
270)143·00000(·52962+
    1350
    ----
      800
      540
      ----
      2600
      2430
      ----
       1700
       1620
       ----
         800
         540
         ---
         260
         ....
```

The remainders, 80, 260, 170, being reproduced, it is evident that the process cannot terminate in a last remainder equal to zero. The approximation to an exact result may, however, be always carried to any degree of accuracy required. In the present instance one unit of the last quotient figure is the $\frac{1}{100000}$ part of 1; and as the value of all the succeeding figures of the quotient, how far soever carried, cannot amount to one unit of this order, (for then the last figure would be, not 2, but 3,) it follows that this quotient falls short of the exact quotient by less than the 100000th part of unity.

If carried one place farther, the approximation would be to within less than the millionth part of unity.

---

* + written after a decimal fraction is employed to indicate that the expression is incomplete, and greater than that which is put down.

The number of zeros required may be at once annexed to the numerator of the fraction. Thus, if the approximation is required to tenths, 1 zero must be annexed to the numerator; if to hundredths, 2 zeros; if to thousandths, 3 zeros . . . . .

Since the integral and fractional parts of the quotient are both obtained by the common process of division, the whole calculation may be made continuously, thus,

$$\begin{array}{l}
270)34703{\cdot}00000(128{\cdot}52962+ \\
\qquad\quad \underline{270} \\
\qquad\quad 770 \\
\qquad\quad \underline{540} \\
\qquad\quad 2303 \\
\qquad\quad \underline{2160} \\
\qquad\qquad 1430 \\
\qquad\qquad \underline{1350} \\
\qquad\qquad\quad 800 \\
\qquad\qquad\quad \underline{540} \\
\qquad\qquad\quad 2600 \\
\qquad\qquad\quad \underline{2430} \\
\qquad\qquad\qquad 1700 \\
\qquad\qquad\qquad \underline{1626} \\
\qquad\qquad\qquad\quad 800 \\
\qquad\qquad\qquad\quad \underline{540} \\
\qquad\qquad\qquad\quad 260 \\
\qquad\qquad\qquad\quad \underline{\ldots\ldots}
\end{array}$$

When the dividend is less than the divisor, the quotient contains no integral part, and the process of division is resolved into that of reducing a vulgar to a decimal fraction.

Let it be required to divide ·0037 by ·042, and to approximate to the exact quotient to within the $\frac{1}{1000000}$ part of unity. ·0037 and ·042, brought to the same denominator, are ·0037 and ·0420; and taken as whole numbers they are 37 and 420.

Since the approximation is to be made to within the 1000000th part of unity, six zeros must be annexed to the numerator, which becomes 37·000000; dividing this number by 420, the quotient is 88095+.

Six zeros having been annexed to the numerator of the fraction $\frac{37}{420}$, the equivalent decimal fraction must contain 6 decimal figures; one, namely, for each zero; to complete this number a zero must be written to the left of the significant figures of the quotient, which, consequently, is ·088095+.

If the reduction of $\frac{37}{420}$ to a decimal is made as in the case of the 1st Example of this Article, it is found that the place of tenths must be filled by a zero, for

$$\tfrac{37}{420} \text{ of 1 simple unit} = \tfrac{370}{420} \text{ of 1 tenth} = 0 \text{ tenths} + \tfrac{370}{420} \text{ of 1 tenth,}$$

$$\text{and } \tfrac{370}{420} \text{ of 1 tenth} = \tfrac{3700}{420} \text{ of 1 hundredth} = 8\tfrac{340}{420} \text{ hundredths, \&c. \&c.}$$

242. If any decimal, expressed as a vulgar fraction, is to be divided by 10, the division can be effected by multiplying the denominator of the dividend by 10 (Art. 149); the denominator of the quotient, therefore, contains one zero more than that of the dividend, and the quotient, expressed decimally, one decimal figure more than the dividend.

Similarly, the divisor being 100, the quotient contains two decimal figures more than the dividend; being 1000, it contains three decimal figures more than the dividend . . . . .

Whence, to divide a decimal by any power of 10, remove the decimal point of the dividend one figure to the left for each zero contained in the divisor; the result is the quotient required.

Example 1. Divide 8497·63 by 100.

The divisor containing two zeros, the quotient is obtained by removing the decimal point of the dividend two figures to the left; the quotient, therefore, is 84·9763.

Example 2. Divide 320·48 by 100000.

The divisor contains five zeros; to obtain the quotient, the decimal point of the dividend must, therefore, be removed five places to the left. As the dividend has only three figures on the left of the decimal point, this cannot be accomplished but by writing two zeros to the left of the number; the quotient obtained in this manner is ·0032048.

Recurring to the principles on which this rule depends,

$$320{\cdot}48=\frac{32048}{100};\quad 100000=\frac{100000}{1},$$

and $\frac{32048}{100}+\frac{100000}{1}=\frac{32048}{100\times100000}=\frac{32048}{10000000}={\cdot}0032048$, the same result as that which has been already obtained.

243. Rule for the division of decimal fractions:

Reduce the dividend and divisor to the same denominator, and divide as in whole numbers; if the division is exact the quotient is a whole number; if the division is not exact, annex to the remainder (or to the dividend) one zero for each decimal place required in the quotient; continue the division till these zeros are exhausted, and point off from the quotient one decimal figure for each zero annexed to the remainder or to the dividend.

If the number of zeros annexed to the dividend is greater than the number of figures contained in the quotient, prefix to the significant figures of the quotient zeros equal in number to the difference.

To divide a decimal by any power of 10, remove the decimal point of the dividend one figure to the left for each zero contained in the divisor.

244. Exercises in the division of decimal fractions:

1st. 2·468÷1·547 ?............*Ans.* 1·595.
2d. 19·545÷2·19 ?............*Ans.* 8·9246.
3d. 17·4244÷13·72 ?............*Ans.* 1·27.
4th. 7·596÷5·37 ?............*Ans.* 1·41452.
5th. 2·8354÷7·95 ?............*Ans.* 0·35665.
6th. ·28475÷·375 ?............*Ans.* 0·75933.
7th. ·0046÷·000254 ?............*Ans.* 18·1102.
8th. ·7854÷3·14159 ?............*Ans.* 0·25.
9th. ·39371÷·80642 ?............*Ans.* 0·4906996.
10th. ·0986÷·00743 ?............*Ans.* 13·2705.
11th. ·00743÷·0986 ?............*Ans.* 0·075355.
12th. 965·4÷396 ?............*Ans.* 2·437.
13th. 396÷965·4 ?............*Ans.* 0·410192.
14th. 5·÷·5 ?............*Ans.* 10·.
15th. 41·÷4182· ?............*Ans.* 0·009803.
16th. 683·4÷100 ?............*Ans.* 6·834.
17th. 7·49÷10000 ?............*Ans.* 0·000749.
18th. 100·÷1000· ?............*Ans.* 0·1.
19th. 143·2÷100 ?............*Ans.* 1·432.
20th. 143·2÷1000000 ?............*Ans.* 0·0001432.
21st. 143·2÷·001 ?............*Ans.* 143200·.
22d. ·01÷10000 ?............*Ans.* 0·000001.

23d. 1000÷·001?……*Ans.* 1000000.
24th. 37·9525÷25000?……*Ans.* 0·0015181·.
25th. ·15÷468·?……*Ans.* 0·0003205.
26th. ·015÷75·2?……*Ans.* 0·0001994.
27th. 30÷·009?……*Ans.* 3333·3333.
28th. ·009÷30?……*Ans.* 0·0003.
29th. 100÷9·9?……*Ans.* 10·101010.
30th. 1·÷9·?……*Ans.* 0·111111.
31st. 1·÷99·?……*Ans.* 0·01010101.
32d. 1·÷999·?……*Ans.* 0·001001001.
33d. 1·÷9999?……*Ans.* 0·00010001.
34th. 1666÷·98?……*Ans.* 1700·.
35th. 16·66÷·98?……*Ans.* 17·.
36th. 1764÷·84?……*Ans.* 2100.
37th. 17·64÷·84?……*Ans.* 21·.
38th. ·1764÷84·?……*Ans.* 0·0021.
39th. 5·4682÷796·85?……*Ans.* 0·0068622.
40th. ·00296÷·098?……*Ans.* 0·03020408.
41st. ·000025÷·7529?……*Ans.* 0·000033204.
42d. 5184÷·36?……*Ans.* 14400.
43d. 986·÷7529·?……*Ans.* 0·13096.
44th. 7529÷986·?……*Ans.* 7·635.
45th. 52297÷·00964?……*Ans.* 5425000.
46th. 52297÷9·64?……*Ans.* 5425.
47th. ·52297÷96·4?……*Ans.* 0·005425.
48th. ·321067÷1600?……*Ans.* 0·0002006668.
49th. 748·2÷60000?……*Ans.* 0·01247.
50th. 7·482÷·0006?……*Ans.* 12470·.
51st. ·001÷10000?……*Ans.* 0·0000001.
52d. ·001÷80000?……*Ans.* 0·0000000125.
53d. ·1÷·1?……*Ans.* 1·.
54th. 1·÷·1?……*Ans.* 10·.
55th. ·1÷1·?……*Ans.* 0·1.
56th. ·0008136÷67·8?……*Ans.* 0·000012.
57th. 81·37÷1000?……*Ans.* 0·08137.
58th. ·006453÷6453·?……*Ans.* 0·000001.
59th. 128÷37·436?……*Ans.* 3·41914.
60th. ·004÷400?……*Ans.* 0·00001.

245. If, in division of decimals, the dividend and divisor are made respectively the numerator and denominator of a vulgar fraction (which may be either proper or improper), and this fraction is decomposed into prime factors, and reduced to the lowest terms;

1st. When the denominator, after reduction, is equal to 1, the quotient is a whole number.

2d. When the only prime factors contained in the denominator are 2 and 5, or either of them, the quotient is composed of a limited number of decimal figures.

The first is manifest; with respect to the second, if the prime factors 2 and 5 are raised to the same power, since $2\times5=10$, $2^2\times5^2=100=10^2$, $2^3\times5^3=1000=10^3$, &c. &c., the denominator is a power of 10, marked by the exponent common to the factors 2 and 5. The expression is consequently a decimal fraction, in which it is only necessary to replace the denominator by the decimal point.

If 2 is the only prime factor contained in the denominator, let the terms of the fraction be multiplied by the corresponding power of 5; if 5 is the only prime factor contained in the denominator, let the terms be multiplied by the corresponding power of 2; and if both 2 and 5 are contained in the denominator, but the one is raised to a higher power than the other, let the terms of the fraction be multiplied by that power of 2 or of 5 (as the case may require) which makes the exponents of both prime factors the same.

Then, as before, the denominator is a power of 10, and the expression a decimal fraction.

The exponent of 2 or 5, when the denominator contains but one of these factors, or both, raised to the same power, and the exponent of the higher, when they are raised to different powers, indicates the number of decimal figures contained in the exact quotient.

3d. When the denominator contains other prime factors than 2 and 5, the quotient is composed of an unlimited number of decimal figures.

For the numerator and denominator being, by hypothesis, prime to each other, the former can only be rendered a multiple of the latter by multiplying it by a number which involves all the prime factors contained in the latter (Art. 111 and 115). But the number by which the numerator is multiplied, namely, 10 or some power of 10 (which multiplication is effected by annexing zeros), contains only the prime factors 2 and 5; the division of the product by the denominator must, consequently, end in a remainder.

There is no limit to the number of zeros which may be annexed to the numerator, and therefore none to the number of quotient figures, that is, of decimals.

246. When a quotient is not expressible by a limited number of decimal figures, it is partly or altogether composed of periods in which the same figures continually recur in the same order.

The numerator being represented by n and the denominator by d, all the remainders which can occur in dividing $n\times10$, $n\times100$, $n\times1000\ldots$ by d, are contained in the series $1, 2, 3\ldots d-1$. When, therefore, at the utmost $d-1$ partial divisions have been made, one of the remainders which has been already obtained must be again found.

Annexing 0 to this remainder, the result is a partial dividend equal to one of the preceding partial dividends. The quotient and the remainder resulting from the division of this dividend by d are equal to the quotient, and the remainder given by the equal partial dividend at the beginning of the first period.

Annexing 0 to the remainder, and dividing by d, a second figure of a second period, corresponding to the second figure of the first period, is found, and similarly a third, &c. (if the period is composed of several figures), till all the figures of the first period are reproduced in their proper order.

This process may, it is evident, be continued indefinitely. Such decimal fractions are termed periodic.

a. When the last figure of the numerator is significant, the remainders, the partial dividends, and the partial quotients, obtained before the significant figures are exhausted, cannot be equal to the corresponding quantities obtained after the exhaustion of the significant figures. Consequently, the period cannot in such a case begin before the decimal point.

b. If the 1, 2, 3 .... last figures of the numerator are zeros, the period may commence 1, 2, 3 .... figures before the decimal point.

c. If the denominator contains either or both of the factors, 2, 5, the period cannot commence immediately after the decimal point.

For the partial dividend, which gives the first decimal figure of the quotient, is a multiple of 10, being made so by the zero annexed to the preceding remainder.

This dividend can, therefore, be divided by 2, 5, 2×5; but such division takes away the zero at the end of the partial dividend, and consequently also the condition by means of which the quotient figure resulting from the division of the reduced partial dividend by the product of the remaining prime factors of the denominator can be the first figure of a period.

For the same reason, if the denominator contains the factors 2^2 or 5^2, the period cannot commence till after the second decimal figure; if 2^3 or 5^3, not till after the third decimal figure....

d. When a period begins at the decimal point the period is termed simple, and the quotient a simple periodic decimal number; when before or after the decimal point, the period is said to be mixed, and the quotient is termed a mixed periodic decimal number.

247. To divide one decimal number by another, and to reduce a vulgar fraction to a number expressed decimally, being in effect the same operation, either form of expression may be employed to describe it.

In the following examples, intended to illustrate the observations contained in Articles 245 and 246, the second is used.

1st Ex. Let it be required to reduce the vulgar fraction $\frac{7}{8}$ to a decimal:

Resolving the terms into simple factors, $\frac{7}{8}=\frac{7}{2^3}$, the fraction containing no common factor is in the lowest terms, and since the denominator is 2 raised to the 3d power, the quotient contains 3 decimal figures;

$$\text{for } \frac{7}{2^3}=\frac{7\times5^3}{2^3\times5^3}=\frac{875}{1000}=\cdot875.$$

2d Ex. Let it be required to reduce the vulgar fraction $\frac{19}{25}$ to a decimal: $\frac{19}{25}=\frac{19}{5^2}$. Therefore the fraction is in the lowest terms, and the equivalent decimal contains 2 decimal figures;

$$\text{for } \frac{19}{5^2}=\frac{19\times2^2}{5^2\times2^2}=\frac{76}{100}=\cdot76.$$

3d Ex. Let it be required to reduce the vulgar fraction $\frac{57}{240}$ to a decimal:

$$\frac{57}{240}=\frac{3\times19}{2^4\times3\times5}=\frac{19}{2^4\times5}=\frac{19}{80};$$

$\frac{19}{80}$ or $\frac{19}{2^4\times5}$ is the fraction expressed in the lowest terms; and since the factors of the denominator, 2, 5, are raised to different powers, and the exponent of that which is raised to the higher power is 4, the decimal must contain 4 decimal figures;

$$\text{for } \frac{19}{2^4\times5}=\frac{19\times5^3}{2^4\times5\times5^3}=\frac{2375}{10^4}=\frac{2375}{10000}=\cdot2375.$$

3d Ex. Let it be required to reduce the vulgar fraction $\frac{13}{37}$ to a decimal:

The terms of the fraction are prime numbers; and since the denominator, which is greater than the numerator, contains a prime factor different from 2 and 5, the result is composed entirely of decimal figures and periodic.

```
37)13·000000(0·351351+
    190
     50
    130
     190
       50
       +
```

In this example the period which begins immediately after the decimal point is composed of three figures.

4th Ex. Let it be required to reduce the vulgar fraction $\frac{1300}{21}$ to a decimal:

$$\frac{1300}{21}=\frac{13\times10^2}{3\times7}.$$

The numerator and denominator containing no common factor, the fraction is expressed in the lowest terms, the decimal is periodic, and the first period may begin before the decimal point.

$$\begin{array}{l}
21)1300{\cdot}0000000000(61{\cdot}9047619047+ \\
\quad\underline{40} \\
\quad\underline{190} \\
\qquad\underline{10} \\
\qquad\underline{100} \\
\qquad\ \underline{160} \\
\qquad\quad\underline{130} \\
\qquad\qquad\underline{40} \\
\qquad\qquad\underline{190} \\
\qquad\qquad\quad\underline{10} \\
\qquad\qquad\quad\underline{100} \\
\qquad\qquad\qquad\underline{+}
\end{array}$$

This result contains 2 integral figures, both of which belong to the first period.

5th Ex. Let it be required to reduce the vulgar fraction $\frac{5}{28}$ to a decimal:

Since $\frac{5}{28}=\frac{5}{2^2\times7}$, this fraction is in the lowest terms, the equivalent decimal is periodic, and the period cannot commence before the 3d decimal figure. For the first and second decimal figures are obtained by annexing two zeros to the numerator, or by multiplying it by $100=10\times10$, and dividing by the denominator;

$$\text{now } \frac{5\times10\times10}{2\times2\times7}=\frac{5\times5\times5}{7}=\frac{125}{7}={\cdot}17+.$$

The first and second decimal figures, being hence obtained from partial dividends whose last figures are significant, cannot belong to the periodic part of the quotient; but the third and succeeding partial dividends being formed by annexing zeros to the successive remainders, the quotient figures, beginning with the third, may be periodic.

Keeping in mind that the expression $\frac{125}{7}$, having been multiplied by 100, must give two decimal figures to the quotient, the reduction may be made as follows:

$$\begin{array}{l}
7)\underline{1{\cdot}25000000\qquad\ } \\
\quad\ \underline{0{\cdot}17857142857142+}
\end{array}$$

The period in this example commences with the 3d decimal figure, and consists of 6 figures.

248. Every decimal fraction which is not periodic may be reduced to an equivalent vulgar fraction by writing the denominator of the decimal (Art. 220), and reducing the fractional expression to the lowest terms.

$$\text{Thus } {\cdot}875=\frac{875}{1000}=\frac{5^3\times7}{2^3\times5^3}=\frac{7}{2^3}=\frac{7}{8}.$$

249. Every periodic decimal fraction may also be reduced to an equivalent vulgar fraction.

a. Let the period be simple. Then, whatever the number of figures contained in the period, it is possible to find another periodic decimal containing the same number of figures in its period, and in which the only significant figure of each period, the last, shall be 1.

This periodic decimal is obtained by dividing 1 by a divisor consisting of 9 as often repeated as the first period contains figures;

$$\begin{array}{r}
\text{for } \frac{1}{9}={\cdot}11111+ \\
\frac{1}{99}={\cdot}010101+ \\
\frac{1}{999}={\cdot}001001+ \\
\frac{1}{9999}={\cdot}00010001+
\end{array}$$

. . . .

Then, since the products of two equal multiplicands by the same multiplier are equal to each other, if $\frac{1}{999...}$ and ·00..100..1+ are both multiplied by a period of the decimal which is to be expressed as a vulgar fraction, the result is, first, a vulgar fraction whose numerator is the given period, and its denominator a number composed of 9 once repeated for each figure contained in the period; second (and equal to this vulgar fraction), the given periodic decimal.

As an application, let it be required to reduce the simple periodic decimal ·351351+ to an equivalent vulgar fraction.

In this example the period contains 3 figures.

To obtain another period consisting also of 3 figures, but of which the last figure only shall be significant and equal to 1, let 1 be divided by 999; then $\frac{1}{999}=·001001+$;

$$\text{therefore } \tfrac{1}{999}\times 351=·001001+\times 351,$$
$$\text{or } \tfrac{351}{999}=·351351+$$

The simple periodic decimal ·351351+ is therefore equivalent to the vulgar fraction $\frac{351}{999}$, which, expressed in the lowest terms, is $\frac{13}{37}$.

2d Example. Let it be required to reduce the simple periodic decimal number 3·4545+ to an equivalent fractional expression.

The period of this example, 45, is equivalent to the vulgar fraction $\frac{45}{99}$, and the given decimal number to $3\frac{45}{99}$, or to $3\frac{5}{11}$.

When the result, as in this instance, is a mixed number, it may become necessary to reduce it to an improper fraction. This reduction can be made by the rule of Article 157; but since the denominator of every vulgar fraction, equivalent to a simple periodic decimal, is composed wholly of nines, and that to multiply one number by another composed of nines is the same thing as to multiply it by unity followed by as many zeros as there are nines, and then to subtract the number itself once from this product, the reduction may also be made thus:

$$3\tfrac{45}{99} \text{ or } 3+\tfrac{45}{99}=\tfrac{3\times 99}{99}+\tfrac{49}{99};$$
$$\text{but } \tfrac{3\times 99}{99}=\tfrac{3(100-1)}{99}=\tfrac{300-3}{99};$$
$$\therefore 3\tfrac{45}{99}=\tfrac{300-3}{99}+\tfrac{45}{99}=\tfrac{300-3+45}{99}=\tfrac{345-3}{99}.$$

The numerator of this result consists of two parts; namely, first, a number (345) composed of the integral part and the first period of the given periodic decimal number; second, the integral part (3) of the same number, the second being subtracted from the first.

The denominator contains two nines, one for each figure contained in the period of the given number.

The reduction of every similar expression may be made in the same manner. Whence it is to be inferred that any simple periodic decimal number is equivalent to a vulgar fraction which has for numerator a number composed of the integral part and first period of the given periodic decimal, less the integral part of the given periodic decimal; and for denominator, a number composed of 9 as often repeated as there are figures in the period of the given periodic decimal.

250. A mixed can always be reduced to the form of a simple periodic decimal number by transferring the decimal point from its given position to the beginning of the first period. This is accomplished, when the period begins before the decimal point, by dividing the given periodic decimal number by 1, with as many zeros annexed to it as there are figures between the beginning of the period and the decimal point; and when the period begins after the decimal point, by multiplying by 1 with as many zeros annexed as there are figures between the decimal point and the beginning of the period.

The simple periodic decimal number thus obtained is reduced to an equivalent vulgar fraction by Article 249; and this fraction being multiplied or divided by that power of 10 by which the given mixed periodic decimal number was divided or multiplied, the result is the fractional expression sought.

1st Example. Given the mixed periodic decimal number, 529·6296+, to find an equivalent vulgar fraction.

In this example the period commences two figures before the decimal point.

$$529{\cdot}6296+=\frac{529{\cdot}6296+}{100}\times 100=5{\cdot}296296+\times 100.$$

$$5{\cdot}296296+=\frac{5296-5}{999}\text{ (Art. 249).}$$

$$\therefore\ 5{\cdot}296296+\times 100=\frac{5296-5}{999}\times 100,$$

$$\text{or } 529{\cdot}6296+=\frac{529600-500}{999}=\frac{529100}{999}.$$

The last figures of the numerator of this result are zeros.

2d Example. Given the mixed periodic decimal number, 27·685454+ (in which the period commences after the decimal point), to find an equivalent vulgar fraction.

$$27{\cdot}6854+=\frac{27{\cdot}6854+}{100}\times 100=\frac{2768{\cdot}54+}{100},$$

$$2768{\cdot}54+=\frac{276854-2768}{99}.$$

$$\therefore\ \frac{2768{\cdot}54+}{100}=\frac{276854-2768}{99}\times\frac{1}{100},$$

$$\text{or } 27{\cdot}6854+=\frac{276854-2768}{9900}=\frac{274086}{9900}.$$

The last figures of the denominator of this result are zeros.

251. The numerator of the vulgar fraction which is equivalent to a simple periodic decimal number cannot end in 0; for, that the last figure of the numerator may end in 0, the last figure of the whole number must be the same as the last figure of the period (see Ex. 2, Art. 249). But this condition carries the beginning of the period one place to the left of the decimal point, and renders mixed the period which by hypothesis is simple.

Likewise, since the denominator of the vulgar fraction which is equivalent to a simple periodic decimal number consists entirely of nines, the factors 2 and 5 cannot enter into the denominator of any such fraction, unless the same factors enter also into the numerator.

Every simple periodic decimal number arises, therefore, from a vulgar fraction which does not contain in its numerator the factor 10, nor in its denominator the factors 2, 5.

A vulgar fraction which satisfies these conditions cannot give rise to a mixed periodic decimal number; for, supposing it to give a mixed periodic decimal number whose period begins before the decimal point, then, conversely, the vulgar fraction obtained as the expression of this mixed periodic decimal contains in its numerator one zero for each figure which is between the beginning of the period and the decimal point. The denominator consisting wholly of nines, the zeros of the numerator cannot be cancelled by any reduction of the expression to lower terms. The fraction must, therefore, contain in its numerator the factor 10, which is contrary to the conditions.

Or, supposing it to give a mixed periodic decimal number whose period commences after the decimal point, then the vulgar fraction obtained as the expression of this mixed periodic decimal number contains in its denominator one zero for each figure between the decimal point and the beginning of the period. The numerator cannot end in zero, for that is contrary to the condition; but it may contain some power of 2 or 5, though not of both 2 and 5. But since the denominator is a multiple of 10 (=2×5), the denominator must contain some power of 5 or 2 not contained in the numerator; which, also, is contrary to the conditions.

It is to be concluded, therefore, that every vulgar fraction, whose denominator contains one or more prime factors different from 2 and 5, leads to a periodic decimal quotient; that this quotient is a simple periodic decimal, when the numerator does not contain the factor 10, or the denominator the factor 2 or 5; that this quotient is a mixed periodic decimal, the period commencing before the decimal point, when the numerator is a multiple of 10 and the denominator not a multiple of 2 or 5; that in this case the exponent of the power of 10 contained in the numerator indicates the number of figures of the period placed before the decimal point; that the quotient is a mixed periodic decimal, the period commencing after the decimal point and after as many figures as there are units in the exponent of the higher of the two factors 2, 5, of the denominator, when the denominator involves either or both of these simple factors.

And, conversely, that every simple periodic decimal number is equal to a vulgar fraction of which the numerator is a whole number obtained by subtracting the figures of the periodic decimal number, which are on the left of the period, from the figures between the beginning of the periodic decimal number and the end of the first period, considered as whole numbers; and the denominator composed of as many nines as there are figures in the period; and that every mixed periodic decimal number is equal to the vulgar fraction which expresses the value of the decimal number rendered simply periodic by the displacement of the decimal point; the said vulgar fraction being subsequently multiplied by the power of 10 marked by the number of periodic figures before the decimal point (when the first period commences before the decimal point); but divided by a power of 10 marked by the number of figures between the decimal point and the first figure of the period, when the first period commences after the decimal point.

252. When a fractional expression leads to an unlimited decimal quotient, the value of any given number of the figures of this quotient is not exactly equivalent to that of the fractional expression. But the approximation to the exact value can be carried to any required degree of nearness by Article 241.

For example, the value of the fraction $\frac{17}{35}$, to within the $\frac{1}{10000000}$ part of unity, is ·4857142.

In some instances, the value, obtained in this manner, admits of closer approximation. Continuing the division by 35, the next figure of the quotient is 8.

The 7th and 8th figures of the quotient, considered with respect to their relative value, are expressed by the fraction $\frac{28}{100000000}$.

$$\text{Now, } \frac{28}{100000000} \text{ differs from } \frac{20}{100000000} \text{ or } \frac{2}{10000000} \text{ by } \frac{8}{100000000};$$

$$\text{and from } \frac{30}{100000000} \text{ or } \frac{3}{10000000} \text{ by only } \frac{2}{100000000}.$$

Whence, the result ·4857142 differs, in defect, from the exact quotient by more than $\frac{8}{10}$ of one unit of the last figure; and the result ·4857143 differs, in excess, by less than $\frac{2}{10}$ of one unit of the last quotient figure, 0·4857143, is, therefore, a nearer approximation to the exact quotient than 0·4857142.

If the 8th figure of this quotient were 5 or any figure greater than 5, the change of the 2 to 3 would be proper; as the value of ·48571425 + is nearer ·4857143 than ·4857142: but if the 8th figure were 1, 2, 3, or 4, the 2 ought to remain unchanged, as ·48571424 is nearer in value to ·4857142 than to ·4857143.

For these reasons, it is usual in practice to find one quotient figure more than is required; and, if this figure is 5 or greater than 5, to increase the last of the required figures by unity of the order of that figure.

SECTION VIII.

OF THE SQUARE, SQUARE ROOT, CUBE AND CUBE ROOT OF NUMBERS.

253. If a number is multiplied by itself the product is termed the second power or square of the number. Thus the product 7×7 or 49 is the square of 7.

The exponent 2 is employed to indicate the multiplication of a number by itself (Art. 75); thus $7^2=7\times 7=49$.

$$\left(\frac{5}{8}\right)^2=\frac{5}{8}\times\frac{5}{8}=\frac{5\times 5}{8\times 8}\left(\text{or }\frac{5^2}{8^2}\right)=\frac{25}{64}\text{ (Art. 194).}$$

$$(\cdot 04)^2=\cdot 04\times\cdot 04=\cdot 0016\text{ (Art. 238).}$$

$$(a+b)^2=(a+b)\,(a+b)=a^2+2ab+b^2\text{ (Art. 76).}$$

254. The product of two factors is equal to the sum of the partial products of every figure of the multiplicand by every figure of the multiplier; and these partial products may be taken in any order if attention is given to the relative values of the partial factors (Art. 62).

Now if the figures expressing the units of the multiplicand and multiplier are multiplied together, their product is the square of the units of the given number; and if the figure expressing the units of the multiplier is multiplied into the other figures of the multiplicand, beginning with the tens, the result is the product of the units of the multiplier into the number composed of the figures on the left of the units of the multiplicand taken with regard to their relative value; likewise if the figure expressing the units of the multiplicand is multiplied into the figures of the multiplier, beginning with the tens, the result is the product of the units of the multiplicand into the number composed of the figures on the left of the units of the multiplier taken with regard to their relative value. The units of the multiplier and multiplicand being equal, as well as the numbers expressed by the figures beginning with the tens of both, it follows that these products are equal; therefore their sum is equal to twice the product of the units of the given number into the number composed of the figures on the left of the units, regard being given to the relative value of the second factor.

All the partial products, therefore, into which the units of the number can enter are,

The square of the figure which expresses the units,

And the double product of this figure into the number expressed by the figures on its left; the second factor being taken with regard to its relative value.

Leaving out of consideration the figure which expresses the units of the number, and reasoning in the same manner on the figure which expresses the tens, it is found that all the partial products into which the tens of the number can enter are,

The square of the figure expressing the tens,

And the double product of this figure into the number which is expressed by the figures on its left; the partial factors of both results being taken with regard to their relative values.

The same process of reasoning can be applied to every figure of the number, and the same result deduced: Whence the square of a number expressed by 2, 3, 4, . . . *n* figures is equal to the sum of the squares of the figures taken individually and with regard to their relative values plus the double product of each figure into the number composed of the figures on its left, both the figures and numbers being taken with regard to their relative values.

The left or highest figure of the number gives its square only to the general result; for there are no figures on its left by which it can be multiplied.

Formed in this manner, 574^2 is equal to $500^2+2\times500\times70+70^2+2\times570\times4+4^2$.

255. If the digits of a number which contains three figures are represented by a, b, c, then $100a+10b+c$ expresses the number; and its square is,

$$(100a)^2+2\times100a\times10b+(10b)^2+2(100a+10b)c+c^2,$$
$$\text{or } a^2\times10000+2\times100a\times10b+b^2\times100+2(100a+10b)c+c^2.$$

This formula gives the means of determining, within certain limits, those figures of the power which contain the significant figures arising from the square of each figure of the number.

For the product of one figure by one figure being expressed by two figures or by one, the numerical value of the square of the units of the number is contained in the tens and units of the power or in the units only; and hence if the two figures which express the units and tens of the power are separated from the other figures by a comma, the separated figures comprehend the numerical value of the square of the units. This may be called the Period of the units of the number.

The product of a number by 100 contains zeros in the places of units and tens. Therefore the square of the tens, represented by $b^2\times100$, gives zeros to the units and tens of the power, and significant figures to the hundreds and thousands or to the hundreds only. The significant figures of the square of the tens cannot, therefore, fall so low as the second or so high as the fifth figure of the power. They are hence comprehended in the two figures on the left of the period of the units. Interposing a comma between the fourth and fifth figures of the power, the two figures intercepted between this comma and the period of the units is the period of the tens of the number.

The product of a number by 10000 contains zeros in the places of units, tens, hundreds, and thousands; therefore the square of the hundreds, represented by $a^2\times10000$, gives zeros to the units, tens, hundreds, and thousands of the power, and significant figures to the ten-thousands and hundred-thousands or to the ten-thousands only. The significant figures of the square of the hundreds cannot, therefore, fall so low as the fourth or so high as the seventh figure of the power; they are consequently comprehended in the two figures on the left of the period of the tens of the number.

Hence it may be inferred, that whatever places the significant figures of the square of any digit occupy, the significant figures of the square of the next higher digit occupy the third and fourth or third places from the last significant figure of the square of the lower digit.

A more general proof of this proposition can be given.

Every figure of a number has to the figure on its right the relative value of 10 to 1. If a, b are any two consecutive digits of a number whose second power is to be found, the relative values to each other of these digits, considered as forming parts of the number, are $10a$ and b; and their squares are $(10a^2)$ and b^2 or $a^2\times100$ and $b^2\times1$. Now the numerical value of b^2 is expressed by not more than two figures; and the numerical value of $a^2\times100$ by not more than four figures, of which the two last are zeros. In combining these partial products to form the square of the number of which a, b are two consecutive digits, the last of the two zeros of $a^2\times100$ falls under the lowest significant figure of the numerical value of b^2; whence the lowest significant figure of the numerical value of a^2 falls two places to the left of the lowest significant figure of the value of b^2; that is, in arranging the partial products for addition, the lowest significant figure of the numerical value of a^2 falls to the left of the highest significant figure of the numerical value of b^2. Consequently, from the units, inclusive of the square of any digit, to the units of the square of the next higher digit, there are two figures of the power; and the significant figures arising from the square of the lower digit are not combined with the significant figures of the square of the next higher digit.

256. From this discussion it follows, that the square of a number is composed of the square of the highest figure of the number (or the first figure reckoning from the left), plus the double product of the first figure into the second, plus the square of the second, plus the double product of the number expressed by the first and second figures into the third, plus the square of the third, plus the double product of the number expressed by the first, second, and third figures into the fourth, plus the square of the fourth, ... each factor being taken with regard to its relative value as ten, hundred, thousand, ... according to its place in the given number:

That the numerical value of the square of the units of the number is comprehended in the units and tens (or in the first and second figures of the power, the figures being taken from right to left); the numerical value of the square of the tens in the third and fourth, that of the hundreds in the fifth and sixth, ...

And that if the second power or square of a number is divided into periods of two figures, taking the figures from right to left, the significant figures arising from the square of the units of the number which has been multiplied by itself are contained in the first period, or the period on the right; the significant figures from the square of the tens in the second; those from the square of the hundreds in the third; . . . . .

There are as many periods in the square as there are figures in the number. Each period, excepting the last, must contain two figures; the last may contain two or one only. For if the number contains m figures, its square contains $2m$* or $2m-1$ (Art. 73); and it is evident, that when the square is composed of $2m$ figures, each of the m periods contains two figures; but that when the square is expressed by $2m-1$ figures, the last or highest period contains 1 figure only.

Since $1^2=1$, $2^2=4$, $3^2=9$, this may occur when the highest figure of the number is less than 4.

257. The factor which is multiplied by itself is termed the square root of the product arising from such multiplication. 7 is the square root of 49; $\frac{5}{8}$ the square root of $\frac{25}{64}$; and ·04 the square root of ·0016.

To express that the square root of a number is to be extracted, or to indicate without extracting the square root, the symbol $\sqrt{\ }$, called the radical sign, is written on the left of the number.

$\sqrt{49}$, for example, is an expression signifying the square root of 49. This being 7, $\sqrt{49}$ and 7 are expressions of the same value.

When the multiplication of a number by itself is indicated by means of the exponent 2, the square root is obtained by removing the exponent.

For as $7^2=49$, $\sqrt{7^2}=\sqrt{49}=7$.

Since $\sqrt{49}=7$, and $7\times7=49$; $\sqrt{49}\times\sqrt{49}=49$; that is, the product of a number, under the radical sign, by itself, is equal to that number without the radical sign.

In effect these consequences follow from the definitions of the terms square root and square.

Generally, if a represent any number, and $\sqrt{a}$, the square root of that number,

$$\sqrt{a}\times\sqrt{a}=a\text{; and }\sqrt{a^2}=a.$$

Also if a, b, represent different numbers, ab the product of these numbers,

$$\sqrt{ab}\times\sqrt{ab}=ab.$$

The product of two equal factors, integral or fractional, is formed in the same manner as that of two unequal factors; but the reverse operation, that namely of returning from the product to the factors, is not the same in both cases.

---

* $2m$ is the case of an even number of figures in the power, $2m-1$, that of an odd number.

When the factors are unequal, the product and one factor are given to find the other factor. This is the problem of Division.

When the factors are equal, the product of one factor by itself is given to find that factor. This is the problem of extraction of the square root.

If in extraction of the square root, as in division, the product of two factors and one of these factors were given; as the two are equal, there could be nothing to discover.

The squares of the nine digits are 1, 4, 9, 16, 25, 36, 49, 64, 81;
and each of the digits 1, 2, 3, 4, 5, 6, 7, 8, 9,
is the square root of that number under which it is placed; that is, it is the square root of its own square.

258. Since 10 is the least number expressed by two figures, and $10^2=100$, it is evident that of the consecutive whole numbers from 1 to 99 inclusive, nine only have whole numbers for their square roots, the others have each a whole number plus a fraction.

Since 5, for example, falls between 4 and 9, the square root of 5 is greater than the square root of 4 ($=2$); and less than the square root of 9 ($=3$); that is, $\sqrt{5}>2$, and $\sqrt{5}<3$; therefore $\sqrt{5}=2+$ a fraction.

In like manner, since 6 and 7 fall between 4 and 9, $\sqrt{6}=2+$a fraction, $\sqrt{7}=2+$a fraction. The fraction which is to be annexed to the whole number 2 is different in each of these instances. The manner of deducing its value shall be explained in the proper place. At present it may be sufficient to show that a whole number, which is not the square of another whole number, cannot have an exact fractional number for its square root.

To prove this generally, let a be the numerator and b the denominator of a fraction, expressed in the lowest terms. Then if $\frac{a}{b}$ is the square root of a whole number, $\frac{a}{b}\times\frac{a}{b}$ or $\frac{a^2}{b^2}$ must be a whole number.

But this is impossible; for a^2, b^2, contain no prime factors which are not also contained in a, b; and since a, b, are prime to each other, so are a^2, b^2. Therefore $\frac{a^2}{b^2}$, being an irreducible fraction, cannot be equal to a whole number.

The square root of a whole number which is not the square of another whole number is, consequently, not expressible by any exact number. $\sqrt{5}$, $\sqrt{6}$, . . . are therefore not expressible by any exact number.

An expression such as $\sqrt{5}$ is termed an incommensurable or irrational or surd number, that is, a number which cannot be measured exactly by means of unity.

A number which has an exact square root is termed a square number, or a perfect square.

259. The square of 1 is 1, and the square of 2, 4; therefore the difference between the squares of the consecutive numbers 1, 2 is $4-1$ or 3; in like manner the difference between the squares of the consecutive numbers 2 and 3 is $9-4$, or 5; and between 3^2 and 4^2, it is $16-9$, or 7.

From these instances the difference between two consecutive square numbers, and consequently the number of not perfect squares falling between them, appears to be greater, as the square roots are greater.

This admits of general proof.

Let a and $a+1$ denote any two consecutive numbers;
then $(a+1)^2=(a+1)\ (a+1)=a^2+2a+1$ (Art. 76), and $a^2=a\times a=a^2$;

$$\therefore (a+1)^2-a^2=2a+1.$$

Whence the difference between the squares of two consecutive numbers is equal to twice the less number $+1$; the greater, therefore, the number a

the greater is the difference between $(a+1)^2$ and a^2, and the greater also the number of imperfect squares between a^2 and $(a+1)^2$.

The difference between the squares of 85 and 86, for example, is $85\times2+1=170+1=171$; and, consequently, between the squares of 85 and 86 there are 170 numbers which are not square numbers.

260. If a number is given, and it is required to find a second number whose product by itself shall be equal to the given number, the operation by which the second number is discovered is termed Extraction of the square root.

. Extraction of the square root of a number is an operation the reverse of that by which a square is formed; the direct operation is explained, Articles 254 and 255, and the process for the reverse operation must be derived from the principles contained in these articles.

To apply these principles, let it be required to extract the square root of 6084.

The first step of the process is the determination of the number of figures contained in the square root of 6084; this is effected by separating 6084 into periods of two figures (commencing with the units) and reckoning one figure of the root for each period of the power (Art. 256); the number 6084 contains two periods consisting of two figures each, these are separated by a comma, thus 60,84; and since the power contains two periods its square root consists of two figures, which express tens and units.

2d. To find the figure which expresses the tens of the root; the square of the tens is contained in the period 60 (Art. 256); consulting the table of square numbers expressed by two figures, it is found that 60 falls between $49=7^2$ and $64=8^2$ (Art. 257).

Since the given number 6084 falls also between 4900 or 70^2 and 6400 or 80^2, and that $\sqrt{4900}=\sqrt{70^2}=70$, it follows that 7 is the number of tens contained in the required root; subtracting 4900, the square of the tens of the root, from 6084, the remainder, 1184, is composed of the double product of the tens of the root by the units plus the square of the units.

3d. To find the figure which expresses the units of the root; the 7 tens of the root are equal to 70 units, therefore double the tens of the root are equal to 140 units.

Dividing the first remainder, 1184, by 140, the quotient is 8; assuming that 8 expresses the units of the root, let the double product of the tens by 8 be subtracted from 1184; this double product is $140\times8=1120$, and $1184-1120=64$. This second remainder, 64, ought to be not less than 8^2, the square of the number which has been assumed to express the units of the root.

Now $8^2=64$, and the second remainder$-8^2=64-64=0$.

The last remainder being zero, $\sqrt{8^2}=8$ expresses the units of the root, and 78 is the square root of 6084.

If 8^2 had been greater than the second remainder it would have been necessary to assume 7 as the figure expressing the units of the root; and if 7^2 had also been greater than the second remainder the next lower figure must have been tried, and so on, till a figure had been found whose square was not greater than the second remainder.

The calculation may be made as follows:

power.
60,84(7=tens of the root.
$(7\text{ tens})^2=(70\text{ units})^2=\underline{4900}$
7 tens or 70 units$\times2=140)1184(8$=units of the root.
$2\times70\times8=\underline{1120}$
64
$8^2=8\times8=\ \underline{64}$
0
7 tens+8 units=78 units=square root of 6084.

261. Every example of extraction of the square root of a whole number can be resolved in this manner; certain details which follow are rendered necessary by the abbreviations of the calculation usual in practice; these do not, however, bear on the principles of the rule.

The chief of these abbreviations are,

1st. The zeros terminating the square of the highest figure of the root are omitted; and the significant figures of the square being subtracted from the proper period of the given number, the following period is annexed to the remainder in the same manner as a figure of the dividend is annexed to a remainder in division.

2d. The double product of the first figure of the root by the second and the square of the second are found by one multiplication; and the defect of this product from the sum of the first remainder and second period obtained by one subtraction.

Denoting by a the first, and by b the second figure of the root, $10a+b$ is the number expressed by the first and second figures; $(10a)^2+2\times10a\times b+b^2$, the square of that number; and $2\times10a\times b+b^2$, the double product of the first figure by the second plus the square of the second figure.

Now $2\times10a\times b+b^2$ may be put under the form $2\times10a\times b+b\times b$ or $(2\times10a+b)\times b$ (Art. 70).

A result showing that, if to double the number expressed by the first figure of the root (considered as tens) the second figure is added, and the sum is multiplied by the second figure, the result is equal to the double product of the first figure by the second plus the square of the second.

3d. The addition of the second figure to double the first is made by annexing the second figure of the root to the double of the first. Its proper relative value as expressing units of the order next higher than those of the second figure is thus preserved to the double of the first. The second figure of the root also is annexed to the first, and with the same result.

If the root contains three figures, the number expressed by the first and second figures has to that expressed by the third the relative value of tens to units.

Denoting the tens by a and the units by b, the consequences just deduced may be extended to a square root expressed by three figures.

And in the same manner it may be extended to a square root expressed by any greater number of figures.

2d Example. Let it be required to extract the square root of 7875986.

This number contains four periods, 7, 87, 59, 86; therefore its square root contains four figures.

To find the first figure of the root.

The first period $7>4=2^2$ and $7<9=3^2$.

$\therefore \sqrt{2^2}$ or 2 is the first figure of the root; and $7-4=3$, is the first remainder.

To find the second figure of the root.

3, the first remainder, with 87, the second period, annexed to it, gives 387, as the number which contains the double product of the first figure of the root by the second plus the square of the second; or $(2\times10a+b)\times b$.

In this example $a=2$, and twice $a=4$. Therefore $(4\times10+b)\times b=387$.

To find b, it is necessary to divide either the whole dividend 387 by $4\times10=40$, or to divide the dividend with the exception of its last figure by 4. In practice, the second alternative is commonly taken.

$38\div4=9$. Assuming that 9 is the second figure of the root $(2\times10a+b)\times b=(2\times10\times2+9)\times9=49\times9=441$.

This number being greater than 387, it follows that 9 is greater than the second figure of the root.

Trying the next lower figure 8.

$$48\times 8=384 \text{ and } 387-384=3.$$

∴ 8 is the second figure of the root, and 3 the second remainder.

To find the third figure of the root.

The number formed of the second remainder and third period is 359.

Double the part of the root already obtained is $28\times 2=56$.

Dividing, as before, the dividend with the exception of its last figure by the double of the root, $35\div 56$—the quotient is zero.

Whence the third figure of the root is 0, and the third remainder 359.

To find the fourth figure of the root.

The number formed of the third remainder and fourth period is 35986.

Double the part of the root, already obtained is $280\times 2=560$.

Dividing the dividend, with the exception of its last figure by 560.

$$3598\div 560=6$$
$$5606\times 6=33636$$
$$\text{And } 35986-33636=2350.$$

The fourth figure of the root is 6; and $\sqrt{7875986}=2806+$ a fraction.

Since the remainder 2350 is less than $2\times 2806+1$ the required square root cannot be equal to $2806+1$ or 2807.

Calculation in an abridged form.

```
      7,87,59,86(2806
      4
      ─
  48\ 387
   8/ 384
      ─────
5606\ 35986
    / 33636
      ─────
       2350 remainder.
```

In practice it is usual to double the first figure of the root; then to add the second figure to the first divisor, the third figure to the second divisor, &c. Of this method, the addition of 8 the second figure of the root to 48 the first divisor is an instance. The sum, 56, is equal to double the number expressed by the first and second figures of the root, or to 2×28.

The number obtained by dividing the tens of any partial dividend by the double of the root already found may be greater than the next figure of the root. This, as in the last example, leads to the subtraction of a greater number from a less. In such cases a lower figure must be taken.

When any remainder is equal to or greater than twice the root obtained plus 1, the root is less than it ought to be by, at least, unity.

If the double of the root, already obtained, is not contained in the tens of any partial dividend, the units of the corresponding order in the root are deficient; and a zero must be written to occupy their place and preserve the proper relative values of the preceding figures of the root.

In extraction of the square root, the calculation is carried from the left to the right of the power, for reasons similar to those already given (Art. 92) in the case of division.

262. Rule for the extraction of the square root of any whole number.

Separate the number into periods of two figures each, beginning with the simple units; the number of periods is equal to the number of figures of the root.

Find the greatest square number contained in the highest period or period on the left, and subtract the square number from the highest period. Take the root of the square number (which is expressed by a single figure) for the first figure of the root, and write it as a quotient figure in division.

To the remainder annex the second period of the number; divide the tens of the result by double the first figure of the root, and annex the quotient (which is the second figure of the root) to the first figure, and also to the divisor. Multiply the new divisor thus formed by the second figure of the root, and subtract the product from the number composed of the first remainder and second period. To this, the second remainder, annex the third period of the number; divide the tens of this result by the double of the root now found; annex the quotient (which is the third figure of the root) to the part of the root already obtained, and also to the divisor. Multiply the divisor, thus formed, by the third figure of the root; and subtract the product from the number composed of the second remainder and third period.

Proceed in this manner till all the periods of the given number are brought down.

If the last remainder is zero, the given number is a perfect square; and the result obtained, its exact square root; if any number less than twice the root plus 1, the number is not a perfect square; and the result is the integral part of its square root.

If the remainder is equal to or greater than $2a+1$ (a representing the root found), the result is inaccurate.

When there is no remainder, a verification of the process is obtained by multiplying the square root by itself; and when there is a remainder, it is obtained by multiplying the integral part of the root by itself, and adding the last remainder to the product. The result, in both cases, ought to be equal to the given power or number whose square root has been extracted.

262′. The square root of a whole number may also be found in the following manner.

Separate the number or power into periods, and find the first figure of the root and the first remainder as in Art. 262; to the first remainder annex the first figure of the second period of the power; divide the dividend thus formed by twice the first figure of the root; annex the quotient to the first figure as the second figure of the root; form the square of the number expressed by these two figures, and subtract it from the first and second periods of the power. But if the square of the first and second figures is greater than the number expressed by the first and second periods of the power, diminish the second figure of the root by one or more units, till a square which can be subtracted from the first and second periods is obtained.

Having subtracted the square of the first and second figures of the root from the number expressed by the first and second periods of the power, annex to the second remainder the first figure of the third period; divide the number thus formed by twice the number composed of the first and second figures of the root; annex the quotient to the root already obtained; subtract the square of the result from the number composed of the first, second, and third periods of the given power; and in this manner proceed till all the periods of the given number or power are exhausted. The result is the square root required.

Example of extraction of the square root of a whole number by the method of Article 262′.

Extract the square root of 123904.

$$\begin{array}{rl}
 & 12,39,04(352 \\
3^2= & \underline{9} \\
3\times 2=\ 6) & 33(5 \\
 & 1239 \\
35^2= & \underline{1225} \\
35\times 2=70) & 140(2 \\
 & 123904 \\
352^2= & \underline{123904} \\
 & 0
\end{array}$$

263. Exercises in extraction of the square root.

Extract the square roots of the following numbers:

1st. 729 ? .... *Ans*. 27.
2d. 2025 ? .... *Ans*. 45.
3d. 133225 ? .... *Ans*. 365.
4th. 5499025 ? .... *Ans*. 2345.
5th. 173056 ? .... *Ans*. 416.
6th. 29506624 ? .... *Ans*. 5432.
7th. 17698849 ? .... *Ans*. 4207.
8th. 698485 ? .... *Ans*. 835+remainder=1260.
9th. 6285049 ? .... *Ans*. 2507.
10th. 144024001 ? .... *Ans*. 12001.
11th. 210358000 ? .... *Ans*. 14503+remainder=20991.
12th. 145037237 ? .... *Ans*. 12043+remainder=3388.

264. There are several tests of easy application, by means of which it is possible to ascertain whether a number is a perfect square.

1st. Every even number being expressible by $2n$, and the square of $2n$ being $4n^2$, the square of every even number is divisible by 4. Whence every even number which is not divisible by 4 is not a perfect square. In like manner the square of any odd number, $2n+1$, being $4n^2+4n+1$, a number which, diminished by unity, is divisible by 4, it follows that every odd number which, being diminished by unity, cannot be divided by 4, is not a square number.

2d. Every number which involves a prime factor a, and which cannot be divided by a^2, is not a square number.

For if the square root of a number which involves the prime factor a is a whole number, it must be of the form an, of which the square a^2n^2 is divisible by a^2 (Art. 104).

Hence a number divisible by 3 or by 5 must be divisible by $3^2=9$, or by $5^2=25$, in order that it may be a square number.

3d. A number whose last figure is 2, 3, 7 or 8, cannot be a perfect square.

For the simple units of the square of a number consisting of several figures arise from the square of the units of the root. But the square of none of the nine digits ends in 2, 3, 7 or 8 (Art. 257).

4th. Every number whose last figure is 5 cannot be a perfect square, unless the figure immediately preceding the last is 2.

For the last figure of the number can only arise from the square of the units of the root; and the only figure whose square ends in 5 is 5. Now the double product of the tens of the root by 5 gives hundreds (for $2\times 5\times 10=100$); the significant figures of the double product of the preceding figure by 5 cannot therefore affect the tens of the last period;

consequently, if the tens are expressed by some other number than 2, the last period is not $25=5^2$, and the number is not a perfect square.

5th. Every number terminated by an odd number of zeros cannot be a perfect square.

For if the square root of such a number is exact, it must be a whole number ending in one or more zeros. The square of this square root must contain twice as many zeros as are contained in the root; and consequently, an even number, which is contrary to the supposition.

265. Manner of obtaining, approximately, the square root of a number which is not a perfect square.

It is proved (Art. 258), that a whole number which is not the square of another whole number cannot have an exact fractional number for its square root.

The fraction which completes the value of the square root can, however, be obtained, by approximation, to any required degree of accuracy.

It follows from the definitions (Art. 253), that $\frac{a}{b}$ being a fraction, its square is $\frac{a^2}{b^2}$; and conversely that the square root of $\frac{a^2}{b^2}$ is $\frac{a}{b}$.

The square root of a fraction is therefore obtained by extracting the square roots of the terms of the fraction, and dividing the square root of the numerator by that of the denominator.

Suppose next that a whole number, a, which is not a perfect square, is given, and that it is required to find its square root to within the $\frac{1}{n}$th part of unity.

$$a=\frac{a}{1}=\frac{a}{1}\times\frac{n^2}{n^2}=\frac{an^2}{n^2}\ \text{(Art. 149).}$$

Let r be the integral part of the square root of an^2.

$$\therefore\ an^2>r^2 \text{ and } an^2<(r+1)^2.$$

$$\therefore\ \frac{an^2}{n^2}>\frac{r^2}{n^2} \text{ and } \frac{an^2}{n^2}<\frac{(r+1)^2}{n^2}.$$

$$\therefore\ \sqrt{\frac{an^2}{n^2}},\ \text{i.e. } \sqrt{a}>\sqrt{\frac{r^2}{n^2}}=\frac{r}{n}.$$

$$\text{And } \sqrt{a}<\sqrt{\frac{(r+1)^2}{n^2}}=\frac{r+1}{n}.$$

that is, the square root of a is comprehended between $\frac{r}{n}$ and $\frac{r+1}{n}$; or $\frac{r}{n}$ expresses the square root of a to within the $\frac{1}{n}$th part of unity.

Whence the following method of approximation.

Multiply the given number (a) by the square of the denominator (n^2) of the fraction which indicates the degree of approximation required; extract the integral part of the square root of this product, and divide this part by the denominator n.

Example. Find the square root of 223 to within the $\frac{1}{40}$th part of unity.

$$223=\frac{223\times40^2}{40^2}=\frac{356800}{1600}.$$

$$\sqrt{356800}=597+ \qquad \sqrt{1600}=40.$$

$$\therefore\ \sqrt{223}=\frac{597}{40}=14\frac{37}{40} \text{ or } 14\frac{38}{40}.$$

each of which values is correct to within less than the $\frac{1}{40}$th part of unity.

This method of approximation consists in making the given number to fall between the squares of two fractional expressions whose common denominator is that of the fraction which determines the degree of approximation, and whose numerators differ from each other by unity.

266. When the fraction $\frac{1}{n}$ is $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, . . . the approximate value is expressed in decimals. To obtain the square root of a whole number which is not a perfect square to within the $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$ part of unity, it is therefore necessary to multiply the given number by $(10)^2$, $(100)^2$, $(1000)^2$, ... or to annex to the number 2, 4, 6, ... zeros, to extract the square root of the product to the nearest unit, and to divide this root by 10, 100, 1000, ...

Whence, to obtain a determinate number of decimal figures in the square root of a number which is not a perfect square,

Rule. Annex to the given number two zeros for each decimal figure required in the root, extract the square root of this result to the nearest unit, and point off from the square root the required number of decimal figures.

Example. Find the square root of 15 to within the $\frac{1}{1000}$th part of unity.

$$(1000)^2=1000000 \text{ and } 15=\frac{15000000}{1000000}.$$

$$\sqrt{15000000}=3872 \text{ and } \sqrt{1000000}=1000.$$

$$\therefore \sqrt{15}=\frac{\sqrt{15000000}}{\sqrt{1000000}}=\frac{3872}{1000}=3{\cdot}872.$$

267. Extraction of the square root of a vulgar fraction.

Let n denote the numerator and d the denominator of a vulgar fraction, neither being square numbers; then $\frac{n}{d}$ is the expression of the fraction. Multiplying both terms by d, $\frac{n}{d}=\frac{nd}{d^2}$. Let r be the integral part of the square root of nd; then nd is comprehended between r^2 and $(r+1)^2$; consequently $\frac{nd}{d^2}$ is comprehended between $\frac{r^2}{d^2}$ and $\frac{(r+1)^2}{d^2}$, and the square root of $\frac{n}{d}$ between $\frac{r}{d}$ and $\frac{r+1}{d}$.

Whence $\frac{r}{d}$ expresses the square root of $\frac{n}{d}$ to within the $\frac{1}{d}$ part of unity.

Wherefore, to obtain the square root of a vulgar fraction,

Rule. Render the denominator a perfect square by multiplying the terms of the fraction by the denominator, extract the square root of the reduced numerator to the nearest unit, and divide the result by the denominator.

Example. Extract the square root of $\frac{6}{11}$.

$$\frac{6}{11}=\frac{6\times 11}{11\times 11}=\frac{66}{(11)^2}.$$

Now $\sqrt{66}=8+$a fractional part of unity; and $\sqrt{11^2}=11$;

$$\therefore \sqrt{\frac{6}{11}}=\frac{\sqrt{66}}{\sqrt{11^2}}=\frac{8}{11};$$

a value correct to within less than the $\frac{1}{11}$th part of unity.

If a more exact degree of approximation is required, it becomes necessary to recur to the fraction $\frac{66}{11^2}$, and to find the square root of the numerator more exactly. Suppose that it is required to find the square root of 66 to within the $\frac{1}{100}$th part of unity. It is found (Art. 266) that,

$$\sqrt{\frac{660000}{100^2}}=\frac{812}{100}=8{\cdot}12;$$

$$\therefore \sqrt{\frac{6}{11}}=\frac{8{\cdot}12}{11}=\frac{812}{1100};$$

a value correct to within the $\frac{1}{1100}$th part of unity.

For it is evident that $\frac{66}{11^2}>\frac{8{\cdot}12^2}{11^2}$ and $\frac{66}{11^2}<\frac{8{\cdot}13^2}{11^2}$; therefore the square root of $\frac{66}{11^2}$ differs from $\frac{8{\cdot}12}{11}$ by a quantity less than the $\frac{1}{11}$th part of $\frac{1}{100}$ or less than the $\frac{1}{1100}$th part of unity.

268. If the denominator of a fraction whose square root is required involves one perfect square factor it is not necessary to multiply the terms of the fraction by the whole denominator, but only by the factor which is not a perfect square.

Let it be required to find the square root of $\frac{23}{48}$. The terms of this fraction may be multiplied by 48, by which means the denominator is rendered a perfect square. But since $48=16\times 3=4^2\times 3$, if the terms of the fraction $\frac{23}{4^2\times 3}$ are multiplied by 3, the result is $\frac{23\times 3}{4^2\times 3^2}=\frac{69}{12^2}$, and the denominator is rendered a perfect square.

The square root of 69 to within the $\frac{1}{1000}$th part of unity is 8·306; therefore $\frac{8{\cdot}306}{12}$ or $\frac{8306}{12000}$ is the square root required to within the $\frac{1}{12000}$th part of unity.

Therefore, to find the square root of a vulgar fraction whose denominator involves a perfect square factor,

Rule. Multiply the terms of the fraction by the factor which is not a perfect square, and extract the square roots of the numerator and denominator as in Art. 262.

269. Extraction of the square root of a decimal fraction.

The shortest process for the extraction of the square root of a decimal fraction is deducible from the last article.

Suppose it required to extract the square root of the decimal fraction 3·425.

$$3{\cdot}425=\frac{3425}{1000}$$

Now the denominator of this expression is not a square number, but it is equal to 100×10 or to $10^2\times 10$. To render the denominator a square number it is sufficient to multiply the terms of the fraction $\frac{3425}{1000}$ by 10.

The result is $\frac{34250}{10000}$ or $\frac{34250}{100^2}$.

Now $\sqrt{34250}=185$ plus a fraction less than 1, and $\sqrt{100^2}=100$;

$\therefore \sqrt{3{\cdot}425}=\frac{\sqrt{3425}}{\sqrt{1000}}=\frac{185}{100}$, $=1.85$; a result true to within the $\frac{1}{100}$th part of unity.

If a more exact degree of approximation is required, it is necessary to annex to 34250 one period of two zeros for each additional decimal figure required in the result.

270. To extract the square root of a decimal fraction.

Rule. First, render the number of decimal figures even and equal to twice the number of decimal figures required in the root, (which is accomplished by annexing the proper number of zeros to the given decimal fraction).

Second, make abstraction of the decimal point in the new number, and extract its square root to the nearest unit.

Third, point off from the right of the root, the required number of decimal figures.

This rule might be inferred from the process for the multiplication of decimal fractions; for the square of a decimal fraction, or the product of the fraction by itself, must contain twice as many decimal figures as are contained in the square root.

When it is required to express decimally the square root of a vulgar fraction;

Rule. Reduce the vulgar fraction to a decimal expressed by twice as many figures as are required in the root; extract the square root of this decimal; the result is the square root of the vulgar fraction expressed decimally.

270. Exercises in extraction of the square root of fractional expressions, and of numbers which are not perfect squares.

1st. Square root of 3 ?……………………………*Ans.* 1·732+.
2d. Square root of 6 ?……………………………*Ans.* 2·449489+.
3d. Square root of 12 ?……………………………*Ans.* 3·464101+.
4th. Square root of $\frac{9}{16}$?……………………………*Ans.* $\frac{3}{4}$.
5th. Square root of $\frac{12}{147}$?……………………………*Ans.* $\frac{2}{7}$.
6th. Square root of $9\frac{43}{49}$?……………………………*Ans.* $3\frac{1}{7}$.
7th. Square root of $17\frac{3}{8}$?……………………………*Ans.* 4·16833.
8th. Square root of $87\frac{1}{9}$?……………………………*Ans.* $9\frac{1}{3}$.
9th. Square root of 25·3009 ?……………………………*Ans*, 5·03.
10th. Square root of $\frac{9216}{12544}$?……………………………*Ans.* $\frac{6}{7}$.
11th. Square root of 5·875 ?……………………………*Ans.* 2·423+.
12th. Square root of ·001225 ?……………………………*Ans.* 0·035.
13th. Square root of $6\frac{2}{5}$?……………………………*Ans.* 2·529822.
14th. Square root of 9 ?……………………………*Ans.* 3.
15th. Square root of ·9 ?……………………………*Ans.* 0·948683+.
16th. Square root of ·09 ?……………………………*Ans.* 0·3.
17th. Square root of ·64 ?……………………………*Ans.* 0·8.
18th. Square root of 6·399 ?……………………………*Ans.* 2·52962.
19th. Square root of ·064 ?……………………………*Ans.* ·2529822.
20th. Square root of .000003418801 ?……………………………*Ans.* ·001849.

OF THE CUBE AND CUBE ROOT OF NUMBERS.

271. If one factor is made to enter three times into a product, the product is termed the third power or cube of that factor, and the factor is called the cube root of the product.

The cube of a number is indicated thus 5^3.

$5^3=125$ expresses that the cube of 5, or $5\times5\times5$, is equal to 125.

The cube root of a number is indicated by the symbol $\sqrt[3]{}$. Thus, $\sqrt[3]{125}=5$, expresses that the cube root of 125 is 5.

The formation of the cube of a whole number, or a fractional expression, depends on the rules for the multiplication of numbers whole or fractional.

The cubes of the numbers from 1 to 10 inclusive are

1, 8, 27, 64, 125, 216, 343, 512, 729, 1000; and
1, 2, 3, 4, 5, 6, 7, 8, 9, 10,

are the cube roots of the corresponding numbers in the upper line.

272. From a comparison of these cube numbers with their cube roots, it appears that of numbers expressed by one, two, and three figures, nine only are perfect cubes or cube numbers; each of the 990 others having for cube root a whole number plus a fraction, which cannot be expressed exactly by means of unity.

For let it be supposed that a whole number, n, has for exact cube root an irreducible fractional expression, $\frac{a}{b}$; then, $\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}=\frac{a^3}{b^3}=n$.

But a, b, being prime to each other, a^3, b^3, are also prime to each other (Art. 113). Whence $\frac{a^3}{b^3}$ cannot be equal to a whole number, nor $\frac{a}{b}$ to $\sqrt[3]{n}$.

The cube roots of whole numbers which are not the exact cubes of other whole numbers cannot, therefore, be obtained exactly, for which reason they are termed incommensurable, irrational, or surd numbers.

If a denote the tens of a number, and b the units, the number is expressed by $a+b$.

Now by Art. 76 $(a+b)(a+b)$ or $(a+b)^2=a^2+2ab+b2$,

$$\begin{array}{ll} \text{multiplying by} \quad - \quad - \quad - & \underline{a+b} \\ \text{and adding the partial products} \quad - & \left\{\begin{array}{l} a^3+2a^2b+ab^2 \\ \qquad a^2b+2ab^2+b^3 \end{array}\right. \end{array}$$

it is found that $(a+b)(a+b)(a+b)$ or $(a+b)^3=a^3+3a^2b+3ab^2+b^3$.

Whence the cube of a number consisting of tens and units is composed of the cube of the tens of the number, plus three times the square of the tens into the units, plus three times the tens into the square of the units, plus the cube of the units,

$$\begin{array}{ll} \text{making} & b=1,\ a^3+3a^2b+3ab^2+b^3 \text{ becomes} \\ & a^3+3a^2+3a+1=(a+1)^3 \\ \text{subtracting} & a^3 \\ \text{the remainder is} & 3a^2+3a+1 \end{array}$$

Whence, if a and $a+1$ represent two consecutive whole numbers, the difference between the cube of $a+1$ and a is equal to three times the square of the less number, plus three times the less number, plus one,

The difference between 91^3 and 90^3 is

$$3\times 90^2+3\times 90+1=24300+270+1=24571.$$

From this example it may be inferred that there is a great difference between two consecutive cube numbers when the roots of these cubes are expressed by large numbers.

273. Process for the extraction of the cube root of a whole number.

If the number contains not more than three figures its cube root is obtained by reference to Art. 271. Thus, the cube root of 27 is 3; the cube root of 235 is 6 plus a fraction; for 235 falls between $216=6^3$ and $343=7^3$; and the cube root of 950 is 9 plus a fraction, for 950 falls between $729=9^3$ and $1000=10^3$.

Let the number whose cube root is required contain more than three figures, as 103823.

103823 falls between $1000=10^3$ and $1000000=100^3$; its root therefore contains two figures, tens and units. Since $10^3=1000$, the three last figures of the number can contain no part of the significant figures of the cube of the tens; it is therefore in the part 103 that these significant figures are found. Employing a comma to separate the last three figures, and referring to Article 271 for the root of the cube number equal to or next less than 103, it is found that 103 falls between $64=4^3$ and $125=5^3$. Since also 103823 falls between $64000=40^3$ and $125000=50^3$, it is evident that the required cube root is composed of four tens and of a certain number of units less than ten.

Subtracting 64, the cube of the tens of the root, from 103, and to the remainder 39 annexing the second period of the given number, the result, which is 39823, contains $3\times40^2\times b+3\times40\times b^2+b^3$.

Next, to find the numerical value of b in this example. The square of the tens giving hundreds, the significant figures of the triple square of the tens by the units must be found in 398, the part on the left of the last figures 23.

$$\begin{array}{l} \text{Now } 3\times4^2=3\times16=48, \\ \text{and } 398\div48=8+\text{a remainder;} \\ \text{or } 39823\div4800=8+\text{a remainder.} \end{array}$$

The quotient 8 is the figure expressing the units (b) of the root, or a figure expressing a larger number.

39823, besides three times the square of the tens of the root into the units, contains also the product of three times the tens by the square of the units together with the cube of the units. These partial products may be formed and subtracted successively from 39823, and the remainders; or their sum may be subtracted at once.

It is, however, a less complicated process to form the cube of 48, and subtract it from the number 103823.

$$\begin{array}{l} 48^3=110592, \text{ which is greater than } 103823, \\ \therefore 48 \text{ is greater than the cube root of } 103823. \\ \text{Trying the lower number } 47,\ 47^3=103823. \\ \therefore 47=\sqrt[3]{103823}. \end{array}$$

Form of calculation.

$$\begin{array}{rl} & 103{,}823(47 \\ & \underline{64} \\ 3\times4^2=48) & 398(7 \\ & \\ & 103823 \\ 47^3= & \underline{103823} \end{array} \qquad \begin{array}{r} 47 \\ \underline{47} \\ 329 \\ \underline{188} \\ 2209 \\ \underline{47} \\ 15463 \\ \underline{8836} \\ \underline{103823} \end{array}$$

2d. Example. Extract the cube root of 47954.

The number 47954 is comprehended between 1000 and 1000000, therefore its cube root falls between 10 and 100; that is, it contains tens and units.

The cube of the tens is contained in the 47 thousands; and since 47 falls between $27=3^3$ and $64=4^3$; and 47954 also falls between 27000 and 64000, or between 30^3 and 40^3, it follows that $\sqrt[3]{3^3}$ or 3 expresses the tens of the root.

Subtracting 27, the cube of 3, from 47, the remainder is 20. To this remainder annexing 9, the first figure of the second period, the number 209 hundreds is composed of the triple square of the tens of the root by the units plus the hundreds arising from the other partial products.

The triple square of the tens$=3\times3^2=27$; dividing 209 by 27, the quotient is 7; 7 is the figure which expresses the units of the root, or it is a greater figure.

Assuming that 7 expresses the units of the root, 37 is the root, and $37^3=50653$.

$$\begin{array}{l} \text{But } 50653 \text{ is greater than } 47954; \\ \text{therefore } 37 \text{ is greater than } \sqrt[3]{47954}. \end{array}$$

Assuming the next lower number, 36, and forming its cube, the result is 46656.

Subtracting 46656 from 47954, the remainder is 1298. Whence 47954 is not a perfect cube; but its cube root to the nearest unit is 36.

In effect the difference between the given number and 36^3 is 1298, and the difference between 37^3 and 36^3 is $3\times36^2+3\times36+1=3888+108+1=3997$, a difference much greater than that between 36^3 and the given number.

Form of calculation:

47,954(36	36
$3^3=27$	36
$3\times3^2=27$)209(6	216
	108
47954	1296
$36^3=46656$	36
1298	7776
	3888
	46656

3d Example. Extract the cube root of 43725658.

How many soever the number of figures by which the required root is expressed, it may be regarded as composed of units and tens only, the tens being expressed by more than one figure.

Now, the cube of the tens gives at least a thousand; therefore the significant figures of the cube of the tens of the root are of necessity found in that part of the number which is on the left of the last three figures, 658.

Next, if the root of the greatest cube contained in 43725, considered as expressing simple units, is extracted, the result is the whole number of tens contained in the required root.

For let a be the root of the greatest cube contained in 43725; it follows that the root required has at least a number, a, of tens, since $a^3\times1000$ can in this case be subtracted from 43725000, and à fortiori from 43725658.

Besides, the root cannot contain $a+1$ tens, for $(a+1)^3$ being greater than 43725, $(a+1)^3\times1000$ must exceed 43725000 by at least 1000; consequently $(a+1)^3\times1000$ is greater than 43725658. Therefore, in conclusion, the cube root required is composed of a tens plus a certain number of units less than 10.

The question is consequently reduced to the extraction of the cube root of 43725. This number consisting of more than three figures, its cube root contains more than one figure; that is to say, it contains tens and units.

To obtain the tens it is necessary to point off the three figures 725, and to extract the root of the greatest cube number contained in 43.

(*Note*. It is evident that if the period on the left, in this instance 43, contained more than 3 figures, it would be necessary to point off three figures again from its right.)

The greatest cube number contained in 43 is 27, the cube root of which is 3. This figure, then, expresses the tens of the cube root of 43725, or the hundreds of the whole root.

Subtracting the cube of 3 from 43, the remainder is $43-27=16$. Annexing the first figure of the 2d period to this remainder, the result is 167.

Dividing 167 by three times the square of 3, or 27, the quotient is 6. 6, therefore, is the figure which expresses the units of the cube root of 43725, or a greater figure than that of the root.

Making trial (by forming 36^3) it is found that the figure 6 is too great. Assuming the next lower figure, 5, and forming the cube of 35, which is 42875, it is found that 35 expresses the total number of tens of the required root, and that the second remainder is $43725-42875=850$.

To obtain the units of the root, the first figure of the 3d period of the given number is annexed to the second remainder, and the result, 8506, is divided by three times the square of the tens of the root, that is, by $3\times35^2=3675$:

$$8506\div3675=2$$

Annexing 2 to 35, 352 is assumed to express the cube root of the given number.

$$\text{Since } 352^3=43614208,$$
$$43725658-43614208=111450;$$

and $111450<3\times352^2+3\times352+1$, or 372769, it follows that 352 is the cube root of 43725658 to the nearest unit.

Form of calculation:

$$\begin{array}{ll} & 43,725,658(352\\ 3^3= & \underline{27}\\ 3\times3^2=27) & \overline{167}(5\\ & 43725\\ 35^3= & \underline{42875}\\ 3\times35^2=3675 & \overline{\ \ 8506}(2 \end{array}$$

$$\begin{array}{r} 35\\ \underline{35}\\ 175\\ \underline{105}\\ 1225\\ \underline{35}\\ 6125\\ \underline{3675}\\ 42875 \end{array} \qquad \begin{array}{r} 352\\ \underline{352}\\ 704\\ 1760\\ \underline{1056}\\ 123904\\ \underline{352}\\ 247808\\ 619520\\ \underline{371712}\\ 43614208 \end{array}$$

$$\begin{array}{rl} & 43725658\\ 352^3= & \underline{43614208}\\ & 111450 \text{ remainder.} \end{array}$$

If any divisor is not contained in the corresponding dividend, zero is written as the proper figure of the root; three figures of the given number are brought down to form a new dividend, and the process of calculation is continued in the same manner as if the zero had been a significant figure.

The process cannot be commenced by finding the units of the cube root of a given number; for the cube of the units may give tens and hundreds, which cannot be distinguished from tens and hundreds arising from $3a^2b$ and $3ab^2$.

274. To extract the cube root of a whole number:

Rule. Separate the number into periods of three figures each (taking the figures from right to left) till a last period of one, two, or three figures is obtained. The number of periods is equal to the number of figures of the root.

Extract the cube root of the greatest cube number contained in the first period of the given number (the period, namely, on the left), and subtract this cube from the first period; to the remainder annex the first figure of the second period; divide the number thus formed by three times the square of the figure of the root already found; annex the quotient to this first figure as the second figure of the root; form the cube of the number expressed by the first and second figures of the root, and subtract it from the first and second periods of the given number. But if the cube of the first and second figures is greater than the number expressed by the first and second periods of the given number, diminish the second figure of the root by one or more units, till a cube which can be subtracted from the first and second periods of the given number is obtained.

The subtraction made, annex to the 2d remainder the 1st figure of the 3d period; divide the number thus formed by three times the square of the number expressed by the 1st and 2d figures of the root; the quotient figure (which, if not too great, is the third figure of the root) ought to be such that, annexing it to the number expressed by the 1st and 2d figures of the root, and forming the cube of the number thus obtained, this cube shall be capable of subtraction from the number composed of the 1st, 2d, and 3d periods of the given number.

This series of operations is continued till all the periods of the given number are brought down.

When, in the course of the operation, it is suspected that one of the quotient figures is too great, and it is in consequence diminished by one or more units, if, after forming the cube of the assumed root and subtracting it from the corresponding periods of the given number, the remainder is so considerable that it appears doubtful whether the last figure of the root is not too small, the following test can be employed: if the remainder is equal to or greater than three times the square of the assumed root plus three times the root plus unity, the root is too small; but if less, the root is accurate to within less than unity of the order of its last figure.

Exercises in extraction of the cube root:

1st. The cube root of 103823?............................*Ans.* 47.
2d. - - 21035?............................*Ans.* 27+.
3d. - - 48228544?........................*Ans.* 364.
4th. - - 12812904?........................*Ans.* 234.
5th. - - 485613?..........................*Ans.* 78+.
6th. - - 5159780352?....................*Ans.* 1728.
7th. - - 343147021001?.................*Ans.* 7011.
8th. - - 673373097125?.................*Ans.* 8765.

275. The following are tests by means of which it may be ascertained whether a number is a perfect cube:

1st. Every even number being expressible by $2n$, and its cube by $8n^3$, the cube of every even number is divisible by 8. Every even number, therefore, which is not divisible by 8 is not a perfect cube.

2d. Every number which involves a prime factor, a, and which cannot be divided by a^3, is not a perfect cube; for, if the cube root of a number which involves the prime factor a is a whole number, it must be of the form an, of which the cube a^3n^3 is divisible by a^3. Hence a number divisible by 7 or by 11 must be divisible also by 343 or by 1331, in order that it may be a cube number.

3d. Every number terminated by a number of zeros not expressible by $3n$ (n being any number of the series 1, 2, 3 . . .) cannot be a cube number; for, if the cube root of such a number is exact, it must be a whole number terminated by one or more zeros. The cube of this cube root must contain three times as many zeros as are contained in the root, and consequently a number expressed by $3n$, which is contrary to the supposition.

276. Extraction of the cube root by approximation.

When the given number is not the cube of another whole number, the preceding operation gives only the integral part of the root. As to the fraction which ought to complete the root, it has been already proved (Art. 272) that this cannot be obtained exactly, but another fraction may be found which shall differ from it by as small a quantity as is desired.

Let it be required to extract the cube root of a (a representing any whole number which is not a perfect cube) to within the $\frac{1}{n}$th part of unity.

The number a may be expressed as $\frac{a \times n^3}{n^3}$.

Then, if r is the root of the greatest cube number contained in $a \times n^3$ that is, the root of an^3 to the nearest unit, the number $\frac{an^3}{n^3}$ or a is comprehended between $\frac{r^3}{n^3}$ and $\frac{(r+1)^3}{n^3}$. Wherefore also the cube root of a is comprehended between the cube roots of these two numbers, or between $\frac{r}{n}$ and $\frac{r+1}{n}$. Consequently $\frac{r}{n}$ is the expression of the cube root of a to within the $\frac{1}{n}$th part of unity.

Whence, to extract the cube root of a number to within the $\frac{1}{n}$th part of unity (n denoting any whole number):

Rule. Multiply the given number by the cube of the denominator n, extract the cube root of the product to the nearest unit, and divide the result by n.

Example. Extract the cube root of 15 to within the $\frac{1}{12}$th part of unity:

$$12^3 = 12 \times 12 \times 12 = 1728,$$
$$15 \times 12^3 = 15 \times 1728 = 25920,$$
$$\sqrt[3]{25920} = 29 + \text{a fraction less than unity.}$$

Whence the required root is $\frac{29}{12} = 2\frac{5}{12}$.

277. Approximation in decimals is a particular case of the method of Article 276.

Suppose that it is required to extract the cube root of 25 to within a smaller part of unity than that which is expressed by the decimal ·001, that is, to within the $\frac{1}{1000}$th part of unity; in this instance it is necessary to multiply 25 by 1000^3 (or to annex nine zeros to 25), to extract the cube root of the product, and to divide the result by 1000.

Now $\sqrt{25000000000} = 2924 +$ a fraction less than 1.

Whence $\frac{2924}{1000}$, or 2·924, is the root required.

To obtain the approximate value in decimals of a whole number which is not a perfect cube:

Rule. Annex to the given number three times as many zeros as the root is required to contain decimal figures; extract the cube root of this number to the nearest unit, and point off the required number of decimal figures from the result.

278. Extraction of the cube root of a vulgar fraction:

If $\frac{n}{d}$ represents a fractional expression whose cube root is required, multiply the terms of the fraction by the square of the denominator; then $\frac{n}{d} = \frac{nd^2}{d^3}$.

Representing by r the integral part of the cube root of nd^2, it follows, from a process of reasoning analogous to that of Article 276, that $\frac{r}{d}$ is the cube of $\frac{n}{d}$ to within less than the $\frac{1}{d}$ part of 1.

If a more exact degree of approximation is required, it becomes necessary to approximate more nearly to the value of $\sqrt[3]{nd^2}$, and to divide the result by d.

When the denominator, without being already a perfect cube, contains factors of which some are perfect cubes and others perfect squares, the reduction to be made upon the fraction is more simple.

Let it be required, as an illustration, to extract the cube root of $\frac{113}{360}$.

The denominator 360 can be decomposed into the prime factors 2^3, 3^2, and 5; whence, if the terms of the given fraction are multiplied by $3\times5^2=75$, it becomes $\frac{113\times75}{2^3\times3^3\times5^3}=\frac{8475}{30^3}$.

Since $\sqrt[3]{8475}=20+$a fraction less than 1, and $\sqrt[3]{30^3}=30$,

$$\therefore \sqrt[3]{\frac{113}{360}}=\sqrt[3]{\frac{8475}{30^3}}=\frac{20}{30}=\frac{2}{3}, \text{ the root required.}$$

If the denominator is a perfect cube, but the numerator not, the cube root of the number, approximated to any required degree of accuracy, is divided by the cube root of the denominator for the cube root of a given fractional expression.

279. Extraction of the cube root of a decimal fraction:

Let it be required to find the cube root of 3·1415.

The denominator of 3·1415, which is 10000, is not a perfect cube, being equal to 1000×10, or $10^3\times10$; it is rendered a perfect cube by the multiplication of the terms of the fraction by 10^2 or 100; this is accomplished by annexing two zeros to the terms of the fraction, which thus becomes $\frac{3141500}{1000000}$ or $\frac{3141500}{100^3}$.

Now $\sqrt[3]{3141500}=146+$a fraction less than 1, and $\sqrt[3]{100^3}=100$,

$$\therefore \sqrt[3]{3{\cdot}1415}=\frac{146}{100}=1{\cdot}46, \text{ a value true to within the } \frac{1}{100}\text{th part of unity.}$$

If a more exact degree of approximation is required, it is necessary to annex to the number three zeros for each additional decimal figure required in the root.

To extract the cube root of a decimal fraction:

Rule. Render the number of decimal figures a multiple of 3, and equal to three times the number of decimal figures required in the root; make abstraction of the decimal point in this number, and extract its cube root to the nearest unit; lastly, point off from the right of the root the required number of decimal figures.

To obtain in decimals the cube root of a vulgar fraction, reduce the vulgar fraction to a decimal, and extract the cube root of the latter.

280. Exercises in extraction of the cube root of fractional expressions, and of numbers which are not perfect cubes:

1st. The cube root of 2 ?............................*Ans.* 1·25992+.
2d. - - 42·875 ?............................*Ans.* 3·5.
3d. - - $28\frac{3}{4}$?............................*Ans.* 3·06.
4th. - - $15\frac{5}{8}$?............................*Ans.* $2\frac{1}{2}$.
5th. - - $\frac{250}{686}$?............................*Ans.* $\frac{5}{7}$.
6th. - - $\frac{4}{7}$?............................*Ans.* ·829.
7th. - - $12\frac{19}{27}$?............................*Ans.* $2\frac{1}{3}$.
8th. - - $31\frac{15}{343}$?............................*Ans.* $3\frac{1}{7}$.
9th. - - $9\frac{1}{8}$?............................*Ans.* 2·092.
10th. - - $405\frac{28}{125}$?............................*Ans.* $7\frac{2}{5}$.
11th. - - $8\frac{5}{7}$?............................*Ans.* 2·057+.
12th. - - 127·263527 ?............................*Ans.* 5·03.
13th. - - 80· ?............................*Ans.* 4·308+.
14th. - - 800· ?............................*Ans.* 9·283+.

15th. The cube root of ·8 ?.... .... *Ans.* ·9283+.
16th. - - ·08 ?.......................... *Ans.* ·4308+.
17th. - - $7\frac{1}{5}$?.......................... *Ans.* 1·93+.
18th. - - 3·00415 ?.......................... *Ans.* 1·4429+.
19th. - - $\frac{14}{25}$?.......................... *Ans.* ·824+.
20th. - - ·0034567 ?.......................... *Ans.* ·1512+.

SECTION IX.

OF DIFFERENT SYSTEMS OF NUMERATION.

281. It has been explained in the Section on Numeration that it is possible to express any finite number by means of ten characters or figures, and of the conventional principle that any figure placed to the left of another shall express units ten times greater than the units expressed by that other figure.

From the number of distinct characters employed, and the principle which fixes the relative values of these characters, this system of numeration has obtained the name of the decimal or denary system.

The decimal is the system used by all civilized nations. The names of numbers, the figures employed to express them, and all the details of arithmetic are so identified with this system, that it may seem difficult to contemplate number otherwise than through its means. Yet innumerable other systems are equally capable of serving the same purposes. The determining cause of the first preference given to the decimal was probably the ready and constant facility afforded by the fingers of both hands to the breaking down of numbers into equal parcels, and the consequent more easy comparison of large numbers with unity or with each other.

Some articles are enumerated by twos or pairs, others by threes or leashes, twelves or dozens, twentys or scores, &c., but in these instances the number of pairs, leashes, dozens, scores, &c. is always expressed by the ordinary names and characters of numbers.

It is, however, possible to express all numbers by means of fewer or of more than ten figures, but there ought to be not fewer than two, and of the number, whatever it is, zero should be one.

282. The number of figures employed is called the base of a system of numeration.

The system in which two figures are employed, or in which the number
two is base, is named binary,
three - - ternary,
four - - quaternary,
five - - quinary,
. . . .
ten - - denary,
. . . .
twelve - - duodenary.
. . . .

In every system of numeration like the denary the convention must be made that any figure placed on the left of another shall express units as many times greater, with regard to the units of that other figure, as there are units in the base, or as there are figures in the system.

Thus, in the binary system each figure obtains a value from two to two times greater, as it is moved one, two, three . . . places towards the left; in the ternary system the value of a figure is from three to three times greater; in the quinary, from five to five times; in the duodenary, from twelve to twelve times greater; and, in general, in a system of which the base is b, each figure acquires a value from b to b times greater.

When a number is written in the system of which the base is b, the extreme figure on the right expresses the units of the first order, the figure immediately to the left the units of the second order, the figure to the left of these two the units of the third order, &c.; and it requires b units of the first order to form one unit of the second order, b units of the second order to form one unit of the third order, &c. &c.

283. With the help of these conventional principles it is possible to express any whole number in figures, whatever be the system adopted.

A base supplied by the fingers of one hand is five. Considering the system with this base, or the quinary system, the figures required are 1, 2, 3, 4, 0, of which 1, 2, 3, 4, express simple units, or units of the first order. Adding unity to four, the result is the number five, or one unit of the second order. Five, in accordance with the preceding principles (Art. 282), is written 10, the zero being annexed to give 1 its proper relative value, namely, that of one unit of the second order.

The numbers six, seven, eight, nine, are written

11, 12, 13, 14.

The number ten, which is equal to two units of the second order, is written 20; eleven, 21; twelve, 22; thirteen, 23; fourteen, 24.

By writing successively the figures 1, 2, 3, 4, in the first and in the second place, all the numbers from 10, = five, to 44, = twenty-four, are expressed.

If to 44 one unit is added, the result is 4 units of the second order + 5 units of the first order; that is, five units of the second order, or one unit of the third order, which is written 100.

By writing again in the first, in the second, and in the third place the different figures of the system, all the numbers between 100, = twenty-five, and 444, = one hundred and twenty-four, inclusive, are provided with expressions.

In like manner may the numbers between 444+1 or 1000, = one hundred and twenty-five, and 4444+1 or 10000, = six hundred and twenty-five, the numbers between 10000 and 44444+1 or 100000, = three thousand one hundred and twenty-five, &c. &c. &c. be written.

This seems sufficient to show that any whole number can be expressed in this system, that is, in a system whose base is less than ten.

284. It is equally possible to express any number by means of a system whose base is greater than ten.

Since to express numbers in the denary system nine significant characters and one zero are required, to express numbers similarly in a system whose base is greater than ten a greater number of significant characters than nine is required.

Suppose the base twelve, characters must be selected to express the numbers ten and eleven. Taking t and e, the initial letters of the names of these numbers, the figures of the duodenary system are,

1,	2,	3,	4,	5,	6,	7,	8,	9,	t,	e.
one,	two,	three,	four,	five,	six,	seven,	eight,	nine,	ten,	eleven.

Twelve $= e+1=10$.

By writing all the figures 1, 2, 3, ... e in the first and in the second place, all the numbers between 10 and $ee+1$ or 100, that is, between twelve and one hundred and forty-four, are supplied with expressions; and similarly between 100 and $eee+1$ or 1000, that is, between one hundred and forty-four and one thousand seven hundred and twenty-eight, &c. &c. &c.

What has been said of the quinary and duodenary systems may suffice to show that any whole number can be expressed by means of figures in systems whose bases are either less or greater than ten, the base of the decimal system.

Whatever the system adopted, the units of different orders are represented respectively by

1, 10, 100, 1000 ..... as in the decimal system.

285. The manner of expressing numbers otherwise than by the decimal system being explained, it becomes a subject of inquiry by what process a number enunciated in ordinary language or written in the decimal system can be changed into an equivalent expression in the system whose base is five, for instance, or twelve, or *b*.

Since *b* units of the first order form one unit of the second order, the given number must contain one unit of the second order of the system *b* for every time it contains *b*. If, therefore, the number is divided by *b*, the quotient expresses the number of units of the second, and the remainder the units of the first order, written in the system of which *b* is the base.

In like manner, since *b* units of the second order in system *b* form one unit of the third order in the same system, if the quotient expressing the number of units of the second order is divided by *b*, the new quotient expresses units of the third order, and the remainder (which must be less than *b*) the units of the second order in system *b*.

The manner of deducing the units of the third, the fourth . . . . ., the *n*th order is manifestly the same. Whence, to express in system *b* any number expressed in common language,

Divide the given number by the base *b*, written in the decimal system; the quotient is composed of units of the 2d order, and the remainder of units of the 1st order in system *b*. Reserve the remainder, and divide the quotient by *b*: the 2d quotient is composed of units of the 3d order, and the remainder of units of the 2d order in system *b*. Write the second remainder on the left of the first, as expressing units of the 2d order, and divide the second quotient by *b*. The third quotient is composed of units of the 4th order, and the remainder of units of the 3d order. This remainder is written on the left of the second remainder, as expressing units of the 3d order, and the 3d quotient is divided by *b*. This series of operations is continued till a quotient less than *b* is obtained. This quotient, which expresses the units of the highest order, is written to the left of all the remainders which have been successively obtained. The result is the given number expressed in system *b*.

Example. Let it be required to express the number 1654 in the quinary and also in the duodenary system.

The base of the quinary, expressed in the denary system, is 5.

5) 1654
5) 330+4, 1st remainder.
5) 66+0, 2d -
5) 13+1, 3d -
2+3, 4th -

Whence 1654, expressed in the quinary system, is 23104.

The base of the duodenary, expressed in the denary system, is 12.

12) 1654
12) 137+10=*t*, duodenary system.
11+5

The first remainder is 10, which, expressed in the duodenary system, is *t*; the second remainder, 5, is written on the left of *t*; and lastly the quotient 11, which in the duodenary system is represented by *e*.

Therefore 1654, expressed in the duodenary system, is *e5t*.

286. The manner of translating numbers from the denary into any other system, *b*, having been explained, let it be required, conversely, to reduce numbers from system *b* to the denary system.

To render the investigation general, let *hgfdc* represent a number in system *b*; *c* expressing the units of the first order, *d* the units of the second order, *f* the units of the third order, &c. &c.

Since d expresses units of the 2d order, it contains b times as many simple units as d, a unit of the 1st order.

Whence d, a unit of the 2d order, contains $d\times b$ units of the 1st order.

Again, f units of the 3d order contain $f\times b$ units of the 2d order, and $f\times b\times b=f\times b^2$ units of the 1st order.

In like manner g units of the 4th order contain $g\times b$ units of the third order, or $g\times b^2$ units of the 2d order, or $g\times b^3$ units of the 1st order;

And h units of the 5th order contain $h\times b^4$ units of the 1st order.

Whence c contains c units of the 1st order.
- d - db -
- f - fb^2 -
- g - gb^3 -
- h - hb^4 -

And the number expressed in simple units is equal to $c+db+fb^2+gb^3+hb^4$.

Assigning to the base b and to the figures c, d, f, g, h, &c. particular values, and making in the denary system the calculations just indicated, a number expressed in any system may be changed into an equivalent number expressed in the denary system.

A practical rule deduced from the expression $c+db+fb^2+gb^3+hb^4$ is:

Form the different powers of the base b, expressed in the decimal system.

Express the figures c, d, f, g, h, &c. also in the decimal system.

Multiply these figures, so expressed, respectively by $1, b, b^2, b^3, b^4, \ldots.$

Add the partial products into one sum; the result is the number expressed in the denary system.

Example. Let it be required to reduce the quinary number 23104, and the duodenary, *e5t*, to the equivalent number in the denary system.

In the quinary system $b=5$, $c=4$, $d=0$, $f=1$, $g=3$, $h=2$.

Also b, b^2, b^3, b^4, are equal to
5, 25, 125, 625.

$$\begin{aligned}
\text{Whence } c\times 1 &= & 4\\
d\times b &= 0\times 5 = & 0\\
f\times b^2 &= 1\times 25 = & 25\\
g\times b^3 &= 3\times 125 = & 375\\
h\times b^4 &= 2\times 625 = & \underline{1250}\\
\text{The sum of these partial products - - } &= & \underline{\underline{1654}}
\end{aligned}$$

The number 23104 in the quinary system is consequently equivalent to 1654 in the denary system.

In the duodenary system,

$$b=12,\ c=10,\ d=5,\ f=11,\ b^2=144.$$

$$\begin{aligned}
c\times 1 &= 10\times 1 = & 10\\
d\times b &= 5\times 12 = & 60\\
f\times b^2 &= 11\times 144 = & \underline{1584}\\
\text{The sum of these partial products is - - } &= & \underline{\underline{1654}}.
\end{aligned}$$

The number *e5t* in the duodenary system is consequently equivalent to 1654 in the denary system.

287. The preceding propositions, viz., to reduce a number from the denary to another system, and to reduce a number from another system to the denary, lead to this more general proposition, to change a number from any system, b, to any other system, c.

To accomplish this change, reduce the given number from the system b to the denary system by Article 286; then from the denary system to system c by Article 285. Thus is obtained the result sought.

Example. Let it be required to change the number 85746 from the nonary to the duodenary system.

1st. Reduction of 85746 from the nonary to the denary system:

$$c=6,\quad d=4,\quad f=7,\quad g=5,\quad h=8$$

$$b=9,\quad b^2=81,\quad b^3=729,\quad b^4=6561$$

$$\begin{aligned} c&=6\times 1 &&= 6\\ db&=4\times 9 &&= 36\\ fb^2&=7\times 81 &&= 567\\ gb^3&=5\times 729 &&= 3645\\ hb^4&=8\times 6561 &&= \underline{52488} \end{aligned}$$

The sum of these partial products $=\underline{56742}$

The number 85746 in the nonary is, consequently, equivalent to the number 56742 in the denary system.

2d. Reduction of 56742 from the denary to the duodenary system:

12)56742
12)4728+6, 1st remainder.
12)394+0, 2d remainder.
12)32+10, 3d remainder, = *t* in duodenary system.
2+8, 4th remainder.

Writing the remainders and last quotient in their proper order, 28*t*06 in the duodenary system is equivalent to 85746 in the nonary system.

288. An obstacle to the easy comprehension of different systems of numeration is caused by the want of corresponding systems of names of numbers, for it is obvious that the nomenclature of the denary system (and no other exists), if employed to name numbers of another system, can only tend to confuse the notions formed of these numbers by its constant reference to the base *ten*, and not to the base of that other system. To be intelligible every system ought to have its own nomenclature.

289. Mankind spoke before they wrote; and the faculty of expressing numbers in words must have preceded that of denoting them by characters. But a notation which is regular and admits of generalisation being formed, the order of procedure may be reversed, and a general nomenclature formed from a general system of notation.

Let b represent the base of a system.

Then the simple units of the system are 1, 2, 3 . . . . . $(b-1)$, of which each must be provided with a name, and denoted by a single character.

The base b (from the name of the number expressed by which, the system obtains its designation,) is always written 10; it is the least number of the second order of units.

The units of the second order are $1\times b$, $2\times b$, $3\times b$, ... $(b-1)\times b$. Each unit of the second order is denoted by the same character as the corresponding unit of the first order; and the b is not written, being indicated by the place which the character occupies on the left of that which expresses the simple units of the number.

The names furnished by the preceding notation for the units of the second order are one b, two b's, three b's, four b's . . . . .

Since $1\times b$ times one unit of the first order make one unit of the second order, and $1\times b$ times one unit of the second order make one unit of the third order, it follows that b times $1\times b$ units of the first order, that is, $1\times b^2$ units of the first order, make one unit of the third order.

The least number of the third order of units is, therefore, b^2, or $1\times b^2$;

And the other units of this order are, $2\times b^2$, $3\times b^2$. . . . . $(b-1)\times b^2$.

Each of these units is denoted (as in the 2d order) by the corresponding unit of the first order; the b^2 being indicated by the place of the character, as second on the left from the character denoting the simple units of the number.

The names given by the notation for the units of the third order are,

One *b*, second power; two *b*'s, second power; three *b*'s, second power . . .; or, omitting the word *power* for the sake of brevity, one *b* second; two *b*'s second; three *b*'s second; . . . . .

In like manner the units of the 4th order are, $1\times b^3$, $2\times b^3$, $3\times b^3$ . . . . $(b-1)\times b^3$; and the names are, one *b* third, two *b*'s third, three *b*'s third . . . .

The units of the 5th order are, $1\times b^4$, $2\times b^4$, $3\times b^4$, ... $(b-1)\times b^4$; and the names are, one *b* fourth, two *b*'s fourth, three *b*'s fourth . . . .

Generally, in the system *b*, if *cde* . . . . . . *pqr* represent the digits of a number, the successive orders of units are, $c\times b^n$, $d\times b^{n-1}$, $e\times b^{n-2}$, ... $p\times b^2$, $q\times b$, r; and the spoken name of the number derived from the notation is, *cb*'s *n*th power, *db*'s $(n-1)$th power, ... *pb*'s second power, *qb*'s, *r*.

If 32421 is the expression of a number written in the quinary system, the spoken name of the number is, three fives fourth, two fives third, four fives second, two fives, one.

9765863 of the denary system expressed in this manner is, nine tens sixth, seven tens fifth, six tens fourth, five tens third, eight tens second, six tens, three.

3 *e* 8 *t* 9 of the duodenary system is, in words, three twelves fourth, eleven twelves third, eight twelves second, ten twelves, nine.

290. The first characters employed to express numbers were probably composed of combinations of perpendicular and horizontal lines or strokes, such as could be easily cut in wood or stone. From this circumstance they have been termed lapidary characters. After the invention of letters these characters were, in some countries, replaced by those letters of the alphabet to which they had the closest resemblance.

Such appears to have been the order of progress in Italy. The Greeks were in possession of the lapidary characters at an early period of their history. Grecian colonists probably communicated these characters to the Italian tribes, and the Romans replaced them by letters of the Latin alphabet about the time of their conquest of Italy.

The Romans communicated their numerals to the nations whom they subdued. Through this means, and, after the fall of the empire of the west, by the employment of Latin as the common language of the learned and of the church of Rome, their system grew into general use throughout Europe, and continued to be employed till the close of the 14th century, when it was supplanted by one more simple and comprehensive, the Arabian.

As the Roman notation was extensively used for so long a time, and is still sometimes employed, especially on monuments, a concise account of it cannot be altogether without utility.

291. The characters employed by the Romans to represent numbers were certain letters of the alphabet, namely, the capitals C, I, L, V, X, which were termed Numerical Letters.

The numbers expressed by single letters were, one by I, five by V, ten by X, fifty by L, a hundred by C. All other numbers were expressed by combinations of these letters.

In these combinations,

1st. By the repetition of a letter its value was repeated; thus, I=1, II=2, III=3, X=10, XX=20, XXX=30, C=100, CC=200, CCC=300.

V and L were not repeated, VV being expressed by X, and LL by C.

2d. According as letters denoting a lower number were placed on the left or on the right of a letter denoting a higher number, the difference or the sum of the numbers expressed by these letters was indicated thus,

V= 5 IV = 4 VI = 6.
X= 10 IX = 9 XI = 11.
L= 50 XL=40 LX= 60.
C =100 XC=90 CX=110.

500 was denoted by the characters IƆ, and 1000 by CIƆ, afterwards contracted, respectively, into D and M.

The character Ɔ (a reversed C) annexed to IƆ rendered its value ten times, and two reversed C's rendered it 100 times, greater,

thus, IƆ=500, IƆƆ=5000, IƆƆƆ=50000.

A C being prefixed and a reversed C annexed to CIƆ, the value of the expression was rendered ten times greater; and two C's being prefixed and two Ɔ's annexed, its value was rendered 100 times greater:

CIƆ=1000, CCIƆƆ=10000, CCCIƆƆƆ=100000.

Numbers larger than 100000 were expressed by repetition, as CCCIƆƆƆ CCCIƆƆƆ for 200000.

Thousands were also sometimes expressed by a horizontal line drawn over the numerical letters, thus $\overline{\text{III}}$=3000, $\overline{\text{X}}$=10000.

292. Besides the lapidary characters, the Greeks had other systems of numerals.

They used the first letter of the name of a number as an abbreviated symbol denoting that number.

For example, Π, the first letter of Πεντε (five), denoted 5; Χ, the first letter of Χιλια (a thousand), denoted 1000; Μ or μ, the first letter of the word Μυρια (a myriad or ten thousand), denoted 10000.

They employed also the 24 letters of their alphabet in two modes:

1st. To denote numbers not greater than 24, they employed the letters in their regular order. In this manner the books of Homer's Iliad are numbered, the first being Α, the second Β, &c.

2d. To denote large numbers, they divided the 24 letters into three classes corresponding to units, tens, and hundreds. As a class contained only 8 letters, a new symbol was introduced into each, to make up the number of nine digits.

With distinct characters for 1, 2, 3, ... 9 units, 1, 2, 3, ... 9 tens, 1, 2, 3, ... 9 hundreds, they possessed the means of expressing all numbers from 1 to 999 inclusive.

To express larger numbers by means of the same characters various expedients were employed.

By the subscribed iota the value of a character was increased a thousand fold; β being the symbol for 2, $\underset{\prime}{\beta}$ denoted 2000.

The value of a character enclosed within a capital Π was increased 5000 fold; η being the symbol for 8, $\overline{\vert\eta\vert}$ denoted 8 × 5000, or 40000.

The value of a character under which was written μ (the expression for 10000 in the abbreviated system) was increased 10000 fold; thus $\underset{\mu}{\beta}$ denoted 2 × 10000, or 20000.

The same effect was produced by writing two points over the character, so that the symbols $\underset{\mu}{\beta}$ and $\ddot{\beta}$ denoted the same number, namely, 20000.

By suitable combinations of these characters large numbers could be expressed with considerable brevity.

From the name of ϛ, the symbol for six (ἐπισημον βαυ), and other reasons, the Greeks are believed to have borrowed the last-described system of notation from the Phenicians.

293. For expressing names of number in an abridged form, both the Greek and Roman systems of notation were sufficient, although in this respect both, and more especially the Roman, were much inferior to the decimal system of modern times.

As instruments of calculation, the ancient systems were very defective. Numbers expressed by them were assemblages of unconnected symbols; each character, whether alone, or on the left or the right of other characters, having the same unchangeable value.

Consequently the processes of addition and subtraction could only be performed by methods similar to those followed in Articles 20 (Example 2) and 30, that of multiplication as in Articles 50, 51, or 62; the collection of the partial products being, moreover, beset with the difficulties of addition.

The process of division was still more laborious.

In all calculations performed by means of the Greek or Roman numerals each step must have appeared separately, and there could be none of that facility which is gained in the decimal system by the distribution of the individuals composing a large number into different orders of units, each expressed by a figure not exceeding nine; or of that concentration which is attained by carrying forward the multiples of ten, and combining them with the higher numbers, so as to obtain at once a result in which there is no occasion for further reductions.

If the processes of Articles 13 and 23, Articles 27 and 39, Articles 50, 51, and 64, Articles 78 and 93 are compared, some notion may be formed of the superiority of the decimal over the ancient systems, even in the most simple and elementary operations. This superiority of the decimal system is derived from the conventions,

That ten units of any order make one unit of the next higher order;

And that the different orders of units ascend regularly from right to left.

294. The knowledge of the ten numeral characters, and of the system of arithmetic connected with them, was communicated to the christian nations of Europe by the Moorish conquerors of Spain. The exact time is uncertain; but dates expressed in figures, instead of Roman numerals, are found occasionally between the middle and the close of the fourteenth century.

Neither the Moors nor their countrymen the Arabians were the inventors of decimal arithmetic; the latter having acquired their knowledge from the Hindoos during their conquest and occupation of India.

There is reason to believe that the Hindoos were acquainted with it long before the invasion of their country by the Arabs; but whether they were the inventors has not been ascertained.

SECTION X.

TABLES OF BRITISH WEIGHTS AND MEASURES. REDUCTION, ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION OF COMPOUND NUMBERS. RULE OF ALIQUOT PARTS.

295. Number, considered without reference to the species of its unit, is termed abstract; but when to the name expressing the number of units is annexed a description of the species of the magnitude, the number is concrete.

The preceding sections contain the elementary principles of arithmetic applied to abstract numbers.

If concrete, like abstract, numbers were always expressed as multiples or parts of the same unit, the arithmetical processes for both would be precisely the same. But when a magnitude is such that it may be requisite to express numerically very large as well as very small portions of it, the practice is to assume some definite part of the magnitude as a principal unit, to divide this into a certain number of equal parts, to consider each of these as a new unit, and to repeat this process through more or fewer subdivisions.

The advantages of introducing simple units of different values into the numerical expression of the same magnitude are, 1st. A more precise idea is obtained of small portions of the magnitude by referring them to a subordinate unit, than can be got by expressing them as fractional parts of the principal unit; and,

2d. Large quantities, being referred to the principal unit, are expressed by smaller numbers than if measured by the subordinate unit.

The disadvantages are: the uniform relation of the different orders of units, as tens or tenths of each other, according as they are taken consecutively from right to left or from left to right, is broken, and troublesome reductions of units of one order into units of another order are rendered necessary.

To illustrate the subject by a familiar case, that of money: the principal unit, a pound sterling or sovereign, is divided into 20 equal parts, of which parts one, under the name of shilling, is taken as a second unit. The shilling, again, is divided into 12 equal parts, each of which, a penny, is a third unit. The penny, lastly, is divided into 4 equal parts, of which each is a farthing.

The pound sterling is therefore equal to 20 shillings,
or to $20 \times 12 = 240$ pence,
or to $20 \times 12 \times 4 = 960$ farthings.

The unit of each order being represented by a particular coin, its individuality is well marked. It is, in consequence, much easier to form a precise notion of the value of a small sum of money, for example, of fivepence, by describing it as a multiple of the subordinate unit, one penny, than as a fractional part of the principal unit, namely, $\frac{5}{240}$ of a pound.

Numbers expressed in this manner are called *Compound*. The principal and subordinate units contained in a compound number are also termed the different denominations of the number.

a. If, beginning with the principal unit, the subdivisions were made from ten to ten times smaller; for example, if the pound were divided into ten shillings, the shilling into ten pence, &c.; it is evident that the successive denominations, being tenths, hundredths, &c. of the principal unit, could be expressed as decimal fractions.

b. Or, if the subdivisions were made from b to b times, or from 20 to 20 times smaller; if the pound, for example, were divided into b shillings or into 20 shillings, the shilling into b pence or 20 pence, &c., the compound number might be regarded as belonging to a system whose base is b or 20, with this irregularity, that the principal unit is referred to the decimal system, and the subordinate units are expressed by the figures adapted to base 10, and not to base b or base 20.

c. If the principal unit is divided into b equal parts, one of these into c equal parts, &c. (as the pound, which is divided into 20 shillings, and the shilling into 12 pence, &c.), the compound number may be regarded as an irregular number, in which the subordinate units are referred each to a different base, and the principal unit to the base 10; the numbers contained in the inferior denominations being, moreover, expressed by the figures of the decimal system.

296. The quantity of certain sorts of magnitude is expressed by compound numbers, not because it is impossible to express their quantity in terms of a single measuring unit, but for convenience alone.

The nature of the principal and subordinate units employed to express the quantity of any thing which is to be measured or weighed necessarily depends on that property, the quantity of which is to be expressed numerically. The properties of body which may be so expressed are length, surface, volume or capacity, weight.

In all compound numbers the principal units and successive subdivisions are arbitrary.

But when a unit or measure of any kind has been chosen it becomes necessary to connect it with some invariable standard, in order that the measure, by comparison with the standard, may be corrected if inaccurate, or recovered if lost.

In British measures the standard of comparison is derived from the measure of time.

The mean interval between mid-day and mid-day is divided into 24 hours; each hour, again, is subdivided into 60 minutes, and each minute into 60 seconds.

The number of seconds in a day is, therefore, $24 \times 60 \times 60 = 86400$.

Now, it has been found possible to determine the length of a pendulum which beats 86400 times in a day.

The number of seconds in a day, and the length of the pendulum which at the same place makes 86400 beats during that time, are presumed to be invariable; so long, therefore, as the duration of a day and the length of the pendulum which beats 86400 times during the day can be measured, so long the exact length of the standard of comparison can be found.

The length of the seconds pendulum in London has been determined with great accuracy, and it would have been desirable to take this as the unit of measures of length. But a previous measure existed, and to change this would have produced much inconvenience. The length of that measure (which is called a yard) was therefore compared with the seconds pendulum, and it was found that, of such equal parts as the yard contains 36, the pendulum contains 39·13929. This relation of the two lengths supplies the means of restoring the standard yard measure, supposing it at any future time to be lost.

The 36th part of the yard is named inch. The ancient standard for determining the length of the inch is said to have been obtained by means of three grains of barley taken from the middle of the ear, and arranged lengthways in a straight line.

A measure equal to one third of the inch is still retained among linear measures, under the denomination of barleycorn.

The measures of surface are square inches, square feet, &c. These are derived from the corresponding linear measures; a square inch being a square whose side is one inch, &c.

Measures of volume or capacity are of two sorts. In the first, called cubic or solid measure, the thing measured is considered as having length, breadth, and thickness, and the measures are cubic inches, cubic feet, &c.; the cubic inch being of the form of a die, and having each of its edges one inch in length.

In determining the quantity of a length, a surface, or a volume, the measurement is made by the application of a rod, cord, chain, or other convenient medium to the substance measured.

The measurement of any substance by a measure of capacity is made in a different manner. The measure being constructed in the form of a hollow vessel, the substance is measured by placing it in this vessel.

The standard measure of capacity is the gallon; the gallon is derived from a measure of extension, the inch, in the manner following:

If a vessel, whose interior is a cubic inch, is filled with distilled water*, its weight receives a certain increase. The increment of weight is divided into 252·458 equal parts, of which one is called a grain.

Seven thousand such grains are a pound avoirdupois weight, and ten pounds avoirdupois weight of water are exactly contained in the gallon measure.

Since the gallon contains $7000 \times 10 = 70000$ grains weight of water, and the cubic inch contains 252·458 grains, the capacity of the gallon in cubic inches, or the number of cubic inches which it holds, is,

$$\frac{70000}{252{\cdot}458} = \frac{70000000}{252458} = 277{\cdot}273844, \text{ or } 277{\cdot}274 \text{ nearly.}$$

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[*] The water is distilled so as to be free from impurities, which might affect the weight: barometer 30 inches, thermometer 62° Fahr.

The standards of weight are two, the pound troy and the pound avoirdupois.

Both are derived from the same part of the cubic inch of water as the gallon is, namely, the grain.

5760 grains make a pound troy weight, and 7000 grains make a pound avoirdupois weight.

The standards of measures and weights, as well as their parts and multiples, were described and their use rendered compulsory by an act of parliament passed in 1835, and which came into operation on the 1st of January 1836.

297. The following tables contain the measures and weights now used throughout the British empire. The quantities in the same horizontal line of the same table are equal to each other.

Measures of length.

Inches. (in.)	Links. (li.)	Feet. (ft.)	Yards. (yd.)	Pole or Perch. (po.)	Chains. (ch.)	Furlongs. (fur.)	Mile. (mi.)
7·92	1	...	...	...	...	...	...
12	1·515	1	...	...	...	...	...
36	4·545	3	1	...	...	...	...
198	25	16·5	5·5	1	...	...	...
792	100	66	22	4	1	...	...
7920	1000	660	220	40	10	1	...
63360	8000	5280	1760	320	80	8	1

Also 12 lines=1 inch, 3 barleycorns=1 inch, 3 inches=1 palm, 4 inches =1 hand, 5 feet=1 pace, 6 feet=1 fathom, 3 miles=1 league, 69⅛ miles (nearly)=1 degree, 60 geographical miles=1 degree.

Cloth measure.

Inches. (in.)	Nails. (nl.)	Quarters. (qr.)	Yards. (yd.)
2·25	1	...	...
9	4	1	...
36	16	4	1

Measures of surface or superficies.

Square Inches. (sq. in.)	Square Links. (sq. li.)	Square Feet. (sq. ft.)	Square Yards. (sq. yd.)	Square Poles. (sq. po.)	Square Chains. (sq. ch.)	Square Roods. (sq. ro.)	Acre. (ac.)
62·726	1	...	...	...	...	...	...
144	2·295	1	...	...	...	...	...
1296	20·661	9	1	...	...	...	...
39204	625	272·25	30·25	1	...	...	...
627264	10000	4356	484	16	1	...	...
1568160	25000	10890	1210	40	2·5	1	...
6272640	100000	43560	4840	160	10	4	1

In artificers work, 144 square lines=1 square inch, 144 square inches= 1 square foot; also, 100 square feet of flooring=1 square, 36 square yards of stone, slate, or brick-work=1 rood.

Measures of volume or solidity.

Cubic Inches. (c. in.)	Cubic Feet. (c. ft.)	Cubic Yard. (c. yd.)
1728	1	...
46656	27	1

40 cubic feet of rough or 50 of hewn timber=1 load or ton. A cubic yard of earth is called a load.

Measures of capacity.

Pounds of Water.	Cubic Inches.	Gills. (gl.)	Pints. (pt.)	Quarts. (qt.)	Pottles. (pot.)	Gallon. (gal.)
1·25	34·659	4	1	...	...	...
2·5	69·318	8	2	1	...	...
5	138·637	16	4	2	1	...
10	277·274	32	8	4	2	1

To gallons, inclusive, the measures for dry and for liquid substances are the same.

In dry measure the multiples of the gallon are,

2 gallons=1 peck (pk.),
8 gallons=4 pecks=1 bushel (bush.),
64 gallons=32 pecks=8 bushels=1 quarter (qr.);

And in liquid measure,

9 gallons=1 firkin (fir.) beer measure,
36 gallons=4 firkins=1 barrel (bar.),
54 gallons=6 firkins=1½ barrels=1 hogshead (hhd.),
63 gallons=1 hogshead, wine measure,
126 gallons=2 hogsheads=1 pipe,
252 gallons=4 hogsheads=2 pipes=1 tun.

WEIGHTS.

Troy weight.

Grains. (gr.)	Pennyweights. (dwt.)	Ounces. (oz.)	Pound. (lb.)
24	1	...	...
480	20	1	...
5760	240	12	1

Gold, silver, platina, pearls, and precious stones are weighed by troy weight. Diamonds are weighed with a weight termed a carat, which is the $\frac{1}{150}$th part of the ounce troy.

The term carat, when applied to gold, denotes its degree of fineness. Any quantity of pure gold, or of gold alloyed with some other metal, being supposed to be divided into 24 equal parts; if the gold is pure it is said to be 24 carats fine; if 23 parts of the whole are pure gold and 1 part is copper (or some other alloy), the gold is said to be 23 carats fine; if 22 parts of the compound are pure gold and 2 alloy, it is 22 carats fine, and so on.

Apothecaries weight.

Troy Grains.	Scruples. (℈.)	Drams. (ʒ.)	Ounces. (℥.)	Pound. (℔.)
20	1	...	...	...
60	3	1	...	...
480	24	8	1	...
5760 .	288	96	12	1

Apothecaries weight is used for medical prescriptions, but the trader weighs drugs with the commercial or avoirdupois weight.

Avoirdupois weight.

Troy Grains.	Drams. (dr.)	Ounces. (oz.)	Pounds. (lb.)	Stones. (st.)	Quarters. (qr.)	Hundred-weights. (cwt.)	Ton.
437·5	16	1	...	...	...	...	...
7000	256	16	1	...	...	...	...
98000	3584	224	14	1	...	...	...
196000	7168	448	28	2	1	...	...
784000	28672	1792	112	8	4	1	...
15680000	573440	35840	2240	160	80	20	1

The troy pound is less than the avoirdupois pound in the proportion of 144 to 175; and the avoirdupois ounce less than the troy ounce in the proportion of 175 to 192.

Measure of time.

Seconds. (s. or ″.)	Minutes. (m. or ′.)	Hours. (h.)	Days. (d.)	Weeks. (wk.)	Year. (yr.)
60	1	...	...	...	...
360	60	1	...	...	...
8640	1440	24	1	...	...
604800	10080	168	7	1	...
31556929·536	525948·8256	8765·81376	365·24224	52·17746	1

Years expressed by numbers which are not multiples of 4 are made to consist of 365 days; and those expressed by numbers which are multiples of 4, of 366 days. Three years, therefore, contain always 365 days each, and the fourth 366. The year of 366 days is called leap-year.

The average length of the year is thus assumed to be 365¼ or 365·25 days, which exceeds the true duration by ·00776 of a day.

In 129 years this amounts to a day, or, in round numbers, it makes 3 days in 4 centuries. The error is corrected by making three of the years which close the centuries common years, and the fourth only leap-year. Thus 1700, 1800, 1900 are common years, and 2000 a leap-year.

Money.

Farthings. (qrs.)	Pence. (d.)	Shillings. (sh.)	Pound. (£.)
4	1	...	...
48	12	1	...
960	240	20	1

In the gold coinage of Great Britain, twelve ounces of the metal of which standard gold coins are made contain 11 oz. of pure gold and 1 oz. of alloy (the alloy used is copper).

The pound troy is coined into $46\frac{89}{120}$ sovereigns; the weight of a sovereign is, therefore, $5760 \div 46\frac{89}{120} = 123\frac{171}{625}$ grs. = 123·25 grs. nearly; of which 113 grains are pure gold, and 10·25 grs. alloy.

In the silver coinage, 12 oz. of the metal of which standard silver coins are made contain 11 oz. 2 dwt. of pure silver, and 18 dwt. of alloy.

The pound troy of this metal is coined into 66 shillings; whence the weight of 1 shilling is $5760 \div 66 = 87\frac{3}{11}$ grains = 87·3 grs. nearly, of which 80·7 grs. are pure silver and 6·6 grs. alloy.

In the present coinage
the actual value of the sovereign is 20 sh.,
that of the shilling is 11·27 d.,
and the relative values of equal weights of gold and silver are nearly as 14·3 to 1.

The penny contains $10\frac{2}{3}$ drams avoird. of copper. The penny, shilling, and sovereign are, respectively, the standard coins of copper, silver, and gold.

Multiplication Table, containing the Products of Numbers from 1 *to* 25.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50
3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60	63	66	69	72	75
4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80	84	88	92	96	100
5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100	105	110	115	120	125
6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114	120	126	132	138	144	150
7	14	21	28	35	42	49	56	63	70	77	84	91	98	105	112	119	126	133	140	147	154	161	168	175
8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128	136	144	152	160	168	176	184	192	200
9	18	27	36	45	54	63	72	81	90	99	108	117	126	135	144	153	162	171	180	189	198	207	216	225
10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200	210	220	230	240	250
11	22	33	44	55	66	77	88	99	110	121	132	143	154	165	176	187	198	209	220	231	242	253	264	275
12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228	240	252	264	276	288	300
13	26	39	52	65	78	91	104	117	130	143	156	169	182	195	208	221	234	247	260	273	286	299	312	325
14	28	42	56	70	84	98	112	126	140	154	168	182	196	210	224	238	252	266	280	294	308	322	336	350
15	30	45	60	75	90	105	120	135	150	165	180	195	210	225	240	255	270	285	300	315	330	345	360	375
16	32	48	64	80	96	112	128	144	160	176	192	208	224	240	256	272	288	304	320	336	352	368	384	400
17	34	51	68	85	102	119	136	153	170	187	204	221	238	255	272	289	306	323	340	357	374	391	408	425
18	36	54	72	90	108	126	144	162	180	198	216	234	252	270	288	306	324	342	360	378	396	414	432	450
19	38	57	76	95	114	133	152	171	190	209	228	247	266	285	304	323	342	361	380	399	418	437	456	475
20	40	60	80	100	120	140	160	180	200	220	240	260	280	300	320	340	360	380	400	420	440	460	480	500
21	42	63	84	105	126	147	168	189	210	231	252	273	294	315	336	357	378	399	420	441	462	483	504	525
22	44	66	88	110	132	154	176	198	220	242	264	286	308	330	352	374	396	418	440	462	484	506	528	550
23	46	69	92	115	138	161	184	207	230	253	276	299	322	345	368	391	414	437	460	483	506	529	552	575
24	48	72	96	120	144	168	192	216	240	264	288	312	336	360	384	408	432	456	480	504	528	552	576	600
25	50	75	100	125	150	175	200	225	250	275	300	325	350	375	400	425	450	475	500	525	550	575	600	625

298. From the formation of compound numbers, it is evident that the same absolute quantity can be expressed by numbers which contain fewer or more units, according as the measure employed is a principal or a subordinate unit. One pound troy weight, for example, can be expressed as 1 lb., or 12 oz., or 240 dwt., or 5760 grs.

In performing the elementary operations of Arithmetic with compound numbers, it may become necessary to reduce numbers expressed in terms of any unit of a weight or measure to equivalent numbers expressed in terms of any other unit of that weight or measure.

Such reduction must be either from a higher denomination to a lower, or from a lower denomination to a higher.

Since the principal unit of any weight or measure is a certain multiple of the first subordinate unit, this a certain multiple of the second, &c., it follows that the number of units of a lower denomination equivalent to any given number of units of a higher denomination must be the same multiple of the given number that the higher unit is of the lower;

And, conversely, that the number of units of a higher denomination equivalent to a given number of units of a lower denomination must be the same part of the given number that the lower unit is of the higher.

The general inference therefore is, that to reduce a number expressed in terms of any unit of a weight or measure to an equivalent number expressed in terms of an inferior unit of the same weight or measure, it is necessary to multiply the given number by a multiplier expressing the number of units of the lower denomination which make 1 unit of the higher denomination;

And that to reduce from any unit to a higher it is necessary to divide the given number by a divisor expressing the number of units of the given which are equivalent to one of the required denomination.

299. Examples of reduction from units of a given denomination to units of the next lower denomination.

1st. Reduce 5 poles, long measure, to yards:

$$1 \text{ po.} = 5\tfrac{1}{2} \text{ yds.} = \frac{11}{2} \text{ yds.},$$

$$\therefore 5 \text{ po.} = \left(5 \times \frac{11}{2}\right) \text{yds.} = \frac{55}{2} = 27\tfrac{1}{2} \text{ yds.}$$

2d. Reduce $\frac{3}{7}$£. to shillings:

$$1£. = 20\text{sh.} \therefore \frac{3}{7}£. = \left(\frac{3}{7} \times \frac{20}{1}\right)\text{sh.} = \frac{60}{7} \text{ sh.} = 8\tfrac{4}{7} \text{ sh.}$$

3d. Reduce ·34 quarters to pounds avoirdupois:

$$1 \text{ qr.} = 28 \text{ lb.} \therefore {·34} \text{ qrs.} = ({·34} \times 28) \text{ lbs.} = 9{·52} \text{ lbs.}$$

4th. Reduce 5sh. 4d. to pence:

$$\text{since } 1\text{sh.} = 12\text{d.}, \ 5\text{sh.} = (5 \times 12)\text{d.} = 60\text{d.};$$
$$\text{and } 60\text{d.} + 4\text{d.} = 64\text{d.}$$
$$\therefore 5\text{sh. } 4\text{d.} = 64\text{d.}$$

The last example differs from the others in this respect, that the given number is compound.

When the 5sh. have been reduced to an equivalent number of pence, in order to obtain the value of the compound expression, it is necessary to add the pence to this number; the sum of both is the result required.

300. The reduction of any number, from units of a given denomination to units of any other denomination lower than the next, is made by successive reductions from one unit to the inferior, then from this to the next inferior, by a repetition of the same process, until the units of the required denomination have been attained.

a. If the number is compound, it is indispensable that the reduction be made thus by successive steps, in order that the units of the denominations successively attained may be combined with the sum of the higher units which have been reduced to their measure.

b. But when the given number is a multiple or part of one unit only of a given weight or measure, it may be reduced to any lower denomination of the same weight or measure by multiplying at once by that number of the required which makes one of the given denomination; for it is equivalent to multiply any number by the factors of a product, or at once by that product (Art. 71).

5th. Reduce 3 £. to farthings:

1 £.=20 sh. ∴ 3 £.=(3 × 20) sh.=60 sh.
1 sh.=12 d. ∴ 60 sh.=(60 × 12) d.=720 d.
1 d.=4 qrs. ∴ 720 d.=(720 × 4) qrs.=2880 qrs.
whence 3 £.=2880 qrs.

Otherwise, since 1 £.=960 qrs.,
3 £.=(3 × 960) qrs.=2880 qrs.

6th. Reduce $\frac{3}{7}$ cwt. to lbs.:

$$1 \text{ cwt.}=4 \text{ qrs.} \therefore \frac{3}{7} \text{ cwt.} = \left(\frac{3}{7}\times\frac{4}{1}\right) \text{qr.} = \frac{12}{7} \text{qrs.}$$

$$1 \text{ qr.}=28 \text{ lb.} \therefore \frac{12}{7} \text{qr.} = \left(\frac{12}{\underset{1}{\cancel{7}}}\times\frac{\overset{4}{\cancel{28}}}{1}\right) = \frac{48}{1} \text{ or } 48 \text{ lbs.}$$

$$\therefore \frac{3}{7} \text{ cwt.} = 48 \text{ lbs.}$$

The preceding calculation may be condensed thus:

$$\frac{3}{7} \text{ cwt.} = \left(\frac{3}{\underset{1}{\cancel{7}}}\times\frac{4}{1}\times\frac{\overset{4}{\cancel{28}}}{1}\right) \text{lbs.} = 48 \text{ lbs.}$$

Otherwise

$$1 \text{ cwt.}=112 \text{ lbs.} \therefore \frac{3}{7} \text{ cwt.} = \left(\frac{3}{\underset{1}{\cancel{7}}}\times\frac{\overset{16}{\cancel{112}}}{1}\right) \text{lbs.} = 48 \text{ lbs.}$$

c. When the number to be reduced is expressed by a vulgar fraction the manner (followed in the 2d solution of the last example) of indicating the successive steps, striking out factors common to the numerator and denominator, and then performing the multiplication (as in multiplication of vulgar fractions), seems preferable to the last method, inasmuch as the common factors are detected in small more readily than in large numbers.

7th. Reduce ·82 poles to inches, long measure:

Calculation by successive reductions:

·82 poles.
Multiply by 5·5 = no. of yards in 1 pole.
410
410
4·510 yards.
Multiply by 3 = no. of feet in 1 yard.
13·53 feet.
Multiply by 12 = no. of inches in 1 foot.
162·36 inches.

Otherwise, multiply ·82 poles by 198, which is the number of inches contained in 1 pole:

·82 × 198=162·36 inches, the same result as before.

8th. Reduce 4 lb. 8 oz. 10 dwt. 15 grs. troy weight to grains:

1 lb.=12 oz. .·. 4 lbs.=(4×12) oz.=48 oz.
and 4 lbs. 8 oz.=48 oz.+8 oz.=56 oz.
1 oz.=20 dwt. .·. 56 oz.=(56×20) dwt.=1120 dwt.
.·. 56 oz. 10 dwt.=1120 dwt.+10 dwt.=1130 dwt.
and 4 lbs. 8 oz. 10 dwt.=1130 dwt.
1 dwt.=24 grs. .·. 1130 dwt.=(1130×24) grs.=27120 grs.
.·. 1130 dwt. 15 grs.=27120 grs.+15 grs.=27135 grs.
and 4 lbs. 8 oz. 10 dwt. 15 grs.=27135 grs.

The reduction may be more concisely made by multiplying the lbs. by 12, and simultaneously combining the oz. with the product; multiplying this result by 20, and combining the dwts. with the product, thus,

lbs. oz. dwts. grs.
4 8 10 15
12
56 oz. =4 lbs. 8 oz.
20
1130 dwts.=4 lbs. 8 oz. 10 dwts.
24
4525
2261
27135 grs.=4 lbs. 8 oz. 10 dwts. 15 grs.

301. Whence, to reduce a compound number to a lower denomination,

Rule. Multiply the number which expresses the highest denomination of the given compound quantity by a multiplier which expresses the number of parts into which one unit of the highest denomination is divided to form one unit of the second; to this product add the units of the second denomination. Multiply the sum by the number expressing the parts into which one unit of the second denomination is divided to form one unit of the third; to the product add the units of the third denomination, and proceed in this manner to multiply and add until the given compound number is reduced to units of the denomination required.

The reduction of a multiple or part of any unit of a weight or measure may, in like manner, be carried by repeated reduction through the units intermediate between the given and the required denominations; or it may be made at once, by multiplying the given number by as many of the required denomination as make one of the given denomination.

302. Exercises in the reduction of compound numbers from higher denominations to lower:

1st. Reduce 23 £. to pence?..................... *Ans.* 5520 pence.
2d. - 351 £. 13 sh. 0¾ d. to farthings? *Ans.* 337587 farthings.
3d. - 17 lbs. apoth. wt. to scruples?... *Ans.* 4896 scruples.
4th. - 72 leagues to yards? .... *Ans.* 380160 yards.
5th. - 2 m. 1 fur. 8 po. 3 yd. 2 in. to inches?... *Ans.* 136334 in.
6th. - 27 lb. 7 oz. 2 dr. 1 sc. 2 gr. apoth. weight to grains? *Ans.* 159022 grains.
7th. - 27 cwt. to lbs.?......................... *Ans.* 3024 lbs.
8th. - 4 tons 3 cwt. 2 qrs. 1 lb. to drams?.. *Ans.* 2394368 dr.
9th. - 5738 lbs. troy to grains?..... *Ans.* 33050880 grains.

10th. Reduce 69 lbs. 11 oz. 12 grs. troy to grains? *Ans.* 402732 grains.

11th. - $697\frac{1}{4}$ acres to square yards?...*Ans.* 3375900 sq. yds.

12th. - 3 acr. 2 ro. 12 po. 10 yds. to square yards? *Ans.* 17313 square yards.

13th. - 519 barrels beer meas. to pints?...*Ans.* 149472 pts.

14th. - 17 hhd. 3 gal. 2 qts. 1 pt. wine measure to pints? *Ans.* 8853 pints.

15th. - 275 yards to nails?......................*Ans.* 4400 nails.

16th. - $\frac{2}{25}$ £. to farthings?.................*Ans.* $\frac{384}{5}$ farthings.

17th. - $\frac{7}{16}$ cwt. to oz.?...............................*Ans.* 784 oz.

18th. - $\frac{13}{15}$ lbs. troy to grains?...............*Ans.* 4992 grains.

19th. - $\frac{5}{12}$ acre to square feet?.......*Ans.* 18150 square feet.

20th. - $\frac{9}{16}$ square poles to square links? *Ans.* $\frac{5625}{16}$ sq. links.

21st. - $3\frac{1}{4}$ furlongs to inches?..............*Ans.* 25740 inches.

22d. - $\frac{2}{3}$ quarter to cubic inches?...*Ans.* 11830·024 cub. in.

23d. - ·25 chains long meas. to inches?...*Ans.* 198 inches.

24th. - 2·3 yards, cloth measure, to nails?...*Ans.* 36·8 nails.

25th. - 1·28 sq. chains to square feet?...*Ans.* 5575·68 sq. ft.

26th. - 2·6 cubic feet to cubic inches?...*Ans.* 4492·8 cubic in.

27th. - ·7 gallons to gills?........................*Ans.* 22·4 gills.

28th. - ·328 weeks to seconds?.......*Ans.* 198374·4 seconds.

29th. - ·625 shillings to farthings?.........*Ans.* 30 farthings.

30th. - ·00945 hogsheads to pints?.........*Ans.* 4·7628 pints.

31st. - ·0125 lbs. troy to grains?...............*Ans.* 72 grains.

32d. - ·199325 miles to links?............*Ans.* 1594·6 links.

33d. - 513·6 poles in length to yards?...*Ans.* 2824·8 yards.

34th. - ·625 cwt. to lbs.?...........................*Ans.* 70 lbs.

35th. - ·009943 miles to inches?.....*Ans.* 629·98848 inches.

36th. - ·3375 acres to square yards? *Ans.* 1633·5 square yds.

37th. - $\frac{7}{16}$ of 5 cwt. 3 qrs. 10 oz. to drams?...*Ans.* 72198 dr.

38th. - $\frac{5}{9}$ of 3 hhd. 12 gal. 1 pt. to pints?...*Ans.* $893\frac{8}{9}$ pints.

39th. - $\frac{2}{9}$ of 2 £. 12 sh. 4 d. to farthings? *Ans.* $558\frac{2}{9}$ farthings.

40th. - $\frac{12}{17}$ of 4 acr. 2 ro. 19 po. to square yards? *Ans.* $15779\frac{14}{17}$ square yards.

303. A fractional expression of any unit of a weight or measure being given, it may be required to reduce the given fraction to a compound number expressed in terms of the lower denominations of that weight or measure.

The compound number obtained as a result is called the value of the fraction.

What is the value of $\frac{3}{7}$ £.?

Since this value is to be expressed in terms of the lower denominations, the reduction of $\frac{3}{7}$ £. to shillings must form the first step of the calculation:

$$\text{Now } \frac{3}{7}\text{£.}=\left(\frac{3}{7}\times\frac{20}{1}\right)\text{ sh.}=\frac{60}{7}\text{ sh.}=8\frac{4}{7}\text{ sh.}$$

For the same reason the reduction of $\frac{4}{7}$ sh. to pence must be the second step:

$$\frac{4}{7}\text{sh.}=\left(\frac{4}{7}\times\frac{12}{1}\right)\text{d.}=\frac{48}{7}\text{d.}=6\frac{6}{7}\text{d.},$$

And that of $\frac{6}{7}$d. to farthings, the third:

$$\frac{6}{7}\text{d.}=\left(\frac{6}{7}\times\frac{4}{1}\right)\text{ qrs.}=\frac{24}{7}\text{ qrs.}=3\frac{3}{7}\text{qrs.}$$

Farthings being the lowest denomination of money, the reduction cannot be carried lower,

Whence the value of $\frac{3}{7}$ £. is 8 sh. 6 d. $3\frac{3}{7}$ qrs.

The process may be represented more compactly thus,

$$\left.\begin{array}{l}\frac{3}{7}\text{ £.}=\left(\frac{3}{7}\times\frac{20}{1}\right)\text{ sh.}=\frac{60}{7}\text{ sh.}=8\frac{4}{7}\text{ sh.}\\ \frac{4}{7}\text{ sh.}=\left(\frac{4}{7}\times\frac{12}{1}\right)\text{ d.}=\frac{48}{7}\text{ d.}=6\frac{6}{7}\text{ d.}\\ \frac{6}{7}\text{ d.}=\left(\frac{6}{7}\times\frac{4}{1}\right)\text{ qrs.}=\frac{24}{7}\text{ qrs.}=3\frac{3}{7}\text{ qrs.}\end{array}\right\}8\text{ sh. }6\text{ d. }3\frac{3}{7}\text{ qrs.}$$

2d Ex. What is the value of ·384 cwt.?

·384 cwt.=(·384 × 4) qrs.=1·536 qrs.

Reserving the 1 qr., and reducing ·536 qrs. to lbs.

·536 qr.=(·536 × 28) lb.=15·008 lb.

Reserving 15 lb., and reducing ·008 lb. to oz.

·008 lb.=(·008 × 16) oz.=0·128 oz.

The value contains 0 oz.; reducing ·128 oz. to drams,

·128 oz.=(·128 × 16) dr.=2·048 drams.

With drams the reduction terminates, and it is found that

·384 cwt.=1 qr. 15 lb. 0 oz. 2·048 drams.

Without detail the calculation may appear thus,

$$\begin{array}{rl}\cdot384 & \text{cwt.}\\ \underline{4} & \\ 1\cdot536 & \text{qrs.}\\ \underline{28} & \\ 4288 & \\ \underline{1072} & \\ 15\cdot008 & \text{lb.}\\ \underline{16} & \\ \cdot128 & \text{oz.}\\ \underline{16} & \\ \underline{\underline{2\cdot048}} & \end{array}$$

And the result, as before, is 1 qr. 15 lb. 0 oz. 2·048 dr.

304. Whence, to find the value of a fractional part of any unit of a given weight or measure as a compound number expressed in terms of the inferior denominations of that weight or measure,

Rule. Reduce the given fraction to an equivalent fraction expressed in terms of the next lower denomination; bring this fraction to a mixed number; reserve the integral part of the mixed number for the highest denomination of the value; reduce the fractional part to an equivalent fraction expressed in terms of the next lower denomination; bring this fraction to a mixed number, reserve the integral part, and reduce the fractional part, as before; and thus proceed till there is no remainder, or till the lowest denomination of the given weight or measure is attained. The compound number formed of the integral parts reserved from the successive reductions, and of the result of the last reduction, is the value sought.

305. Exercises on the preceding rule:

1st. Find the value of $\frac{3}{4}$ £.?................................*Ans.* 15 sh.

2d. - - $\frac{7}{9}$ of a crown (the crown=5 sh.)? *Ans.* 3 sh. 10 d. $2\frac{2}{3}$ qrs.

3d. - - $\frac{9}{16}$ lb. troy?..................*Ans.* 6 oz. 15 dwt.

4th. Find the value of $\frac{10}{11}$ cwt. ?....*Ans.* 3 qr. 17 lb. 13 oz. $1\frac{5}{11}$ dr.
5th. - - $\frac{29}{33}$ mile ?.........*Ans.* 7 fur. 1 po. 1 yd. 6 in.
6th. - - $\frac{79}{144}$ barrel ?................*Ans.* 19 gal. 3 qts.
7th. - - $\frac{49}{55}$ acre ?..*Ans.* 3 r. 22 po. 16 yds. 4 ft. 72 in.
8th. - - $\frac{1}{27}$ bushel ?..................*Ans.* 1 qt. $0\frac{10}{27}$ pt.
9th. - - $\frac{11}{16}$ fathom ?.............*Ans.* 1 yd. 1 ft. $1\frac{1}{2}$ in.
10th. - - $\frac{17}{39}$ hhd. ?...........*Ans.* 27 gal. 1 qt. $1\frac{9}{13}$ pt.
11th. - - $\frac{19}{118}$ yd. ?...........................*Ans.* $5\frac{47}{59}$ in.
12th. - - $\frac{15}{28}$ year ?......*Ans.* 195 d. 12 h. 51′. 25″ $\frac{5}{7}$.
13th. - - ·625 fathom ?.................*Ans.* 1 yd. 9 in.
14th. - - ·08 mile ?.........*Ans.* 25 po. 3 yds. 10·8 in.
15th. - - ·142857 hhd. ?.....................*Ans.* 9 gal.
16th. - - ·8665 £. ?.........*Ans.* 17 sh. 3 d. 3·84 qrs.
17th. - - ·425 half-crown ?............*Ans.* 1 sh. $0\frac{3}{4}$ d.
18th. - - ·249825 pole, in length ?
Ans. 1 yd. 1 ft. 1·4653 in.
19th. - - ·024 day ?............*Ans.* 34 min. 33·6 sec.
20th. - - ·4694 lb. troy ?.*Ans.* 5 oz. 12 dwt. 15·744 gr.
21st. - - ·076 cub. yds. ?.........*Ans.* 2 ft. 89·856 in.
22d. - - 2·354 acres ?
Ans. 2 ac. 1 ro. 16 p. 19 yds. 3·24 ft.
23d. - - 4·4226 £. ?...*Ans.* 4 £. 8 sh. 5 d. 1·696 qr.
24th. - - ·065 acre ?............*Ans.* 10 po. 12·1 yds.

306. A number which expresses units of a given denomination of any weight or measure is reduced to a number expressing the equivalent units of any other denomination of that weight or measure, if the given number is divided by a divisor expressing the number of units of the given denomination, which make one unit of the required denomination (Art. 298).

To apply this principle, let it be required to reduce 135 sh. to pounds sterling:

Since 20 sh.=1 £., it follows that $\frac{135}{20}$£. is the unreduced expression of the required result.

When the numerator in such a case is a multiple of the denominator, the same result, and in the same form (namely, a whole number), is obtained, in whatever manner the reduction of the fraction is made.

But 135 is not a multiple of 20.

Now, it may be required to reduce 135 sh. to a fractional expression, vulgar or decimal, of a pound, or to a compound number consisting of pounds and shillings.

a. If to a vulgar fraction, the calculation is

$$135\text{ sh.}=(135\div20)\text{£.}=\left(\frac{\overset{27}{\cancel{135}}}{1}\times\frac{1}{\underset{4}{\cancel{20}}}\right)\text{£.}=\frac{27}{4}\text{£.}=6\frac{3}{4}\text{£.}$$

b. If to a decimal fraction, it is

$$20\overline{)135}\text{ sh.}$$
$$\underline{\underline{6{\cdot}75\text{ £.}}}$$

c. If to a compound number consisting of pounds and shillings; dividing 135 by 20, the quotient is pounds, and the remainder (which is composed of units of the same value as the dividend) is shillings; now, the quotient arising from the division of 135 by 20 is 6, and the remainder 15; therefore 6 £. 15 sh. is the result required.

The fractional result is - -	$6\frac{3}{4}$ £.
The decimal - - -	6·75 £.
The compound number - -	6 £. 15 sh.

307. In this example the reduction is from one denomination to that which is next higher. If several denominations are interposed between that of the given number and the higher denomination to which it must be reduced; then, if it is required to express the result in terms of the highest denomination only, the reduction may be made, either by successive steps, each like the last example, through all the intermediate denominations; or by dividing at once by the number of units of the given denomination which make one unit of the required denomination. But if the result is to be expressed as a compound number, the reduction must be made step by step.

In this, as in the similar cases of Articles 285, 286, 287, the remainders, taken in order, and the last quotient form the ascending orders of units of the compound number.

2d Example. Let it be required to reduce 4665 qrs. to the vulgar and to the decimal fraction of a pound sterling; also to a compound number expressed in pounds, shillings, &c.

The denominations intermediate between qrs. and pounds are pence and shillings. The reduction of 4665 qrs. to pounds may, therefore, be made thus:

$$4665\text{ qrs.} = \left(\frac{4665}{1}\times\frac{1}{4}\right)\text{d.} = \frac{4665}{4}\text{ d.}$$

$$\frac{4665}{4}\text{ d.} = \left(\frac{\overset{1555}{\cancel{4665}}}{4}\times\frac{1}{\underset{4}{\cancel{12}}}\right)\text{ sh.} = \frac{1555}{16}\text{ sh.}$$

$$\frac{1555}{16}\text{ sh.} = \left(\frac{\overset{311}{\cancel{1555}}}{16}\times\frac{1}{\underset{4}{\cancel{20}}}\right)\text{£.} = \frac{311}{64}\text{ £.} = 4\,\frac{55}{64}\text{ £.}$$

And more concisely thus (A),

$$4665\text{ qrs.} = \left(\frac{\overset{311}{\overset{\cancel{1555}}{\cancel{4665}}}}{1}\times\frac{1}{4}\times\frac{1}{\underset{4}{\cancel{12}}}\times\frac{1}{\underset{4}{\cancel{20}}}\right)\text{£.} = \frac{311}{64}\text{ £.} = 4\frac{55}{64}\text{ £.}$$

or (since 960 farthings = 1 £.), thus,

$$4665\text{ qrs.} = \frac{4665}{960}\text{ £.} = \frac{311}{64}\text{ £.} = 4\,\frac{55}{64}\text{ £.}$$

The reduction of 4665 qrs. to the decimal of a pound is made as follows:

$$\begin{array}{r} 4)\underline{4665\text{ qrs.}} \\ 12)\underline{1166{\cdot}25\text{ d.}} \\ 20\left\{\begin{array}{r} 10)\underline{97{\cdot}1875\text{ sh.}} \\ 2)\underline{9{\cdot}71875} \end{array}\right. \\ \underline{\underline{4{\cdot}859375\text{ £.}}} \end{array}$$

And the reduction to a compound number thus,

$$\left.\begin{array}{r} 4)\underline{4665\text{ qrs.}} \\ 12)\underline{1166\text{ d.}+1\text{ qr.}} \\ 20)\underline{97\text{ sh.}+2\text{ d.}} \\ \underline{\underline{4\text{ £.}+17\text{ sh.}}} \end{array}\right\} 4\text{ £. }17\text{ sh. }2\tfrac{1}{4}\text{d.}$$

The fractional result is - -	$4\frac{55}{64}$ £.
The decimal - - -	4·859375£.
The compound - - -	4 £. 17 sh. $2\frac{1}{4}$ d.

308. Whence, to reduce a number expressing units of a given denomination of any weight or measure to a number which shall express the same value in terms of units of a higher denomination of that weight or measure,

Rule. Divide the number by a divisor expressing the number of units of the given denomination which are equivalent to one unit of the required denomination; or divide the number by a divisor expressing the number of units of the given denomination which are equivalent to one unit of the next higher denomination; the quotient expresses units of the next higher denomination. Divide the quotient by a divisor expressing the number of its units which are equivalent to one unit of the higher denomination, and continue this series of divisions till the required denomination has been attained.

The result in either case is that which is sought.

When the result is to be expressed in terms of one denomination only, and the reduction is made by steps, the remainder from the first division must be divided by the divisor, and the result, whether a vulgar or a decimal fraction, annexed to the integral part of the quotient.

This mixed number must be divided by the next divisor, &c. Employing decimals, the calculation of the integral and fractional parts of the result is uniform, but by vulgar fractions the reductions are troublesome; and the method (A) of the last example seems preferable to this process.

When the result is to be expressed as a compound number, the reduction is made step by step; and the remainder from each partial division is composed of units of the same order as the dividend.

309. Exercises in the reduction of numbers from lower denominations to higher:

1st. Reduce 172800 farthings to pounds?..............*Ans.* 180£.
2d. - 33050880 grs. troy to lbs.?............*Ans.* 5738 lbs.
3d. - 3057120 feet to miles?...............*Ans.* 579 miles.
4th. - 4400 nails to yds.?.......................*Ans.* 275 yds.
5th. - 1192 pecks to bush.?...................*Ans.* 298 bush.
6th. - 12500 links to poles in length?.......*Ans.* 500 poles.
7th. - 260480 feet to miles?................*Ans.* 49⅓ miles.
8th. - 3375900 sq. yds. to acres?..........*Ans.* 697½ acres.
9th. - 592216 drs. to cwt., &c.? *Ans.* 20 cwt. 2 qrs. 17 lb. 5 oz. 8 dr.
10th. - 347592⅓ grs. troy to oz., &c.? *Ans.* 724 oz. 3 dwt. 0⅓ gr.
11th. - 1497600 sq. yds. to acres, &c.? *Ans.* 309 ac. 1 r. 27 p. 13¼ yd.
12th. - 87915423 pints to hhds, &c.? *Ans.* 174435 hhds. 22 g. 3 q. 1 pt.
13th. - 16573 qrs. to pounds, &c.?.....*Ans.* 17 £. 3 sh. 5¼ d.
14th. - 159022 grs. apoth. wt. to lbs., &c.? *Ans.* 27 lb. 7 oz. 2 dr. 1 sc. 2 gr.

15th. Reduce 63359 grs. apoth. wt. to lbs., &c.?
Ans. 10 lb. 11 oz. 7 dr. 2 sc. 19 gr.

16th. - 22253 grs. troy to lbs., &c.?
Ans. 3 lb. 10 oz. 7 dwt. 5 gr.

17th. - 72015 hours to weeks, &c.?
Ans. 4228 weeks 4 days 15 hrs.

18th. - 783456$\frac{3}{4}$ sq. ft. to acres, &c.?
Ans. 17 ac. 3 ro. 37 po. 21 yds. 4$\frac{1}{2}$ ft.

19th. - $\frac{2}{5}$ d. to the fraction of a £.?.................. *Ans.* $\frac{1}{600}$.
20th. - $\frac{5}{9}$ scruple - - lb. apoth.?......*Ans.* $\frac{5}{2592}$ lb.
21st. - $\frac{43}{47}$ yd. - - mile?..........*Ans.* $\frac{43}{82720}$ m.
22d. - 26$\frac{2}{5}$ sq. yds. - acre?..............*Ans.* $\frac{3}{550}$ ac.
23d. - 7$\frac{2}{3}$ sq. chain - acre?...............*Ans.* $\frac{23}{30}$ ac.
24th. - 11$\frac{2}{3}$ pints - - hhd?............*Ans.* $\frac{5}{216}$ hhd.
25th. - 250 sq. yds. - chain?............*Ans.* $\frac{125}{242}$ ch.
26th. - 4$\frac{3}{4}$ pence - - £.?.................*Ans.* $\frac{19}{960}$ £.
27th. - 16$\frac{4}{21}$ grs. - - lb. troy?.........*Ans.* $\frac{17}{6048}$ lb.
28th. - 134 poles - - acre?................*Ans.* $\frac{67}{80}$ ac.
29th. - 3$\frac{1}{11}$ yds. sq. - pole?............*Ans.* $\frac{136}{1331}$ po.
30th. - 150 dwts. - - lb. troy?.............*Ans.* $\frac{5}{8}$ lb.
31st. - ·126 d. to the decimal of a £.?.........*Ans.* ·000525 £.
32d. - 30 sq. yds. - - pole?.......*Ans.* ·99173+ po.
33d. - 125·5 links l. m. - yd.?............*Ans.* 27·61 yd.
34th. - 864 sq. ft. - - acre?.......*Ans.* ·0198347 ac.
35th. - 10$\frac{1}{2}$ inches - - fathom? *Ans.* ·14583+ fath.
36th. - 9 d. - - - £.?...............*Ans.* ·0375 £.
37th. - 1 dwt. - - lb. troy?...*Ans.* ·004166+ lb.
38th. - 7 drams - - lb. av.?....*Ans.* ·02734375 lb.
39th. - 24 yds. - - mile?...... *Ans.* ·013636+m.
40th. - ·056 po. - - acre?..........*Ans.* ·00035 ac.
41st. - 1·2 pints - - hhd.?.......*Ans.* ·00238 hhd.
42d. - 14 min. - - day?....*Ans.* ·009722+ day.
43d. - 1594·6 links - mile?.........*Ans.* ·199325 m.
44th. - $\frac{5}{9}$ of 4 d. - - sixp.?........*Ans.* 3703+6 d.

310. It may be required to reduce a compound number to a fraction expressed in terms of the highest denomination contained in that compound number, or in the weight or measure to which it belongs; as, for example, to reduce 9 sh. 8$\frac{3}{4}$ d. to the decimal of a pound.

The reduction of a number expressed in terms of the units of any denomination to an equivalent number expressed in terms of the units of the next higher denomination is made by dividing the given number by a divisor which is the same multiple of unity as one unit of the denomination required is of that which is given.

Whence 3 qrs. = (3 ÷ 4) d. = ·75 d.,
and 9 sh. 8$\frac{3}{4}$ d. = 9 sh. 8·75 d.
In like manner 8·75 d. = (8·75 ÷ 12) sh. = ·729166 sh.
∴ 9 sh. 8·75 d. = 9·729166 sh.
And 9·729166 sh. = (9·729166 ÷ 20) £. = ·48645833 £.
∴ 9 sh. 8$\frac{3}{4}$ d. = ·4864583$\dot{3}$ £.

The calculation may be made thus,

$$\begin{array}{r|l} 4 & 3 \text{ qrs.} \\ \hline 12 & 8{\cdot}75 \text{ d.} \\ \hline 20\left\{\begin{array}{r} 10 \\ 2 \end{array}\right. & \begin{array}{l} 9{\cdot}729166+\text{sh.} \\ \hline {\cdot}9729166 \end{array} \\ \hline & {\cdot}48645833+\pounds. \end{array}$$

If it is required to reduce the same compound number to the vulgar fraction of a £, the reduction may be made in the same manner, thus,

$$3 \text{ qrs.} = \frac{3}{4}\text{d.}$$

$$8\frac{3}{4}\text{d.} = \frac{35}{4}\text{d.} = \left(\frac{35}{4}\times\frac{1}{12}\right)\text{sh.} = \frac{35}{48}\text{sh.}$$

$$9\frac{35}{48}\text{sh.} = \frac{467}{48}\text{sh.} = \left(\frac{467}{48}\times\frac{1}{20}\right)\pounds. = \frac{467}{960}\pounds.$$

$$\therefore 9\text{ sh. } 8\frac{3}{4}\text{d.} = \frac{467}{960}\pounds.$$

Or, by reducing the given compound number to the lowest denomination contained in it by Article 301, and reducing this result to the fractional expression of the required denomination by Article 308, as follows:

$$\begin{array}{l} 9\text{ sh. } 8\frac{3}{4}\text{ d.} \\ \underline{12} \\ 116\text{ d.} \\ \underline{\ \ 4} \\ 467\text{ qrs.} \end{array}$$

$$467\text{qrs.} = \left(\frac{467}{1}\times\frac{1}{4}\times\frac{1}{12}\times\frac{1}{20}\right)\pounds. = \frac{467}{960}\pounds.$$

311. Whence, to reduce a compound number to an equivalent fraction expressed in terms of the highest denomination contained in that compound number, or in the weight or measure to which it belongs;

Rule. Reduce the lowest denomination of the given number to a fraction of the next higher denomination, prefix the units of this next higher denomination, reduce the resulting mixed number to the next higher denomination, prefix the units of this denomination, and in this manner proceed to the denomination required.

Or, reduce the given compound number to units of the lowest denomination contained in it, and reduce this number to the denomination required.

312. Exercises on this Rule.

1st. Reduce 3 sh. 4 d. to the fraction of a £. ?............*Ans.* $\frac{1}{6}$£.
2d. - 12 sh. 6 d. - - - £.?...............*Ans.* $\frac{5}{8}$£.
3d. - 3 qrs. 19¼ lb. - - cwt. ?........*Ans.* $\frac{59}{64}$ cwt.
4th. - 3 yds. 2 ft. 6 in. - - mile ?.....*Ans.* $\frac{23}{10560}$ m.
5th. - 100 yds. 6¼ ft. - - acre ?.....*Ans.* $\frac{1813}{87120}$ ac.
6th. - 12 cwt. 2 qrs. 14 oz. - ton ?.........*Ans.* $\frac{1601}{2560}$ t.
7th. - 25 po. 4⅔ yds. - - mile ?.....*Ans.* $\frac{853}{10560}$ m.
8th. - 3 sh. 7½ d. - - crown ?.......*Ans.* $\frac{29}{40}$ cr.
9th. - 3 ro. 12½ yds. - - acre ?......*Ans.* $\frac{1457}{1936}$ ac.

10th. Reduce 56 gal. $1\frac{9}{19}$ qts. to the fraction of a hhd.?...*Ans.* $\frac{17}{19}$ hhd.
11th. - 3 sh. 6·15 d. - - £.?......*Ans.* $\frac{281}{1600}$ £.
12th. - 1 f. 16 po. 3 y. 2 ft. - - mile?..*Ans.* $\frac{187}{1056}$ m.
13th. - $11\frac{13}{18}$ lines long mea. - foot?....*Ans.* $\frac{211}{2592}$ f.
14th. - 7 in. $5\frac{2}{3}$ l. l. m. - - ft.?........*Ans.* $\frac{262}{432}$ f.
15th. - $27\frac{1}{9}$ sq. in. - - sq. ft.?...*Ans.* $\frac{61}{324}$ f.
16th. - 5 dwt. 9 grs. - - lb.?.....*Ans.* $\frac{43}{1920}$ lb.
17th. - 4 bush. 1 pk. 1 gal. $2\frac{2}{9}$ qts. to the fraction of a quarter?......................................*Ans.* $\frac{5}{9}$ qr.
18th. - 7 hours 12 min. to the fraction of a day?.*Ans.* $\frac{3}{10}$ day.
19th. - 5 £. 17 sh. $4\frac{1}{4}$ d. to the decimal of a £? *Ans.* 5·8677083+£.
20th. - 10 oz. 6 gr. to the decimal of a lb. troy? *Ans.* ·8343 lb.
21st. - 20 cwt. 2 qr. 14 lb. 10 dr. to the decimal of a ton? *Ans.* 1·0312674+t.
22d. - 6 fur. 30 po. $2\frac{1}{2}$ yds. to the decimal of a mile? *Ans.* ·84517 m.
23d. - 2 r. 15 p. 21 y. $7\frac{3}{8}$ ft. to the decimal of an acre? *Ans.* ·598258 ac.
24th. - 3 bush. 3 pks. 1 pt. to the decimal of a quarter? *Ans.* ·470703 qr.
25th. - 25 po. 2 yds. 2 ft. 9 in. to the decimal of a furlong? *Ans.* ·63825+fur.
26th. - 15 sh. $9\frac{3}{4}$ d. to the decimal of a £.?..*Ans.* ·790625 £.
27th. - 19 £. 17 sh. $3\frac{1}{4}$ d. to the decimal of a £.? *Ans.* 19·8635416 £.
28th. - 7 oz. 3 dwt. 12 gr. to the decimal of a lb.? *Ans.* ·597916 lb.
29th. - 3 qrs. $19\frac{3}{4}$ lb. to the decimal of a cwt.? *Ans.* ·92639 cwt.
30th. - 3 ro. 14 po. to the decimal of an acre?.*Ans.* ·8375 ac.
31st. - 46 gal. 2 pts. to the decimal of a hhd.? *Ans.* ·73412 hhd.
32d. - 5 h. 48 m. 48 sc. to the decimal of a day? *Ans.* ·2422+d.
33d. - 4 po. 5 yd. $4\frac{1}{2}$ ft. to the decimal of an acre? *Ans.* ·026136 ac.
34th. - $\frac{1}{8}$ of 2 cwt. 17 lb. 10 oz. to the decimal of a cwt.? *Ans.* ·2696707 cwt.
35th. - $\frac{2}{9}$ of 5 oz. 3 dwt. 10 gr. to the decimal of a lb. troy? *Ans.* ·095756 lb.

313. Lastly, it may be required to reduce a number expressed in terms of one denomination, or of the successive denominations of one weight or measure, to an equivalent number expressed in terms of some other weight or measure, with which the former has a fixed relation.

For instance, it may be required to reduce any given weight from troy or from apothecaries weight to avoirdupois weight, and conversely; or to reduce cubic measure to the measure of capacity, and conversely.

Let it be required to reduce 6 lbs. 9 oz. 15 dwts. troy weight to lbs. oz., &c. avoirdupois.

Among the different relations between troy and avoirdupois weight, given in Article 297, it is stated that 7000 grains troy=1 lb. avoirdupois.

If, therefore, the given weight 6 lb. 9 oz. 15 dwts. is reduced to grains, and the number of grains is divided by 7000, the result is pounds avoirdupois.

$$6\text{ lbs. } 9\text{ oz. } 15\text{ dwts.}=39240\text{ grs.}$$

$$\therefore 6\text{ lbs. } 9\text{ oz. } 15\text{ dwts. troy}=\frac{39240}{7000}\text{ lbs. av.}$$

$$\text{And } \frac{39240}{7000}\text{ lb. av.}=5{\cdot}60571\dot{4}\text{ lbs.}=5\text{ lb. } 9\text{ oz. } 11{\cdot}0628\text{ drs.}$$

$$\therefore 6\text{ lbs. } 9\text{ oz. } 15\text{ dwts. troy}=5\text{ lbs. } 9\text{ oz. } 11{\cdot}0628\text{ drs. av.}$$

314. Whence the rule for such reductions is,

Reduce the given quantity to that denomination which is at once common to the weight or measure by which the quantity is expressed and to the weight or measure to which its expression is to be changed; then reduce from this common denomination to the required denomination in the changed weight or measure.

315. Exercises in the reduction of compound numbers expressed in terms of one weight or measure to equivalent numbers expressed in terms of some other weight or measure.

1st. Reduce 12 lbs. 10 oz. troy to lbs. avoir.?...*Ans.* $10\frac{14}{25}$ lbs. avoir.

2d. - $10\frac{1}{3}$ oz. avoir. to grs. troy?.........*Ans.* $4520\frac{5}{6}$ grs. troy.

3d. - 1000 lbs. troy to lbs. avoir.?........*Ans.* $822\frac{6}{7}$ lbs. avoir.

4th. - 1000 lbs. avoir. to lbs. troy?...*Ans.* 1215 lb. 3 oz. 6d. 16 gr.

5th. - $\frac{7}{8}$ lb. troy to the fraction of 1 lb. avoir.?
Ans. $\frac{18}{25}$ lbs. avoir.

6th. - $\frac{7}{8}$ lb. avoir. to the fraction of 1 lb. troy?
Ans. $\frac{1225}{1152}$ lb. troy.

7th. - ·4785 lb. troy to the decimal of 1 lb. avoir.?
Ans. ·393737 lb. avoir.

8th. - ·006432 lb. avoir. to the decimal of 1 lb. troy?
Ans. ·007816 lb. troy.

9th. - 1 cubic foot to gallons? ....*Ans.* 6·232106 gallons.

10th. - 1 gallon to cubic feet?........*Ans.* 0·160459 cubic feet.

11th. The solid measure or volume of a cistern being $6\frac{3}{4}$ cubic feet, it is required to find how many gallons of water it is capable of containing?................................*Ans.* 42·066715 gallons.

12th. How much heavier is the cistern when filled with water than it is when empty; the weight of 1 gallon being 10 lbs. avoir.?
Ans. 420·66715 lbs.=3 cwt. 3 qrs. 0·66715 lbs.

316. Compound numbers expressed in terms of the weights and measures used in any country may be reduced to equivalent expressions in terms of the weights and measures used in any other country by the rule of Article 314.

To render such reductions possible, fixed relations must be given between the weights and measures of the countries from which and into which the reductions are to be made.

For tables of foreign weights and measures, and of the relations between these and the British, reference may be made to Buchanan on Weights and Measures, the Encyclopædia Britannica, article Weights and Measures, Kelly's Metrology and Universal Cambist, M'Culloch's Commercial Dictionary, &c.

10th. Reduce 56 gal. 1 … of weights and measures,
11th. - 3 sh … or 3·2808992 feet, or 1·093633
12th. - 1
13th. - 1 … yards=·0247114 acres.
14th. - … cubic yards.
15th. - … cubic inches=·22009687 gallons.
16th. - … grains troy.
… pence.
17th. … grammes=2·679514 lbs. troy or 2·204857 lbs.

18th. … of length,=76·73511 inches, or 6·3945925 feet, or
19th.

… it is found that
20th … 1 inch=·02539954 metres.
1 foot=·3047945 metres.
1 yard=·91438353 metres.
2 … 1 square yard=·00831 ares.
1 acre=40·46715 ares.
1 cubic foot=·028315301 stere.
1 cubic yard=·764513135 stere.
1 gallon=4·543454 litres.
1 grain troy=·06479202 grammes.
1 lb. troy=373·2020352 grammes.
1 lb. avoirdupois=453·54414 grammes.
1 lb. avoirdupois=·45354414 kilogramme.
1 penny=·1031991744 franc.
1 shilling=1·238390093 franc.
1£.=24·76780186 francs.

To show the application of these relative numbers, let it be required,

1st. *To* find in British money the value of 1000000 francs.

1000000 f.=(1000000×9·69) d.=9690000 d.=40375£.

2d. To reduce 1 cwt. to kilogrammes.

1 cwt.=112 lb.=(112×·45354414) kilogr.=50·79694368 kilogr.

3d. To express 1 mile, in metres.

1 mile=1760 yds.=(1760×·91438353) metres=1609·3150128 metres.

4th. Reduce 1000 ares to acres, &c.?.....*Ans.* 24 ac. 2 r. 33 p. 24 y. 8·334 ft.
5th. - 75£. 10 sh. 4 d. to francs?.............*Ans.* 1870·381838 francs.
6th. - 1 hhd. 21 gall. to litres?..................*Ans.* 287·752087 litres.
7th. - 1 mile to toises?..............................*Ans.* 825·6977 toises.
8th. - 20 francs to shillings, &c.?...............*Ans.* 16 sh. 1 d. 3¼ qrs.

317. By the rules of reduction a compound number can be expressed in terms of any single denomination contained in the weight or measure to which that compound number belongs.

Therefore compound numbers, upon which any of the elementary operations of arithmetic are to be performed, may be reduced to the same denomination; and the calculation being made by the rules for whole or fractional numbers, the form of expression of the result, as a compound number, may be restored.

But these reductions are tedious, especially when the numbers are large. Besides, any calculation can, in general, be more easily made with the unreduced compound numbers.

Every denomination of a compound quantity, individually considered, is referred to the decimal system; hence any calculation, made with numbers of one denomination, is made as with whole numbers.

The several denominations also are so arranged, that the lowest being written first, the others are made to occupy successive places on the left of the lowest and of each other, according to their relative values. The relation of these values being fixed by an arbitrary weight or measure, which involves the principle (common to it with other numbers) that a certain number of units of the lowest denomination make one unit of the second, which denomination is written immediately on the left of the lowest; a certain number of units of the second make one unit of the third, which is written on the left of the second, &c. &c.

Whence, if each denomination is considered as forming a distinct order of units, it is evident that the only difference between an arithmetical operation performed upon whole and upon compound numbers lies in the different manner of passing from one order of units to another. In whole numbers the transitions are by tens and tenths; in compound numbers by multiples and parts, which vary from one denomination to another, and which are fixed for each denomination by the table of the weight or measure to which a given compound number belongs.

The elementary operations of arithmetic, with whole numbers and fractions, are Addition, Subtraction, Multiplication, and Division. The same may be performed with compound numbers.

Addition of Compound Numbers.

318. Since any arithmetical operation with numbers composed of different denominations differs from the same operation with numbers composed of different orders of units in respect only that 1 is carried to a higher denomination, not for 10 of the next lower, but for some other number fixed by the table of an arbitrary weight or measure, it follows that, in all other respects, the addition of compound is made like that of whole numbers.

Examples of the addition of compound numbers.

1st. What is the sum of 3 cwt. 3 qrs. 12 lbs.; 5 cwt. 1 qr. 24 lbs; 2 cwt. 3 qrs. 18 lbs.; and 6 cwt. 2 qrs. 19 lbs.?

In forming the sum of compound numbers it is convenient, as in the case of whole numbers, to begin with the lowest denomination; in this instance lbs. avoirdupois.

$$(19+18+24+12)\text{ lbs.}=73\text{ lbs.}$$

As 28 lbs.=1 qr. 27 is the greatest number which can be properly written in the place of lbs., the sum of the lbs., 73, must therefore be reduced to qrs. and lbs. Dividing by 28 (the number of lbs. which make 1 qr.), it is found that 73 lbs. are equal to 2 qrs. and 17 lbs. The 17 lbs. are reserved, as expressing the lbs. of the result, and the qrs. are carried to the denomination of qrs.

$$\text{Next, 2 qrs. carried}+(2+3+1+3)\text{ qrs.}=11\text{ qrs.}$$

$$4\text{ qrs.}=1\text{ cwt.}\therefore 11\text{ qrs.}=2\text{ cwt. }3\text{ qrs.}$$

The 3 qrs. are reserved for the qrs. of the sum, and the 2 cwts. are carried to the denomination of cwts.

Lastly 2 cwt. carried +(6+2+5+3) cwt.=18 cwt.

With cwts. the calculation ends; and the sum of the given compound numbers is 18 cwt. 3 qrs. 17 lbs.

The process of addition being explained, it remains to be noticed, that if the different denominations of any given compound numbers are regarded as different orders of units, the arrangement of these numbers for addition may be made by the rule of Article 23.

The arrangement and sum of the preceding compound numbers are as follow;

cwts.	qrs.	lbs.
3	3	12
5	1	24
2	3	18
6	2	19
18	3	17
	↓	↓

2d. What is the sum of 5 lb. 3 oz. 18 dwt.; 16 lb. 11 oz.; 283 lb. 15 dwt. 9 gr.; 9 oz. 16 dwt. 5⅓ gr.; and 2 lb. 8 oz. 17⅝ gr.

The most direct course is, first, to arrange the given numbers as in the last example, and second, to find their sum. It is often convenient and sometimes necessary to make certain changes or reductions of the given numbers to render them capable of a proper arrangement for addition. Of the first description is the writing of zeros to occupy the places of grains in the first and of dwts. and grs. in the second of the given numbers.

Of the second, are the writing of zeros to occupy the places of the deficient ounces in the third and the deficient dwts. in the last of the given numbers, and the reduction to the same denominator of the fractions $\frac{1}{2}$ gr. and $\frac{5}{8}$ gr. in the fourth and fifth of the given numbers.

The more distinctly to exhibit these reductions, and the use or cause of making them, the given and the reduced numbers are subjoined in horizontal lines, the sum being found by addition of the latter.

		lb.	oz.	dwt.	gr.
5 lb. 3 oz. 18 dwt.	-	5	3	18	0
16 lb. 11 oz. -	-	16	11	0	0
283 lb. 15 dwt. 9 gr.	-	283	0	15	9
9 oz. 16 dwts. $5\frac{1}{2}$ gr.	-	0	9	16	$5\frac{4}{8}$
2 lb. 8 oz. $17\frac{5}{8}$ gr.	-	2	8	0	$17\frac{5}{8}$
		308	9	10	$8\frac{1}{8}$
			$\cancel{2}$	$\cancel{2}$	$\cancel{1}$

Detail of the addition.

$(\frac{5}{8}+\frac{4}{8})$ gr. $=\frac{9}{8}$ gr. $=1\frac{1}{8}$ gr;
$\frac{1}{8}$ is written under the fractions, and 1 carried to the grains.
$(1+17+5+9)$ gr. $=32$ gr. $=1$ dwt. 8 gr.;
8 is written under the grains, and 1 carried to the dwts.
$(1+16+15+18)$ dwt. $=50$ dwt. $=2$ oz. 10 dwt.;
10 is written under the dwts., and 2 carried to the ounces.
$(2+8+9+11+3)$ oz. $=33$ oz. $=2$ lb. 9 oz.;
9 is written under the ounces, and 2 carried to the pounds.
$(2+2+283+16+5)$ lb. $=308$ lb.;
the number 308 is written under the pounds; and with the addition of this, the last denomination, the process is ended.

3d. What is the sum of 34 £. 6 sh. $8\frac{3}{4}$ d.; 8 £. $15\frac{4}{5}$ sh.; 59 £. 17 sh. $6\frac{5}{8}$ d.; 128 £. 5·324 d.; and 538 £. 19 sh. $7\frac{1}{2}$ d.?

Reduction and arrangement of the given numbers.

		£	sh.	d.	qrs.
34 £. 6 sh. $8\frac{3}{4}$ d.	-	34	6	8	3
8 £. $15\frac{4}{5}$ sh. -	-	8	15	9	2·4
59 £. 17 sh. $6\frac{5}{8}$ d.	-	59	17	6	2·5
128 £. 5·324 d.	-	128	0	5	1·296
538 £. 19 sh. $7\frac{1}{2}$ d.	-	538	19	7	2
		770	0	1	3·196
		$\cancel{1}\cancel{4}$	$\cancel{3}$	$\cancel{3}$	$\cancel{2}\cancel{1}$

Remarks on this example.

The farthings in this example are written as a distinct denomination. They are so written because fractional parts of a farthing occur in some of the addenda. When this does not happen, farthings are expressed as fractional parts of a penny.

In the second of the given compound numbers $\frac{4}{5}$ sh. is reduced to its value in pence and farthings. This is a necessary reduction; for fractional parts of a shilling cannot be left unreduced in any compound number which contains pence as well as shillings.

$\frac{2}{5}$ of a farthing are found to form part of the value of $\frac{4}{5}$ sh. This $\frac{2}{5}$ qr. is reduced to a decimal fraction, because the fourth of the given numbers contains a decimal fraction, and it is required to find the sum of the fractional parts as well as of the several denominations of the given numbers.

In the third of the given compound numbers $\frac{5}{8}$ d. is reduced to farthings and decimal parts of a farthing.

In the fourth, zero is written to fill the place of the deficient denomination, shillings; and ·324 d. is reduced to farthings and decimal parts of a farthing.

In the last, $\frac{1}{2}$ d. is written as 2 qrs.

When these reductions have been made, the compound numbers arranged as in the preceding examples, and the decimal fractions as directed Art. 226, the addition is commenced with the decimals.

The sum of the decimal fractions is 1·196 qrs.; of this ·196 is written under the decimal figures of the addenda, and 1 qr. carried to the farthings.

The sum of the column of farthings is 10, which, with 1 qr. carried,= 11 qrs.=2 d. 3 qrs. 3 is written under the column of qrs. and 2 carried to the column of pence.

The sum of the column of pence is 35, which, with 2 d. carried,=37 d.= 3 sh. 1 d. 1 d. is written under the column of pence, and 3 sh. carried to the column of shillings.

The sum of the column of shillings is 57, which, with 3 sh. carried,=60 sh. =3 £. 0 sh. 0 is written under the column of shillings, and 3 £. carried to the column of pounds.

The sum of the column of pounds is 767 £., and 767 £.+3 £. carried= 770 £., which sum is written under the column of pounds.

Therefore the sum of the given compound numbers is 770 £. 0 sh. 1 d. 3·196 qrs.

Rule for the addition of compound numbers.

Replace fractional parts of any denomination higher than the lowest which is contained in the given numbers by the values of such fractional parts expressed in terms of these lower denominations, and fractions whose denominators are different by equivalent fractions reduced to the same denominator, and render the lowest denomination of all the compound numbers the same by writing zeros in the places of such denominations as are wanting.

Then arrange the addenda in such a manner that the numbers of the lowest denomination may fall in the same vertical column; the numbers of the denomination immediately superior to the lowest in a second vertical column on the left of the former; and so on, to the highest denomination, which will be on the left of all the others.

Find the sum of the numbers, whole or mixed, in the right column; reduce this sum to a compound number expressed in terms of the last and the next higher denominations; write the whole or mixed number which expresses the units of the last denomination directly under the column to which it belongs, and carry the units of the next higher denomination to the second vertical column. Find the sum of the numbers contained in the second column plus the units carried from the first; reduce this sum to a compound number expressed in terms of the second and the next higher denominations; write the units of the second denomination in their proper place; and carry the units of the third denomination to the third vertical column. And in this manner proceed till the sum of all the columns has been found.

The result is the sum of the given compound numbers.

Exercises in addition of compound numbers.

1st. Required the sum of 74 £. 18 s. $11\frac{3}{4}$ d.; 96 £. 9 s. $10\frac{1}{2}$ d.; 58 £. 17 s. $8\frac{1}{4}$ d.; and 63 £. 11 s. $9\frac{1}{2}$ d.?

Ans. 293 £. 18 s. 4d.

2d. Required the sum of 84 £. 18 s. $11\frac{1}{2}$ d.; 63 £. 16 s. $10\frac{1}{4}$ d.; 95 £. 4 s.; and 78 £. 10 s. $9\frac{3}{4}$ d ?....*Ans.* 322 £. 10 s. $7\frac{1}{2}$ d.

3d. Required the sum of 74 tons 19 cwt. 3 qrs. 27 lb. 15 oz.; 85 tons 17 cwt. 2 qrs. 24 lbs. 14 oz.; 68 tons 13 cwt. 1 qr. 20 lb. 12 oz.; 52 tons 18 cwt. 3 qrs. 19 lb. 8 oz.; 50 tons 10 cwt. 2 qrs. 18 lb. 6 oz.; 48 tons 9 cwt. 3 qrs. 16 lb. 10 oz.; and 97 tons 5 cwt. 1 qr. 3 lb. 15 oz.? *Ans.* 478 tons 15 cwt. 3 qrs. 20 lbs.

4th. Required the sum of 48 lbs. 11 oz. 18 dwts. 21 grs.; 42 lbs. 10 oz. 14 dwts. 18 grs.; 40 lbs. 9 oz. 16 dwts. 20 grs.; 36 lbs. 8 oz. 15 dwts. 22 grs.; 38 lbs. 10 oz. 10 dwts. 14 grs.; 53 lbs. 6 oz. 17 dwts. 13 grs.? *Ans.* 261 lbs. 10 oz. 14 dwts. 12 grs.

5th. Required the sum of 26 ac. 3 ro. 23 po. $20\frac{1}{4}$ yds.; 40 ac. 1 ro. 39 po. 29 yds.; 15 ac. 2 ro. 17 po. $30\frac{1}{8}$ yds.; 10 ac. 1 ro. 30 po. $10\frac{1}{2}$ yds.?...*Ans.* 93 ac. 1 ro. 31 po. $29\frac{3}{8}$ yds.

6th. Required the sum of 28 mi. 7 fur. 23 po. $3\frac{3}{4}$ yds.; 117 mi. 5 fur. $5\frac{1}{4}$ yds.; 29 mi. $3\frac{2}{3}$ yds.; and $5016\frac{1}{2}$ yds.? *Ans.* 178 mi. 3 fur. 17 po. $1\frac{1}{8}$ yds.

7th. Required the sum of $24\frac{1}{3}$ ac. $5067\frac{1}{2}$ sq. yds. and 50000 sq. links? ....*Ans.* 25 ac. 3 ro. 20 po. $25\frac{5}{8}$ yds.

8th. Required the sum of $25\frac{2}{7}$ hhds.; $1159\frac{1}{2}$ gal.; $13\frac{1}{5}$ pipes; and 4728 qts.?.........*Ans.* 88 hhds. 53 gals. 2 qts. $1\frac{3}{5}$ pts.

9th. Required the sum of 25 cwt. 2 qrs. 17 lbs.; 6 cwt. 86 lbs.; and 15 cwt. 3 qrs. 12·7 lbs.?...*Ans.* 48 cwt. 1 qr. 3·7 lbs.

10th. Required the sum of 126·75 £.; 13·025 sh.; 16 £. 17·75 sh.; and 11·98 d.? ....*Ans.* 144 £. 6 sh. 2·32 d.

11th. Required the sum of 2·15 £.; 31·017 £.; and 19 sh. $11\frac{1}{4}$ d.? *Ans.* 34 £. 3 sh. 3·33 d.

12th. Required the sum of $\frac{5}{12}$ £. and $\frac{3}{4}$ crown (1 crown=5 sh.)? *Ans.* 12 sh. 1 d.

13th. Required the sum of $\frac{3}{4}$ lb. troy and $\frac{1}{6}$ oz. troy? *Ans.* 9 oz. 3 dwts. 8 grs.

14th. Required in troy weight the sum of ·0678 lb. troy and ·9 lbs. avoir.?.........*Ans.* 1 lb. 1 oz. 18 dwts. 18·528 grs.

15th. Required the sum of 1 £. 16 sh. 9 d.; $8\frac{5}{6}$ £.; and 10.58 £.? *Ans.* 21 £. 5 sh. $0\frac{1}{5}$ d.

16th. Required the sum of 16 mi. $30\frac{1}{2}$ po.; $107\frac{1}{3}$ fathoms; and $509\frac{1}{6}$ yds.? ....*Ans.* 16 mi. 4 fur. 2 po. $0\frac{7}{12}$ yds.

17th. Required the sum of 4 £. $13\frac{1}{3}$ sh.; 2 £. $3\frac{1}{8}$ sh.; and $\frac{5}{8}$ of 3 £. 12 sh. 6 d.?.......................*Ans.* 9 £. 1 sh. $6\frac{3}{4}$ d..

18th. Required the sum in avoirdupois weight of $3\frac{2}{9}$ lbs. troy and $16\frac{1}{3}$ lbs. avoirdupois? *Ans.* 18 lbs. 15 oz. $12\frac{52}{525}$ dr. avoir.

SUBTRACTION OF COMPOUND NUMBERS.

319. The difference of two whole numbers is obtained by taking the units of each order in the subtrahend, beginning with the lowest, from the units of the corresponding order in the minuend, borrowing, when the partial subtrahend exceeds the minuend, from the next higher order of units of the minuend, to render the subtraction possible (Art. 39).

The difference of two compound numbers is obtained in the same manner (Art. 317).

Examples of the subtraction of compound numbers.

1st. From 35 lbs. $8\frac{1}{7}$ ozs. troy weight, take 19 lbs. 10 ozs. 13 dwts. $8\frac{1}{4}$ grs.

The reduction of fractional expressions to compound numbers and to common denominators, with other preparations requisite for the addition of compound numbers, are also required for the subtraction of one compound number from another.

In the present example it is necessary to find the value of $\frac{1}{7}$ oz. which by Article 304=2 dwts. $20\frac{4}{7}$ grs.

It is also necessary to reduce $\frac{4}{7}$ gr. and $\frac{1}{4}$ gr. to the same denominator. The fractions thus reduced are $\frac{16}{28}$ gr. and $\frac{7}{28}$ gr.

	lbs.	ozs.	dwts.	grs.
∴ 35 lbs. $8\frac{1}{7}$ ozs. - - - - -	35	8	2	$20\frac{16}{28}$
19 lbs. 10 ozs. 13 dwts. $8\frac{1}{4}$ grs. - -	19	10	13	$8\frac{7}{28}$
	15	9	9	$12\frac{9}{28}$

Detail of the process.

The subtraction is commenced with the lowest denomination, grains.

1st. $(20\frac{16}{28}-8\frac{7}{28})$ grs.$=12\frac{9}{28}$ grs. (Art. 184).

The remainder, $12\frac{9}{28}$, is written under the column of grs.

2d. (2—13) dwts., without borrowing the subtraction is not possible; 1 of the next higher denomination (ounces) is borrowed:

1 oz.=20 dwts. (20+2) dwts.=22 dwts. and (22—13) dwts.=9 dwts.

The remainder, 9, is written under the column of dwts., and 1, for that borrowed, is carried to the ounces of the subtrahend.

3d. 10 ozs.+1 oz. borrowed=11 ozs.

(8—11) ozs., without borrowing the subtraction is not possible; 1 of the next higher denomination (lbs.) is borrowed:

1 lb.=12 ozs. and (12+8) ozs.=20 ozs. and (20—11) oz.=9 ozs.

9 is written under the column of ozs. and 1 carried to the simple units of the lbs. for the pound borrowed.

4th. (1+19) lbs.=20 lbs. and (35—20) lbs.=15 lbs.

15 is written under the column of lbs.; and the difference of the given compound numbers is found to be equal to 15 lbs. 9 ozs. 9 dwts. $12\frac{9}{28}$ grs.

In subtraction of whole numbers, when the partial subtrahend exceeds the minuend, 1 of the next higher order of units (and equal to 10 units of the partial subtrahend or minuend) is borrowed, and added to the partial minuend, to render the subtraction possible.

In subtraction of compound numbers, when the partial subtrahend is greater than the minuend, 1 is borrowed from the next higher denomination, reduced to the denomination of the partial minuend, and added to it. The number so added is not 10 or any constant number, but depending on the number of units of the partial minuend which make 1 unit of the next higher denomination of that weight or measure by which the given compound numbers have been valued.

In this consists the difference between subtraction of whole and subtraction of compound numbers.

2d Example. From 579 £. 12 sh. subtract 289 £. 17 sh. 6 d. $2\frac{3}{7}$ qrs.

Writing zeros to fill the places of the deficient pence and farthings of the minuend,

	£	sh.	d.	qrs.
579 £. 12 sh. - -	579	12	0	0
From which subtract	289	17	6	$2\frac{3}{7}$
The remainder is -	289	14	5	$1\frac{4}{7}$

Detail of the process.

1st. Subtraction of the farthings:

0 qrs.—$2\frac{3}{7}$ qrs., the subtraction is not possible; one of the next higher denomination (pence) is borrowed:

1 d.=4 qrs. and 4 qrs.—$2\frac{3}{7}$ qrs.=$1\frac{4}{7}$ qrs.

$1\frac{4}{7}$ is written under the column of farthings, and 1 is carried to the pence of the subtrahend for the penny borrowed.

2d. Subtraction of the pence:
1 d.+6 d.=7 d., partial subtrahend.
0 d.−7 d., to render the subtraction possible, 1 sh.=12 d. is borrowed from the next higher denomination.

12 d.−7 d.=5 d.

5 is written under the column of pence, and 1 for that borrowed is carried to the shillings of the subtrahend.

3d. Subtraction of the shillings:
1 sh.+17 sh.=18 sh., partial subtrahend.
12 sh.−18 sh.; to render the subtraction possible, 1 £.=20 sh. is borrowed from the next higher denomination.

20 sh.+12 sh.=32 sh. and 32 sh.−18 sh.=14 sh.

14 is written under the column of shillings, and 1, for 1 £. borrowed, is carried to the pounds of the subtrahend.

4th. Subtraction of the pounds:
1 £.+289 £.=290 £., and 579 £.−290 £.=289 £.
289 is written under the column of pounds, and the result is found to be 289 £. 14 sh. 5 d. 1¾ qrs.

These examples and remarks seem sufficient to direct the process of subtracting one compound number from another.

The rule for such subtraction is,

Prepare the quantities in the same manner as for addition of compound numbers.

Write the minuend in a horizontal line, and under its several denominations the corresponding denominations of the subtrahend.

Subtract the number in the lowest denomination of the subtrahend from the number in the lowest denomination of the minuend, if the subtraction can be effected; but if it cannot, borrow one of the next higher denomination, reduce this to units of the lowest denomination, and add the result to the number in the lowest denomination of the minuend. Subtract the number in the subtrahend from the sum; carry one for that borrowed to the next denomination of the subtrahend; subtract the number thus augmented from the corresponding denomination of the minuend, if the subtraction is possible; if not, augment the number in the minuend by as many units of the denomination to which it belongs as make one unit of the next higher denomination; subtract the number in the subtrahend from the sum.

And proceed in this manner till the calculation is completed.

Exercises in subtraction of compound numbers.

1st. From 73 £. 15 sh. 10½ d. subtract 48 £. 12 sh. 6½d.?
Ans. 25 £. 3 sh. 4 d.

2d. From 708 £. 14 sh. 6¼ d. subtract 278 £. 17 sh. 10¾ d.?
Ans. 429 £. 16 sh. 7½ d.

3d. From 95 £. 8 sh. 7 d. subtract 68 £. 13 sh. 6½ d.?
Ans. 26 £. 15 sh. 0½d.

4th. From 17 cwts. 1 qr. 21 lbs. subtract 13 cwt. 2 qrs. 24 lbs?
Ans. 3 cwt. 2 qrs. 25 lbs.

5th. From 13 lb. 5 oz. 13 dwt. 17 gr. subtract 9 lb. 7 oz. 14 dwt. 21 grs.? ....*Ans.* 3 lbs. 9 oz. 18 dwt. 20 gr.

6th. From 18 tuns 2 hhds. 48 gals. subtract 13 tuns 2 hhds. 54 gals. 2 qts? ....*Ans.* 4 tuns 13 hhds. 56 gals. 2 qts.

7th. Required the difference between 17·92 £. and 26 £. 12 sh.? *Ans.* 8 £. 13 sh. $7\frac{1}{5}$ d.

8th. Required the difference between $1\frac{2}{7}$ acres and $3769\frac{2}{3}$ sq. yds.? *Ans.* 2 ro. 1 po. $2\frac{79}{84}$ yds.

9th. Required the difference between 20 mi. 3 fur. 25 po. $2\frac{1}{3}$ yds. and 13 mi. 7 fur. 37 po. $3\frac{7}{8}$ yds.? . *Ans.* 6 mi. 3 fur. 27 po. $3\frac{23}{24}$ yds.

10th. Required the difference between $\frac{1}{7}$ of 65·2 £. and 456·625 crowns?. .... *Ans.* 104 £. 16 sh. $10\frac{1}{14}$ d.

11th. From $2\frac{5}{9}$ miles take $1275\frac{1}{3}$ yds.? *Ans.* 1 mi. 6 fur. 25 po. $4\frac{17}{18}$ yds.

12th. Required the difference between 46 ac. 3 ro. 17 po. $17\frac{2}{3}$ yds. and $230\frac{5}{7}$ square chains?.... *Ans.* 23 ac. 3 ro. 6 po. $4\frac{59}{84}$ yds.

13th. Required the difference between $10075\frac{1}{5}$ links long measure and $20\frac{2}{7}$ miles? .... *Ans.* 19 mi. $8\frac{368}{875}$ po.

14th. From $\frac{3}{4}$ £. subtract $\frac{3}{4}$ sh.? .... *Ans.* 14 sh. 3 d.

15th. Find, in troy weight, the difference between $\frac{3}{8}$ lb. troy and $\frac{3}{8}$ lb. avoirdupois?............. *Ans.* 19 dwts. 9 gr. troy.

16th. Required the difference between 26·5 po. and $70\frac{3}{5}$ yds. sq. measure? .... *Ans.* 24 po. $5\frac{1}{40}$ yds.

17th. Required the difference between $2\frac{71}{80}$ £. and 3·125 £.? *Ans.* 4 sh. 9 d.

18th. Two pieces whose lengths are 25 feet $7\frac{1}{2}$ inches and 19 feet $10\frac{3}{5}$ inches are cut from a plank whose length is 17 yds. 2·7 feet. How much is left? *Ans.* 2 yds. 2 feet 2·3 in.

MULTIPLICATION OF COMPOUND NUMBERS.

320. In multiplication of whole numbers, if the multiplicand is repeated as often as the multiplier contains unity, the sum of these repetitions is the product of the numbers.

The repetitions of the multiplicand are made by repeating the units of each order contained in the multiplicand.

Consequently, if each of the denominations of a compound number is repeated as often as a multiplier contains unity, the result is the product of the compound number by that multiplier.

Examples of the multiplication of compound numbers.

1st. Multiply 3 miles 6 fur. 32 po. by 9.

The multiplicand and multiplier may be arranged as in whole numbers, thus,

$$\begin{array}{rrrl}
\text{m.} & \text{fur.} & \text{po.} & \\
3 & 6 & 32 & \text{multiplicand.} \\
 & & 9 & \text{multiplier.} \\
\hline
34 & 5 & 8 & \text{product.}
\end{array}$$

The product is equal to 32 po. × 9 + 6 fur. × 9 + 3 mi. × 9.

32 po. × 9 = 288 po. = 7 fur. 8 po.

8 is written under the poles of the multiplicand, and 7 reserved for addition to the product of the furlongs by 9.

6 fur. × 9 = 54 fur. and 54 + 7 carried = 61 fur. = 7 miles 5 fur.

5 is written under the furlongs of the multiplicand, and 7 carried to the product of the miles by 9.

3 mi. × 9 = 27 mi. and 27 + 7 carried = 34 miles.

34 is written under the miles of the multiplicand; and the required product is 34 miles 5 fur. 8 po.

The partial products of the several denominations of the multiplicand by the multiplier may be formed separately, and afterwards combined into one sum by the rules for the reduction and addition of compound numbers.

But the method taken in the preceding example, namely, that of making the reduction and the combination of the partial products to proceed simultaneously, is less laborious and more concise.

a. This method is, however, advantageous so long only as the multiplier is a number so small that the multiplications and reductions can be made mentally.

To extend the limits of mental calculation, a multiplication table to 12 × 12 is generally employed in the multiplication of compound numbers.

In this treatise a table of products from 1 × 1 to 25 × 25 is given. The labour of committing to memory the whole of this table would be considerable; but the part of it to 12 × 12 may be easily learnt.

2d Example. Multiply 8 £. 15 sh. 9¼ d. by 132.

The product required is equal to

$$\tfrac{1}{4}\text{ d.}\times 132+9\text{ d.}\times 132+15\text{ sh.}\times 132+8\text{ £.}\times 132.$$

But the multiplications and reductions of these partial products cannot be made mentally; therefore, unless some other process is devised, the partial products must be formed separately, and the complete product composed by addition of these partial products into one compound number.

What is required is to find a compound number equal to 132 repetitions of 8 £. 15 sh. 9¼ d; and this without the necessity of multiplying by a number too great to allow the partial multiplications and reductions to be made mentally.

Now, it is proved in Articles 69 and 71 that, whether a number is multiplied by any multiplier, or by the factors of that multiplier taken in any order, the product is the same,

As often, therefore, as a multiplier can be resolved into factors, each not greater than the greatest factor contained in the multiplication table which is used (generally 12), the product of a multiplicand which is a compound number by that multiplier can be obtained by repetitions of the method followed in Example 1.

132, the multiplier of Example 2, is composed of the prime factors 2^2, 3, and 11; but as any factor not greater than 12 can be used, it will be sufficient to decompose 132 into the factors 11 and 12, and to multiply the multiplicand by 11, and this product by 12.

The calculation is subjoined. A lengthened detail, after the explanation of the first example, seems unnecessary.

£	sh.	d.	
8	15	9¼	
		11	
96	13	5¾	=11 repetitions of multiplicand.
		12	
1160	1	9	=11 × 12 or 132 repetitions of the multiplicand.

The product required is 1160 £. 1 sh. 9 d. It could have been obtained by multiplying 8 £. 15 sh. 9¼ d. by 2, this product by 2, this by 3, and this again by 11; 2, 2, 3, 11, being the prime factors of 132. But it is more concise to multiply at once by 12 than successively by 2, 2, 3, the prime factors of 12.

b. Any multiplier which contains among its simple factors a prime number greater than 12 cannot be decomposed into factors of which none shall exceed 12, and whose continual product shall be equal to the given multiplier.

The method of Article 132 consequently fails for such examples.

But although the product cannot, in these cases, be obtained by continual multiplication of the multiplicand by the factors of the multiplier, it can be formed by breaking down the multiplier into parts which are either composed of factors not greater than 12, or are themselves not greater than 12, forming the separate products of the multiplicand by these parts of the multiplier, and taking the sum of the partial products (Art. 70).

To illustrate these remarks, let it be required.

3d Example. To multiply 12 lb. 3 oz. 16 dwt. troy by 59, the multiplier is a prime number (Art. 103); it cannot therefore be decomposed into any factors whose product shall be 59; but it can be broken into the parts 3 and 56, of which 3 is less than 12, and 56 is the product of the factors 7 and 8, both of which are also less than 12. And (denoting the multiplicand by m for the sake of brevity)

$$m\times7\times8+m\times3 \text{ or } m\times(56+3)=m\times59 \text{ (Art. 70).}$$

That is, if to the continual product of the multiplicand by the factors 7 and 8, the product of the multiplicand by 3 is added, the result is equal to 59 repetitions of the multiplicand, which is the number required.

Calculation.

$$\begin{array}{rrrl}
\text{lb.} & \text{oz.} & \text{dwt.} & \\
12 & 3 & 16 & \times 3 \\
 & & 7 & \\
\hline
86 & 2 & 12 & =7 \text{ repetitions of the multiplicand.} \\
 & & 8 & \\
\hline
689 & 8 & 16 & =7\times8 \text{ or } 56 \text{ repetitions of multiplicand.} \\
36 & 11 & 8 & =3 \text{ repetitions of multiplicand.} \\
\hline
726 & 8 & 4 & =56+3 \text{ or } 59 \text{ repetitions of multiplicand.} \\
\hline
\end{array}$$

$$\begin{aligned} \text{Since } 59&=60-1=6\times10-1, \\ m\times59 &=m\times6\times10-m, \end{aligned}$$

that is, if m is multiplied by 6, this product by 10 and once m is subtracted from the result, the remainder contains 60—1 or 59 repetitions of m.

Whence, in general, if a product is found in excess, and the number of repetitions of the multiplicand in excess are subtracted, the remainder contains the number of repetitions required.

The product of 12 lb. 3 oz. 16 dwt. by 59 is found by this method as follows,

$$\begin{array}{rrrl}
\text{lb.} & \text{oz.} & \text{dwt.} & \\
12 & 3 & 16 & \\
 & & 6 & \\
\hline
73 & 10 & 16 & =m\times6\,; \\
 & & 10 & \\
\hline
739 & 0 & 0 & =m\times6\times10=m\times60\,; \\
12 & 3 & 16 & =m\times1, \text{ which subtract;} \\
\hline
726 & 8 & 4 & =m\times(60-1)=m\times59. \\
\hline
\end{array}$$

c. Any whole number can be decomposed into as many parts as it contains different orders of units. Of these parts (regard being had to the relative values of the different units), the first is less than 10, the second is a multiple of 10, the third a multiple of 10^2, the fourth a multiple of 10^3, &c.

Decomposing a multiplier in this manner, the sum of the partial products of the multiplicand by the several orders of units of the multiplier is equal to the complete product of the multiplicand and multiplier.

The number which expresses the relative value of the units of any order, n, higher than the first, being one of the series,

$$1\times10^n;\ 2\times10^n;\ 3\times10^n;\ .\ .\ .\ .\ .\ 9\times10^n;$$

it is evident that the partial products can be formed without the necessity of multiplying by any factor greater than 10; and these products are, in practice, easily derived from each other.

To indicate the manner, let m be put for any compound multiplicand; and since this method is most advantageous when the multiplier is a large number, let it be required to multiply m by 57684.

$$57684=50000+7000+600+80+4.$$

$$\therefore m\times57684=m\times50000+m\times7000+m\times600+m\times80+m\times4.$$

Form of calculation.

$$\begin{array}{rl}
m & \times4 \\
10 & \\ \hline
m\times10 & \times8 \\
10 & \\ \hline
m\times100 & \times6 \\
10 & \\ \hline
m\times1000 & \times7 \\
10 & \\ \hline
m\times10000 & \\
5 & \\ \hline
m\times50000 & \\
m\times\ \ 7000 & \\
m\times\ \ \ \ 600 & \\
m\times\ \ \ \ \ \ 80 & \\
m\times\ \ \ \ \ \ \ \ 4 & \\ \hline
m\times57684 & \\ \hline
\end{array}$$

Giving to m the particular value 5 £. 9 sh. 7¾ d., and retaining the same multiplier, 57684, the preceding calculation is replaced by the following.

$$\begin{array}{rrrl}
\text{£} & \text{sh.} & \text{d.} & \\
5 & 9 & 7\frac{3}{4} & \times4 \\
 & & 10 & \\ \hline
54 & 16 & 5\frac{1}{2} & \times8 \\
 & & 10 & \\ \hline
548 & 4 & 7 & \times6 \\
 & & 10 & \\ \hline
5482 & 5 & 10 & \times7 \\
 & & 10 & \\ \hline
54822 & 18 & 4 & \\
 & & 5 & \\ \hline
224114 & 11 & 8 & = \\
38376 & 0 & 10 & = \\
3289 & 7 & 6 & = \\
438 & 11 & 8 & = \\
21 & 18 & 7 & = \\ \hline
266240 & 10 & 3 & = \\ \hline
\end{array}
\qquad
\left.\begin{array}{r}
50000 \\
7000 \\
600 \\
80 \\
4 \\
57684
\end{array}\right\}\text{repetitions of the multiplicand.}$$

In this calculation 50000 repetitions of the multiplicand are obtained by multiplying consecutively by 10, 10, 10, 10, 5.

7000 repetitions are obtained by multiplying the partial product 1000m by 7.

600 repetitions are obtained by multiplying the partial product 100m by 6.

80 repetitions, by multiplying the partial product 10m by 8.

And 4 repetitions by multiplying the multiplicand itself by 4.

The sum of these partial products is the complete product required.

d. When the multiplicand is a compound number, and the multiplier a fraction, if the multiplicand is reduced to units of the lowest denomination contained in it, then multiplied by the fractional multiplier, and the product reduced to a compound number, the result is the product of the given compound multiplicand and fractional multiplier.

Otherwise, since the product of the multiplicand by a fraction must contain some part or parts of the multiplicand as often as the multiplier contains the same part or parts of unity, the product may be found by resolving the multiplicand into as many equal parts as the denominator of the multiplier contains simple units, and taking of these equal parts the number expressed by the numerator.

To obtain the product of a compound number by a fraction, without reduction of the former to a single denomination, it is therefore necessary to divide a compound number by a whole number.

e. In like manner it follows from Article 205, that to divide a compound number by a fraction, it is necessary to multiply the dividend by the denominator, and to divide the product by the numerator of the fractional divisor.

Consequently, to complete the principles for either multiplication or division of compound numbers, the principles of both must be developed.

DIVISION OF COMPOUND NUMBERS.

321. The process of dividing a compound quantity into any number of equal parts, for the purpose of obtaining one of these parts, differs from that of dividing one whole number by another in respect only of the multipliers employed to reduce the successive remainders from any denomination to the denomination which is next lower.

In whole numbers, 1 of any order of units is equal to 10 of the next lower, and by annexing a figure of the dividend to a remainder for the purpose of forming a new partial dividend, the multiplication of the remainder by 10 and the addition of the units of the next lower denomination are both accomplished.

In compound numbers, the multiplication of the remainder, and addition of the units of the next lower denomination to the product, cannot be effected by mere annexation of these units. But the principle of reduction is the same; the remainder is multiplied by a factor which expresses that number of units of the next lower denomination which make one unit of the remainder, and the units of that lower denomination are added to the product.

Examples of division of compound numbers.

1st. Divide 27£. 14sh. $8\frac{3}{4}$d. by 6.

(The quotient is obtained by dividing the number in the highest denomination by 6, reducing the compound number composed of the remainder and the number in the second denomination to units of the second denomination, dividing this result by 6, &c.)

1st. To find the pounds of the quotient.
27£. ÷ 6 = 4£. quotient + 3£. remainder.

2d. To find the shillings of the quotient.
3£. remainder + 14sh. dividend = 74sh.
74sh. ÷ 6 = 12sh. quotient + 2sh. remainder.

3d. To find the pence of the quotient.
2sh. remainder + 8d. dividend = 32d.
32d. ÷ 6 = 5d. quotient + 2d. remainder.

4th. To find the farthings of the quotient.
2d. remainder + $\frac{3}{4}$d. dividend = 11 qrs.
11 qrs. ÷ 6 = $1\frac{5}{6}$ qrs. quotient.

Whence the complete quotient is 4£. 12 sh. 5 d. $1\frac{5}{6}$ qrs.

The calculation may be represented as in Article 86 thus,

```
       £  sh.  d. qrs.
     6)27  14   8   3
     _________________
        4  12   5   1⅚
```

The manner of passing from any denomination to the denomination next lower is the same for divisors either not greater than 12 or greater than 12; but in the latter case the calculation must be made as in Article 88 or Article 91.

2d Example. Divide 576 miles 4 furlongs 24 poles 4 yards by 168.

```
      Mi.  fur. po. yd. Mi. fur. po. yd. ft.
  168)576   4   24   4 (3    3   18   1   1
      504
      ___
       72
        8
      ___
  168)580(3 furlongs.
      504
      ___
       76
       40
      ____
  168)3064(18 poles.
      168
      ____
      1384
      1344
      ____
        40
         5½
      ____
        20
       204
      ____
  168)224(1 yard.
      168
      ___
       56
        3
      ___
  168)168(1 foot.
      168
      ___
```

Performing the multiplications and subtractions mentally, and recording the remainders only, as in Article 91, the preceding calculation is reduced to this:

```
      Mi.  fur. po. yd. Mi. fur. po. yd. ft.
  168)576   4   24   4 (3    3   18   1   1
      ___
       72
        8
      ___
  168)580(3 furlongs.
      ___
       76
       40
      ____
  168)3064(18 poles.
      ____
      1384
      ____
        40
         5½
      ____
  168)224(1 yard.
      ___
       56
        3
      ___
  161)168(1 foot.
      ___
        0
```

Since 168=4×6×7, the quotient may be obtained by dividing successively by these factors of 168 (Art. 95).

	Mi.	fur.	po.	yd.	ft.
168 { 4)	576	4	24	4	0
6)	144	1	6	1	0
7)	24	0	7	3	2½
	3	3	18	1	1

Nothing in the division by 4 or in that by 6 seems to require explanation. But the division by 7 presents a partial dividend which, although not the last, contains a fraction.

The process of the division by 7 seems therefore to require development.

Detail of the division of 24 miles 0 furlongs 7 poles 3 yards 2½ feet by 7.

24 miles ÷ 7=3 miles quotient+3 miles remainder.
3 miles remainder+0 furlongs dividend=24 furlongs.
24 furlongs ÷ 7=3 furlongs quotient+3 furlongs remainder.
3 furlongs remainder+7 poles dividend=127 poles.
127 poles ÷ 7=18 poles quotient+1 pole remainder.
1 pole remainder+3 yards dividend=8½ yards.
8½ yards ÷ 7=1 yard quotient+1½ yard remainder.
1½ yard remainder+2½ feet dividend=7 feet.
7 feet ÷ 7=1 foot quotient.

The complete quotient (as before) is 3 miles 3 furlongs 18 poles 1 yard 1 foot.

322. It has been already explained (Art. 320 d), that to multiply a compound number by a fraction, it is necessary to multiply the compound number by the numerator of the multiplier, and to divide the product by its denominator.

This is the process to be followed in the case of a decimal as well as in that of a vulgar fraction.

In most instances of multiplication by the former some advantage may be gained by writing the decimal as a vulgar fraction and reducing it to the lowest terms.

When the multiplier is a mixed number, the products of the multiplicand by the integral and fractional parts of the multiplier may be formed separately, and the results added together.

323. Division by a fraction being reduced to multiplication by the reciprocal of the fraction, these principles for the multiplication serve also for the division of a compound number by a fraction.

When the divisor is a mixed number it must be reduced to an improper fraction. The reciprocal is, of course, a proper fraction.

Examples of the multiplication and division of compound numbers by fractional expressions.

1st. Multiply 104£. 7 sh. 6⅛ d. by ⅝.

To obtain the required product it is necessary to multiply 104£. 7 sh. 6⅛ d. by 5, and to divide the result by 8.

		£.	sh.	d.
Calculation.		104	7	6⅛
				5
	8)	521	17	6⅝
		65	4	$8\frac{21}{64}$

Wherefore (104£. 7 sh. 6⅛ d.) × ⅝=65 £. 4 sh. $8\frac{21}{64}$ d. or 65 £. 4 sh. 8 d. $1\frac{5}{16}$ qrs.

2d Example. Multiply 227 £. 9 sh. 11¼ d. by 25·6.

25·6=$25\frac{6}{10}$=25⅗. What is required, therefore, is to find the partial product of the multiplicand, first by 25, and second by ⅗; and to take the sum of these partial products for the complete product of the multiplicand by $25\frac{3}{5}$ or 25·6.

£.	sh.	d.	qrs.	
227	9	11	1	
			5	
1137	9	8	1	
			5	
5687	8	5	1	=product of multiplicand by 25.
136	9	11	$2\frac{1}{5}$	- - - by $\frac{3}{5}$.
5823	18	4	$3\frac{1}{5}$	=product of multiplicand by 25·6.

£.	sh.	d.	qrs.
227	9	11	1
			3
5)682	9	9	3
136	9	11	$2\frac{1}{5}$

3d Example. Divide 24 £. 15 sh. $8\frac{13}{16}$ d. by $\frac{3}{4}$.

To obtain the required quotient it is necessary to multiply 24 £. 15 sh, $8\frac{13}{16}$ d. by the reciprocal of $\frac{3}{4}$, that is, by $\frac{4}{3}$.

£.	sh.	d.	
24	15	$8\frac{13}{16}$	
		4	
3)99	2	$11\frac{1}{4}$	
33	0	$11\frac{3}{4}$	=quotient required.

4th Example. Divide 12 miles 4 furlongs 37 poles $3\frac{8}{35}$ yards by 4·05.

$$4·05=4\tfrac{5}{100}=4\tfrac{1}{20}=\tfrac{81}{20}.$$

The reciprocal of $\frac{81}{20}$ is $\frac{20}{81}$. It is therefore necessary to multiply the dividend by 20, and to divide the result by 81.

	Mi.	fur.	po.	yds.	
	12	4	37	$3\frac{8}{35}$	
				4	
	50	3	30	$1\frac{32}{35}$	
				5	
81 {	9)252	2	31	$4\frac{1}{14}$	=product of dividend by 20.
	9)28	0	12	$2\frac{2}{7}$	
	3	0	36	$5\frac{1}{7}$	=quotient required.

∴ 12 mi. 4 fur. 37 po. $3\frac{8}{35}$ yds. ÷ 4·05 = 3 mi. 36 po. $5\frac{1}{7}$ yds.

324. Rules for the multiplication and division of compound numbers.

a. To multiply a compound number by a whole number.

1st. When the multiplier does not exceed the greatest factor contained in the multiplication table.

Multiply the number in the lowest denomination of the multiplicand by the multiplier. Reduce the product to a compound number expressed in terms of the denomination multiplied and of that which is next higher (the 2d). Write the units of the lowest denomination in their proper place, and reserve the units of the next higher denomination. Multiply the number in the next higher denomination by the multiplier; to the product add the units of this denomination reserved from the product of the lowest denomination; reduce the sum to a compound number expressed in terms of this and the next higher denomination; write the units of this (the 2d) denomination in their proper place, and reserve the units of the next higher denomination.

Proceed in this manner to multiply, reduce, and carry from one denomination to the next higher, until all the denominations of the compound number are multiplied by the multiplier. The result is the product required.

2d. When the multiplier exceeds the greatest factor contained in the multiplication table.

If the multiplier can be resolved into factors each not exceeding the greatest factor contained in the multiplication table, multiply the given compound number by one of these factors, this product by a second, this by a third, &c. The continual product of the multiplicand by the factors of the multiplier is the product of the given compound number by the undivided multiplier.

3d. When the multiplier cannot be resolved into suitable factors.

Assume the nearest number (which may be either less or greater than the multiplier) which can be resolved into such factors. Multiply by the factors of this number, as in the 2d case, and add to or subtract from the result as many repetitions of the multiplicand as the assumed number is less or greater than the multiplier. The sum or difference of these products is the product of the given compound number by the unbroken multiplier.

b. To divide a compound number by a whole number.

Divide the number in the highest denomination of the dividend by the divisor, the quotient (if not =0) expresses units of the order of the partial dividend; reduce the compound number composed of the remainder and the second denomination of the dividend to units of the second denomination, and divide the result by the divisor; the quotient expresses units of the second denomination of the dividend. Reduce the second remainder and third denomination to units of the third denomination, divide the result by the divisor, and continue this process of partial reduction and division to the inclusion of all the terms of the dividend. The compound number composed of the successive quotients is the result required.

c. To multiply a compound number by a fractional expression, vulgar or decimal.

Multiply the compound number by the numerator of the multiplier, and divide the product by its denominator.

d. To multiply a compound number by a mixed number.

Find separately the partial products of the multiplicand by the integral and fractional parts of the multiplier; and take the sum of these partial products for the complete product required.

e. To divide a compound number by a fractional expression.

Having reduced the divisor, if a decimal or a mixed number, to the form of a vulgar fraction, multiply the dividend by the reciprocal of the divisor.

325. Exercises in the multiplication and division of compound numbers.

1st. Multiply 3 £. 8 sh. 4 d. by 6 ?................*Ans.* 20 £. 10 sh.
2d. - - 64 £. 8 sh. 10½ d. by 7 ?......*Ans.* 451 £. 2 sh. 1½ d.
3d. - - 2 £. 14 sh. 9¾ d. by 8 ?.........*Ans.* 21 £. 18 sh. 6 d.
4th. - - 2 cwt. 3 qr. 14 lb. by 9 ?...*Ans.* 25 cwt. 3 qrs. 14 lb.
5th. - - 3 yds. 1 ft. 7·9 in. by 12 ? *Ans.* 42 yds. 1 ft. 10·8 in.
6th. - - 1 £. 13 sh. 4 d. by 20 ?.........*Ans.* 33 £. 6 sh. 8 d.
7th. - - 19 sh. 6½ d. by 100 ?...........*Ans.* 97 £. 14 sh. 2 d.
8th. - - 18 £. 16 sh. 8½ d. by 64 ?...*Ans.* 1205 £. 9 sh. 4 d.
9th. - - 37 cwt. 3 qrs. 18 lb. by 28 ?...*Ans.* 1061 cwt. 2 qrs.

10th. Multiply 3 sh. 0$\frac{3}{4}$ d. by 26 ?............ *Ans.* 3 £. 19 sh. 7$\frac{1}{2}$ d.
11th. - - 2 £. 5 sh. 3$\frac{1}{4}$ d. by 29 ?.....*Ans.* 65 £. 12 sh. 10$\frac{1}{4}$ d.
12th. - - 9 mi. 4 fur. 39 po. 2 ft. 8 in. by 39 ?
Ans. 375 mi. 2 fur. 7 po. 2 yds. 1 ft. 2 in.
13th. - - 1 ac. 1 po. by 51 ?............*Ans.* 51 ac. 2 ro. 3 po.
14th. - - 3 sh. 4 d. by 274 ?..............*Ans.* 45 £. 13 sh. 4 d.
15th. - - 9 sh. 6$\frac{1}{2}$ d. by 365 ?...........*Ans.* 174 £. 2 sh. 8$\frac{1}{2}$ d.
16th. - - 19 sh. 6$\frac{1}{2}$ d. by 8604 ?......*Ans.* 8406 £. 16 sh. 6 d.
17th. - - 18 sh. 7$\frac{1}{4}$ d. by 9760 ?......*Ans.* 9078 £. 16 sh. 8 d.
18th. Divide 430 £. 11 sh. 6 d. by 3 ?.....*Ans.* 143 £. 10 sh. 6 d.
19th. - - 3 £. 18 sh. 2 d. by 4 ?................*Ans.* 19 sh. 6$\frac{1}{2}$ d.
20th. - - 43 cwt. 2 qr. 8 lb. by 5 ?....*Ans.* 8 cwt. 2 qr. 24 lb.
21st. - - 526 £. 12 sh. 5 d. by 6 ? *Ans.* 87 £. 15 sh. 4 d. 3$\frac{1}{3}$ qrs.
22d. - - 24 lb. 3 oz. 14 dwt. 5 gr. by 7 ?
Ans. 3 lb. 5 oz. 13 dwt. 11 gr.
23d. - - 109 qrs. 5 bush. by 8 ?
Ans. 13 qrs. 5 bush. 2 pks. 1 gal.
24th. - - 49 mi. 20 po. 5 yds. by 9 ?
Ans. 5 mi. 3 fur. 24 po. 3 yds.
25th. - - 72 hhd. 12 gal. 3 qt. by 10 ?
Ans. 7 hhd. 13 gal. 3 qt. 1 pt.
26th. - - 91 ac. 3 ro. 36 po. 2$\frac{3}{4}$ yds. by 11 ?
Ans. 8 ac. 1 ro. 17 po. 25 yds.
27th. - - 41 ft. 7 in. 6 lines by 12 ?...*Ans.* 3 ft. 5 in. 7$\frac{1}{2}$ lines.
28th. - - 30 £. 18 sh. 8 d. by 16 ?.......*Ans.* 1 £. 18 sh. 8 d.
29th. - - 1061 cwt. 2 qrs. by 28 ?...*Ans.* 37 cwt. 3 qrs. 18 lb.
30th. - - 571 yds. 2 qrs. 1 nail by 47 ?
Ans. 12 yds. 0 qr. 2 nails.
31st. - - 3648 £. 15 sh. 4$\frac{1}{2}$ d. by 73 ?
Ans. 49 £. 19 sh. 7 d. 3$\frac{59}{73}$ qrs.
32d. - - 36 £. 13 sh. 3 d. by 84 ?.............*Ans.* 8 sh. 8$\frac{3}{4}$ d.
33d. Multiply 25 £. 3 sh. 5 d. 2$\frac{1}{5}$ qrs. by $\frac{1}{8}$?
Ans. 3 £. 2 sh. 11 d. 0$\frac{31}{40}$ qr.
34th. - - 15 £. 13 sh. 11$\frac{1}{4}$ d. by $\frac{7}{9}$? *Ans.* 12 £. 4 sh. 2 d. 0$\frac{1}{13}$ qr.
35th. - - 100 tons 17 cwt. 3 qr. 6 lb. by $\frac{3}{4}$?
Ans. 75 tons 13 cwt. 1 qr. 11$\frac{1}{2}$ lb.
36th. - - 13 £. 9 sh. 6$\frac{1}{2}$ d. by $\frac{5}{7}$?......*Ans.* 9 £. 12 sh. 6$\frac{5}{14}$ d.
37th. - - 7 fur. 29 po. 3$\frac{1}{4}$ yds. by 10$\frac{3}{8}$?
Ans. 10 mi. 2 fur. 1 po. 3$\frac{13}{20}$ yds.
38th. - - 227 £. 9 sh. 11$\frac{1}{4}$ by 25·6 ?
Ans. 5823 £. 18 sh. 4 d. 3$\frac{1}{5}$ qr.
39th. - - 5 mi. 3 fur. 37 po. 4$\frac{1}{8}$ yds. by 46$\frac{5}{7}$?
Ans. 256 mi. 4 fur. 32 po. 3$\frac{7}{10}$ yds.
40th. - - 6 ac. 2 ro. 17 po. 25$\frac{1}{3}$ yds. by 57$\frac{3}{5}$?
Ans. 380 ac. 3 ro. 11 po. 13$\frac{1}{4}$ yds.
41st. - - 20 £. 17 sh. 5·23 by 203$\frac{5}{6}$?
Ans. 4254 £. 7 sh. 4 d. 0$\frac{29}{150}$ qrs..
42d. - - 3 mi. 36 po. 5$\frac{1}{7}$ yds. by 4·05 ?
Ans. 12 mi. 4 fur. 37 po. 3$\frac{8}{35}$ yds.

43d. Divide 107 £. 11 sh. $8\frac{3}{4}$ d. by $1\frac{10}{11}$?
Ans. 56 £. 7 sh. 1 d. $0\frac{13}{21}$ qrs.

44th. - - 497 £. 17 sh. $7\frac{3}{4}$ d. by ·9 ?
Ans. 553 £. 4 sh. 0 d. $2\frac{4}{5}$ qrs.

45th. - - 176 ac. $19\frac{2}{3}$ yds. by $15\frac{1}{6}$?
Ans. 11 ac. 2 ro. 16 po. $22\frac{52}{91}$ yds.

46th. - - 17 hhd. 37 gal. 1 pt. by $20\frac{2}{3}$? *Ans.* 53 gal. $2\frac{59}{124}$ qts.

47th. - - 793 ac. 2 ro. 33 po. $28\frac{1}{140}$ yds. by $29\frac{1}{7}$?
Ans. 27 ac. 37 po. $19\frac{1}{5}$ yds.

48th. - - 9753 £. 14 sh. $8\frac{1}{4}$ d. by 234·5 ?
Ans. 41 £. 11 sh. $10\frac{1}{2}$ d.

49th. - - 79 ac. 1 ro. 26 po. $17\frac{7}{8}$ yds. by $81\frac{1}{2}$?
Ans. 3 ro. 35 po. $27\frac{1}{2}$ yds.

50th. - - 598 cwt. 3 qrs. 17 lb. by $13\frac{1}{3}$?
Ans. 44 cwt. 3 qrs. $18\frac{31}{40}$ lb.

51st. - - 3741 ac. 1 ro. 19 po. 8·5 yds. by 79·75 ?
Ans. 46 ac. 3 ro. 26 po. $5\frac{629}{638}$ yds.

52d. - - 182 £. 10 sh. $0\frac{3}{8}$ d. by 365·25 ?
Ans. 9 sh. 11 d. $3\frac{987}{1461}$ qrs.

53d. - - 61 £. 4 sh. $7\frac{5}{16}$ d. by 267·1875 ?......*Ans.* 4 sh. 7 d.

54th. Multiply 3 ac. 1 ro. $17\frac{1}{5}$ yds. by 137·2 ?
Ans. 446 ac. 1 ro. 22 po. $18\frac{19}{30}$ yds.

55th. Divide 146 £. 13 sh. $7\frac{5}{8}$ d. by $117\frac{1}{6}$?
Ans. 1 £. 5 sh. 0 d. $1\frac{1747}{2109}$ qrs.

56th. Multiply 3 mi. 5 fur. $3\frac{1}{7}$ yds. by $23\frac{5}{8}$?
Ans. 85 mi. 3 fur. 17 po. $4\frac{41}{42}$ yds.

57th. Divide 289 ac. 25 po. 28 yds. by 123·24 ?
Ans. 2 ac. 1 ro. 15 po. $12\frac{1}{2}$ yds.

58th. Multiply 15 sh. 7·56 d. by $309\frac{2}{3}$?...*Ans.* 242 £. 0 sh. 1·08 d.

59th. Divide 6 £. 6 sh. 3 d. $3\frac{1}{2}$ by $\frac{1}{8}$?........ *Ans.* 50 £. 10 sh. 7 d.

60th. If 9 cwt. cost 10 £. 12 sh. 6 d., what is the price of a ton ?
Ans. 23 £. 12 sh. $2\frac{2}{3}$ d.

61st. If 27 gallons of wine cost 35 £. 13 sh. 4 d., what is the price of a hhd ?..................................*Ans.* 83 £. 4 sh. $5\frac{1}{3}$ d.

62d. If the whole pay of 729 men for 718 days be 25364 £. 17 sh. 7 d. 3·768 qrs., what is the daily pay of each man ?
Ans. 11 d. 2·521+qrs.

63d. If 2500 men have each 50 rounds of ball cartridge, what is the whole weight of lead, each ball weighing ·99 oz. av. ?
Ans. 69 cwt. 6 lb. 6 oz.

64th. A common pasture containing 54 ac. 3 ro. 35·5 po., another containing $54\frac{1}{2}$ ac., and another containing 39 ac. $13\frac{3}{4}$ po., are to be divided into 60 equal shares, after deducting 11 ac. $2\frac{3}{8}$ ro. for tithes ; what is one share ? *Ans.* 2 ac. 1 ro. 5 po. $7\frac{177}{960}$ yds.

326. In the preceding rules for the multiplication and division of quantities expressed by compound numbers, and also in the examples of multiplication and division, the multipliers and divisors are abstract numbers.

It remains to consider the cases in which it is possible to multiply or divide one compound number by another.

The quantities proposed for multiplication or division may be of the same kind or of different kinds.

a. The multiplication of a compound number by a compound number of the same kind can be effected when the numbers express measures of length; and that of a compound number by a compound number of another kind, when one factor is the expression of a surface and the other that of a line.

In the first case the product expresses the quantity of a surface whose length and breadth are the given factors; and in the second it expresses the volume or solid content of a body, the surface of whose base is expressed by one of the given factors, and its height by the other.

The rules for the formation of these products belong to another part of this course, the mensuration of surfaces and solids.

As far as the arithmetical process is concerned, the product of two compound numbers, which express quantities either of the same kind or of different kinds, can be formed; but the results, with the exceptions already mentioned, have no intelligible meaning.

For, if a quantity expressed numerically by, for example, lbs. oz. and drs. is multiplied by another quantity expressed also by lbs. oz. dr., the product is composed of numbers which cannot be referred to a weight, or to any sort of measuring unit, and which is unintelligible.

And if a quantity expressed numerically by yds. ft. and in. is multiplied by another expressed by quarters, bushels, and pecks, the result is also unintelligible.

b. The division of one compound number by another compound number of the same kind is an operation which it is often necessary to perform.

Two sums of money, or two quantities measured by the same weight or the same measure, and expressed by compound numbers, may be given; and it may be required to determine how often the one, or some part of the one, is contained in the other.

The process supposes the division of the one compound quantity into parts each equal to the whole or to some part of the other compound quantity; and the result is composed of unity, or a part of unity, as often repeated as the dividend contains the divisor or some part of the divisor.

The result is, consequently, an abstract number.

A compound number expressed by several denominations can be divided by a smaller compound number, also expressed by several denominations, without preparatory reduction of the dividend or the divisor. But as the inferior denominations of the divisor must be multiplied by the successive quotient figures, considered with respect to their relative as well as absolute values, it is necessary to determine both for every quotient figure before forming the partial product of the divisor by it.

As an instance, let it be required to divide 58 £. 13 sh. 4 d. by 3 £. 6 sh. 8 d.

Having arranged the divisor and dividend in the usual manner, the first step of the process is the division of 50 by 3.

The partial quotient (without regarding the units in it) is 10; the product of the divisor by 10 is 33 £. 6 sh. 8 d.; and the difference between 58 £. 13 sh. 4 d. and 33 £. 6 sh. 8 d. is 25 £. 6 sh. 8 d.

The second step is the division of 25 by 3. The partial quotient obtained, without considering the inferior denominations of the divisor, is 8. But, on trial, 8 is found too great. Taking the next lower number, 7, the product of the divisor by 7 is 23 £. 6 sh. 8 d., and the remainder left by the subtraction of this number from 25 £. 6 sh. 8 d. is 2 £.

The integral part of the quotient is therefore 10+7=17.

The fractional part is to be obtained from the division of 2 £. by 3 £. 6 sh. 8 d.

This division is effected by reducing both divisor and dividend to the lowest denomination contained in either of them, writing the dividend as the numerator and the divisor as the denominator of a vulgar fraction, and reducing this fraction to the lowest terms.

In this instance 2 £.=480 d.
3 £. 6 sh. 8 d.=800 d.

$$\text{Therefore } \frac{2\text{ £.}}{3\text{ £. } 6\text{ sh. } 8\text{ d.}}=\frac{480}{800}=\frac{3}{5};$$

and the complete quotient is $17\frac{3}{5}$.

The calculation may be made as follows:

$$\begin{array}{rrrcrrrl}
\text{£.} & \text{sh.} & \text{d.} & & \text{£.} & \text{sh.} & \text{d.} & \\
3 & 6 & 8 &) & 58 & 13 & 4 & (\ 10+7+\frac{3}{5}=17\frac{3}{5}. \\
 & & & & 33 & 6 & 8 & \\ \cline{5-7}
 & & & & 25 & 6 & 8 & \\
 & & & & 23 & 6 & 8 & \\ \cline{5-7}
 & & & & \multicolumn{4}{l}{\dfrac{2\quad 0\quad 0}{3\quad 6\quad 8}=\dfrac{480}{800}=\dfrac{3}{5}.}
\end{array}$$

Although it is possible in this manner to divide the greater of two compound numbers of the same kind by the less, the quotient is found more readily by reducing both dividend and divisor to the same denomination (which must be the lowest contained in either of them), and dividing the former by the latter, by the rules for the division of whole or of fractional quantities, according as the reduced numbers have the integral or fractional form.

Employing this method to discover how often 3 £. 6 sh. 8 d. is contained in 58 £. 13 sh. 4 d.

58 £. 13 sh. 4 d.=14080 d.
3 £. 6 sh. 8 d.= 800 d.

$$\therefore \frac{58\text{ £. } 13\text{ sh. } 4\text{ d.}}{3\text{ £. } 6\text{ sh. } 8\text{ d.}}=\frac{14080}{800}=17\frac{3}{5}.$$

The second method is applicable to the division of any concrete number by a concrete number of the same kind, either being expressed in terms of any single denomination or of several denominations of the weight or measure appropriated to the kind of quantity which is considered in the question.

When the divisor is expressed by one denomination only, if the dividend is reduced to this denomination and divided by the divisor, the result is the quotient required.

Example. Into how many parts, each weighing 2 qrs., can 12 cwt. 3 qrs. 18 lb. be divided?

$$12\text{ cwt. } 3\text{ qrs. } 18\text{ lb.}=51\text{ qrs. } 18\text{ lb.}=51\frac{9}{14}\text{ qrs.}$$

$$\therefore 51\frac{9}{14}\div 2=25\frac{23}{28}=\text{the number of parts required.}$$

If the divisor is 1, the division may be omitted; but it must be kept in mind that the result (in this case the reduced dividend) expresses the number of units of a particular denomination contained in the dividend, and that it is consequently an abstract number.

Example. How many musket balls, weighing one ounce each, can be cast from 10 cwt. 2 qrs. 14 lb. of lead?

10 cwt. 2 qrs. 14 lb.=42 qrs. 14 lb.=1190 lb.=19040 oz.

Whence, since the given quantity of lead contains 19040 oz., it follows that 19040 ounce balls is the number which can be made from it.

c. The division of a concrete quantity of one kind by a concrete quantity of another kind, whether expressed by one or by several denominations, cannot be made; for a quantity consisting, for instance, of yards, feet, and inches, cannot be divided into parts each equal to another quantity composed of lbs., oz., and dwts., or to any fractional part of such a quantity.

327. Most of the commodities which are bought and sold are weighed or measured, and the quantity of their weight or capacity or extension is expressed in terms of one or more denominations of an appropriate weight or measure.

Besides these measures of the quantity of articles of commerce, there is a general measure of their value—money. One unit of the principal denomination, or of any other denomination of the compound number, which expresses the quantity of a commodity, being valued at a certain sum of money, 2, 3, ... n times this unit are valued at 2, 3, ... n times the value of the unit, and $\frac{1}{2}, \frac{1}{3}, \ldots\ldots \frac{1}{n}$ part at $\frac{1}{2}, \frac{1}{3}, \ldots\ldots \frac{1}{n}$ part of the value of the unit; so that whatever multiple, part or parts of the given unit, are expressed by a compound number, the same multiple, part or parts of the value of one unit, are taken for the value of the quantity expressed by this compound number; the money value or price of a commodity has, therefore, a constant relation to the quantity of that commodity.

This constant relation between quantity and price in the same object supplies the means of resolving many questions of commercial arithmetic.

These questions are, chiefly, reducible to two classes; one in which two quantities, and the money value of one of these quantities, are given, and the money value of the other quantity is to be found; the other, in which the money values of two quantities and one of these quantities are given, and the other quantity is to be found.

Examples.

1st. The value of silver being 5 sh. 4 d. per ounce, what is the worth of 9 lb. 8 oz.?

$$9\text{ lb. }8\text{ oz.}=116\text{ oz.} \therefore \frac{9\text{ lb. }8\text{ oz.}}{1\text{ oz.}}=\frac{116}{1}=116 \text{ (Art. 326 b).}$$

9 lb. 8 oz. being equal to 116 times 1 oz., the price of 9 lb. 8 oz. is equal to 116 times the price of 1 ounce; that is to (5 sh. 4 d.) × 116 = 30 £. 18 sh. 8 d.

2d Ex. The price of 64 yds. 3 qrs. 2 nls. of cloth being 60 £. 0 sh. 2¼ d., what is the price of 1 yard?

The price of 1 yard must be the same part of 60 £. 0 sh. 2¼ d. that 1 yard of 64 yds. 3 qrs. 2 nls.

To find what part 1 yard is of 64 yds. 3 qrs. 2 nls. it is necessary to divide the former quantity by the latter:

Now 1 yd. = 16 nls. and 64 yds. 3 qrs. 2 nls. = 1038 nls.

$\therefore \frac{1\text{ yd.}}{64\text{ yds. }3\text{ qrs. }2\text{ nls.}}=\frac{16}{1038}=\frac{8}{519}$, that is, one yard is equal to $\frac{8}{519}$ths of the whole quantity; therefore the price of 1 yd. is equal to

$$\frac{8}{519}\text{ths of }60\text{ £. }0\text{ sh. }2\tfrac{1}{4}\text{ d.}=\frac{(60\text{ £. }0\text{ sh. }2\tfrac{1}{4}\text{ d.})\times 8}{519}=18\text{ sh. }6\text{ d.}$$

3d Ex. 8 qrs. 5 bush. 2 pks. of wheat cost 22 £. 16 sh. 6½ d.; what must be given for 186 qrs. 6 bush. 1 pk. at the same rate?

The price of 186 qrs. 6 bush. 1 pk. must contain some part or parts of 22 £. 16 sh. 6½ d., as often as 186 qrs. 6 bush. 1 pk. contain the same part or parts of 8 qrs. 5 bush. 2 pks.

$$\text{Now, by Article 326 b, }\frac{186\text{ qrs. }6\text{ bush. }1\text{ pk.}}{8\text{ qrs. }5\text{ bush. }2\text{ pks.}}=\frac{43}{2};$$

Whence the price required is

$$22\text{ £. }16\text{ sh. }6\tfrac{1}{2}\text{ d.}\times\frac{43}{2}=22\text{ £. }16\text{ sh. }6\tfrac{1}{2}\text{ d.}\times 21\tfrac{1}{2}=490\text{ £. }15\text{ sh. }7\tfrac{3}{4}\text{ d.}$$

From the preceding discussion and examples it is evident that two quantities, expressed by compound numbers, and the money value of one of these quantities being given, the rule for finding the money value of the other quantity is,

Find what multiple part or parts the quantity whose money value is required is of that whose money value is given, and take the same multiple part or parts of the given money value for the result which is required.

Example of the second class of questions:

If the price of 1 cwt. 2 qrs. 10 lb. of gunpowder is 11 £. 17 sh. 4 d., what quantity can be bought for 180 £. 19 sh. 4 d. at the same rate?

The quantity bought for 180 £. 19 sh. 4 d. must be the same multiple part of 1 cwt. 2 qrs. 10 lb. that 180 £. 19 sh. 4 d. is of the price of 1 cwt. 2 qrs. 10 lb., namely, 11 £. 17 sh. 4 d.

Now $\frac{180\text{ £. } 19\text{ sh. } 4\text{ d.}}{11\text{ £. } 17\text{ sh. } 4\text{ d.}} = 15\frac{1}{4}$ (Art. 326 b).

Therefore the required quantity is equal to $15\frac{1}{4}$ times 1 cwt. 2 qrs. 10 lb., and 1 cwt. 2 qrs. 10 lb. × $15\frac{1}{4}$ = 24 cwt. 26 lb. 8 oz., which is the quantity required.

Hence, when the money values of two quantities expressed by compound numbers and one of these quantities are given, and it is required to find the other quantity,

Rule. Find what multiple part or parts the money value of the required quantity is of the money value of the given quantity, and take the same multiple part or parts of the given quantity for that which is to be found.

Examples.

1st. How many packages of 1 cwt. 3 qrs. 12 lb. each will load a vessel of 1000 tons burthen?................*Ans.* $10769\frac{3}{13}$.

2d. What cost 438 yds. 3 qrs. 2 nls. of linen, at the rate of 2 £. 15 sh. 0 d. $1\frac{1}{3}$ qrs. for 12 yds.?
Ans. 100 £. 12 sh. 7 d. $3\frac{23}{64}$ qrs.

3d. How many gallons of wine can be bought with 25 £. 4 sh. $6\frac{3}{4}$ d. at the rate of 17 sh. $6\frac{3}{8}$ d. for 1 gallon?...*Ans.* 28 gals. 3 qts.

4th. The charge for the carriage of 5 cwt. 14 lb. of goods being 11 sh. $4\frac{1}{2}$ d., what must be paid for the carriage of $1\frac{1}{2}$ tons at the same rate?.......................*Ans.* 3 £. 6 sh. $7\frac{1}{41}$ d.

5th. What is the rent of 269 ac. 2 ro. 32 po. of land at the rate of 2 £. 12 sh. 6 d. for 2 acres?.......*Ans.* 353 £. 19 sh. 7 d.

6th. The price of wine being 65 £. 10 sh. 6 d. per hhd., what quantity can be bought with 1715 £. 17 sh. 5 d. $0\frac{2}{21}$?
Ans. 26 hhds. 11 gals. 3 qts.

328. When the price of one unit of any denomination of a quantity, expressed by a compound number, is given to find the price of the quantity expressed by the compound number, a method different in form from the preceding, and known as the rule of Aliquot Parts or Practice, is often employed.

The process by aliquot parts may involve both multiplication and division of compound numbers, or division only.

If the compound number contains a denomination the money value of one simple unit of which is the given price, it is necessary to multiply this price by the number of units of this denomination.

Then, if the number which expresses the units of the next lower denomination of the compound number or multiplier is $\frac{1}{2}$ or $\frac{1}{3}$ or $\frac{1}{4}$........ or $\frac{1}{12}$th part of that number of units of this (the 2d) denomination, which make one unit of the first denomination, a corresponding part of the price (or multiplicand) is taken for the product of the multiplicand by the units of the second denomination of the multiplier.

But if the number in the second denomination of the multiplier is not reducible to such a part of one unit of the first denomination as can be expressed by some fraction in the series $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$,...... $\frac{1}{12}$, it is broken into parts, of which at least one is so expressed; the other parts being similarly expressed either in terms of one unit of the first denomination, or of that and the first part of the second denomination, or of the first part of the second denomination only, or of either or both, and of each other in succession.

The relation of the parts of the second denomination of the multiplier being arranged, corresponding parts of the whole multiplicand (or price of one unit of the multiplier) and of the parts which have to it and to each other that relation which the parts of the number in the second denomination of the multiplier have to one unit of the first denomination, and to each other, are taken as the respective products of the multiplicand by those parts of the multiplier.

The units of the third denomination of the multiplier are expressed in terms of the units of the second denomination, or of any of the parts into which the number expressing the units of the 2d denomination is divided, in the same manner as the units of the second denomination are expressed in terms of one unit of the first.

And the corresponding parts of the product are formed from the preceding parts and from each other in the same manner as before.

This process is repeated for the 4th and every lower denomination of the compound multiplier; and the sum of all the partial results is the product required.

When a compound multiplier contains only denominations inferior to that which is pointed out by the conditions of a question as expressing the principal units of the multiplier, the result is found by division, and the first step of the process in this case coincides with the second in the preceding discussion; that, namely, for obtaining the product of the multiplicand by the 2d denomination of the multiplier.

For decomposing the inferior denominations of a compound multiplier into convenient parts of the highest denomination and of each other no general rule can be given.

The condition of subdivision, namely, that each part shall be $\frac{1}{2}$ or $\frac{1}{3}$ or $\frac{1}{4}$........ or $\frac{1}{12}$ of some preceding part is imposed in order that, in finding the corresponding parts of the multiplicand, it may not be requisite to multiply, but only to divide, and that by a number not greater than 12.

Examples of questions resolved by the method of aliquot parts.

1st. What is the price of 5 cwt. 3 qrs. 18 lb. of sugar at the rate of 3£. 17 sh. 8 d. per cwt.?

First, to find the price of 5 cwt.;

1 cwt. costs 3£. 17 sh. 8 d. ∴ 5 cwt. cost 3£. 17 sh. 8 d. × 5 = 19£. 8 sh. 4 d.

Second, to find the price of 3 qrs.;

3 qrs., being $\frac{3}{4}$ of 1 cwt., must cost $\frac{3}{4}$ of the price of 1 cwt.

But, to avoid multiplying by 3, the numerator of this fraction, the 3 qrs. are separated into parts of 2 qrs. and 1 qr.

Then, since 2 qrs. = $\frac{1}{2}$ cwt., the price of 2 qrs. is equal to half the price of 1 cwt. = $\frac{1}{2}$ of 3£. 17 sh. 8 d. = 1£. 18 sh. 10 d.;

And since 1 qr. = $\frac{1}{2}$ of 2 qrs., the price of 1 qr. is equal to half the price of 2 qrs. = $\frac{1}{2}$ of 1£. 18 sh. 10 d. = 19 sh. 5 d.

Third, to find the price of 18 lb.;

18 lb.$=\frac{18}{28}$ qr.$=\frac{9}{14}$ qr. This expression being unsuitable, the 18 lb. are broken into the parts 14 lb. and 4 lb., of which 14 lb.$=\frac{14}{28}$ qr.$=\frac{1}{2}$ qr. and 4 lb.$=\frac{4}{28}$ qr.$=\frac{1}{7}$ qr.

Then, since 1 qr. costs 19 sh. 5 d., 14 lb., being the half of 1 qr., must cost $\frac{1}{2}$ of 19 sh. 5 d.=9 sh. $8\frac{1}{2}$ d.;

And 4 lb. or $\frac{1}{7}$ of 1 qr. must cost $\frac{1}{7}$ of 19 sh. 5 d.=2 sh. 9 d. $1\frac{1}{7}$ qrs.

Collecting the partial results,

cwts.	qrs.	lb.		£	sh.	d.	qrs.
5	0	0	- at 3£. 17 sh. 8 d. - =	19	8	4	0
0	2	0	- - - - - -	1	18	10	0
0	1	0	- - - - - -	0	19	5	0
0	0	14	- - - - -	0	9	8	2
0	0	4	- - - - - -	0	2	9	$1\frac{1}{7}$
5	3	18	- at 3£. 17 sh. 8 d. per cwt.=£.	22	19	0	$3\frac{1}{7}$

The calculation of a product by aliquot parts is generally made as follows:

		£	sh.	d.	qrs.
2 qrs. =	$\frac{1}{2}$	3	17	8	0
					5
		19	8	4	0
1 - =	$\frac{1}{2}$	1	18	10	0
14 lb. =	$\frac{1}{2}$	0	19	5	0
7 - =	$\frac{1}{7}$	0	9	8	2
		0	2	9	$1\frac{1}{7}$
		22	19	0	$3\frac{1}{7}$

2d Ex. What is the price of 2 lb. 4 oz. 18 grs. of silver plate at 10 sh. 6 d. per ounce?

The denomination indicated by the question as that of the principal units is ounces.

Now 2 lb. 4 oz.=28 oz.; whence the multiplier is 28 oz. 18 grs.

The price of 28 oz. is 10 sh. 6 d.×4×7=14£. 14 sh.

To find the price of 18 grs. at 10 sh. 6 d. per ounce;

18 grs.$=\frac{18}{24}$ dwts.$=\frac{18}{480}$ oz.$=\frac{3}{80}$ oz., an unsuitable expression;

Since 12 grs.$=\frac{1}{2}$ dwt., and 6 grs.$=\frac{1}{2}$ of 12 grs., it is evident that if the price of 1 dwt. is found, that of 18 grs. can be obtained from it.

But as 1 dwt.$=\frac{1}{20}$ of an oz., to avoid the necessity of dividing by 20, it is preferable to find the price of 2 dwts.$=\frac{1}{10}$ oz., and from this the price of 12 grs.$=\frac{1}{4}$ of 2 dwts.

1 oz. cost 10 sh. 6 d. ∴ 2 dwts. cost $\frac{1}{10}$ of 10 sh. 6 d.=1 sh. 0 d. $2\frac{2}{5}$ qrs.

12 grs. cost $\frac{1}{4}$ part of the price of 2 dwts., or $\frac{1}{4}$ of 1 sh. 0 d. $2\frac{2}{5}$ qrs.= 3 d. $0\frac{3}{5}$ qrs.

6 grs. cost half as much as 12 grs., or $\frac{1}{2}$ of 3 d. $0\frac{3}{5}$ qrs.=1 d. $2\frac{3}{10}$ qrs.

The partial results are,

lb.	oz.	grs.		£	sh.	d.	qrs.
2	4	0	- at 10 sh. 6 d. per oz.=	14	14	0	0
0	0	12	- - - - -	0	0	3	$0\frac{3}{5}$
0	0	6	- - - - -	0	0	1	$2\frac{3}{10}$
2	4	18	- - - - -	14	14	4	$2\frac{9}{10}$

The price of 2 dwts. (which is found as affording the simplest means of obtaining that of 18 grs.) is not added into the general result.

The price of a quantity not contained in the compound multiplier, but introduced to facilitate the calculation, is termed a subsidiary product, or false product.

When the calculation is written in the second form, the subsidiary product is recorded among the other partial results, but a line is drawn through it to give warning that it forms no part of the complete product or answer to the question.

dwts.	oz.	£.	sh.	d.	qrs.
2 =	$\frac{1}{10}$	0	10	6	0
					4
		2	2	0	0
					7
		14	14	0	0
grs. 12 =	$\frac{1}{4}$		~~1~~	~~0~~	~~2$\frac{2}{5}$~~
6 =	$\frac{1}{2}$			3	0$\frac{3}{5}$
				1	2$\frac{3}{10}$
		14	14	4	2$\frac{9}{10}$

The method of aliquot parts is frequently employed in questions in which the quantity whose money value is to be found is not a compound number.

For example, the product of 1 £. 16 sh. 10½ d. by 365¼ may be found by aliquot parts in the following manner:

If 1 £. is taken 365¼ times, the result is 365¼ £.=365 £. 5 sh.

10 sh. being the half of 1 £., 10 sh. taken 365¼ times is the half of 365 £. 5 sh.=182 £. 12 sh. 6 d.

5 sh.=½ of 10 sh. ∴ 5 sh. taken 365¼ times=½ of 182 £. 12 sh. 6 d.= 91 £. 6 sh. 3 d.

1 sh. = ⅕ of 5 sh. ∴ 1 sh. taken 365¼ times = ⅕ of 91 £. 6 sh. 3 d. = 18 £. 5 sh. 3 d.

6 d. = ½ of 1 sh. ∴ 6 d. taken 365¼ times = ½ of 18 £. 5 sh. 3 d. = 9 £. 2 sh. 7½ d.

3 d. = ½ of 6 d. ∴ 3 d. taken 365¼ times = ½ of 9 £. 2 sh. 7½ d. = 4 £. 11 sh. 3¾ d.

1½ d. = ½ of 3 d. ∴ 1½ d. taken 365¼ times = ½ of 4 £. 11 sh. 3¾ d.= 2 £. 5 sh. 7 d. 3½ qrs.

Collecting the partial products,

	£.	sh.	d.	qrs.		£.	sh.	d.	qrs.
365¼ at	1	0	0	0	=	365	5	0	0
365¼ at	0	10	0	0	=	182	12	6	0
365¼ at	0	5	0	0	=	91	6	3	0
365¼ at	0	1	0	0	=	18	5	3	0
365¼ at	0	0	6	0	=	9	2	7	2
365¼ at	0	0	3	0	=	4	11	3	3
365¼ at	0	0	1	2	=	2	5	7	3½
365¼ at	1	16	10	2	=	673	8	7	0½

Form of calculation by aliquot parts:

sh.	£.	£.	sh.	d.	qrs.
10 =	½	365	5	0	0
					1
		365	5	0	0
5 =	½	182	12	6	0
1 =	⅕	91	6	3	0
6 d. =	½	18	5	3	0
3 d. =	½	9	2	7	2
1½ d.=	½	4	11	3	3
		2	5	7	3½
		673	8	7	0½

Exercises. Find answers to the following questions by the method of aliquot parts.

1st. 112 lb. of gunpowder at 1 sh. 8 d. per lb.?
Ans. 9 £. 6 sh. 8 d.

2d. 365 days wages at 2 sh. 4 d. per day?
Ans. 42 £. 5 sh. 10 d.

3d. 56 lb. of tea at 5 sh. 10 d. per lb.?......*Ans.* 16 £. 6 sh. 8 d.

4th. 479 oz. of silver plate at 7 sh. 9 d. per oz.?
Ans. 185 £. 12 sh. 3 d.

5th. $117\frac{1}{4}$ yds. of cloth at 12 sh. 6 d. per yd.?
Ans. 73 £. 5 sh. $7\frac{1}{2}$ d.

6th. $7251\frac{1}{4}$ at 14 sh. $8\frac{1}{4}$ d. each?
Ans. 5325 £. 2 sh. 8 d. $3\frac{1}{16}$ qrs.

7th. $373\frac{3}{4}$ at 2 £. 13 sh. 4 d. each?......*Ans.* 996 £. 13 sh. 4 d.

8th. $520\frac{1}{2}$ at 13 sh. $8\frac{1}{2}$ d. each?.........*Ans.* 356 £. 15 sh. $2\frac{1}{4}$ d.

9th. $617\frac{3}{5}$ at 1 £. 17 sh. 9 d. $2\frac{2}{3}$ qrs. each?
Ans. 1167 £. 0 sh. 2 d. $1\frac{5}{8}$ qrs.

10th. 36 ac. 2 ro. 25 po. at 3 £. 13 sh. 6 d. per acre?
Ans. 134 £. 14 sh. 2 d. $3\frac{1}{4}$ qrs.

11th. 19 cwt. 2 qrs. $16\frac{1}{4}$ lb. at 10 £. 17 sh. $8\frac{1}{2}$ d. per cwt.?
Ans. 213 £. 16 sh. 10 d. $3\frac{41}{224}$ qrs.

12th. 7 hhds. $13\frac{1}{2}$ gals. at 50 £. 13 sh. $9\frac{3}{4}$ d. per hhd.?
Ans. 365 £. 13 sh. $11\frac{11}{56}$ d.

13th. 100 ac. 2 ro. 25 po. at 1 £. 15 sh. per acre?
Ans. 176 £. 2 sh. $11\frac{5}{8}$ d.

14th. 8 cwt. 2 qrs. 12 lb. at 4 £. 5 sh. 6 d. per cwt.?
Ans. 36 £. 15 sh. 10 d.

15th. 17 lb. 3 oz. $13\frac{1}{3}$ dwts. at 3 sh. $11\frac{1}{4}$ d. per oz.?
Ans. 40 £. 17 sh. 7 d. 3·74 qrs.

16th. 7 hhds. $10\frac{1}{2}$ gals. at 97 £. 15 sh. 7·25 d. per hhd.?
Ans. 700 £. 15 sh. 1·9583 d.

17th. 7 ac. 3 ro. $13\frac{1}{3}$ po. at 5 £. 13 sh. 6 d. per acre?
Ans. 44 £. 9 sh. 1 d.

18th. What is $\frac{3}{4}$ of 21 mi. 7 fur. 39 po. 4 yds?
Ans. 16 mi. 3 fur. 39 po. $4\frac{3}{8}$ yds.

19th. What is $\frac{5}{9}$ of 7 hhds. 49 gals. 3 qts.?
Ans. 4 hhds. 20 gals. $2\frac{5}{9}$ qts.

20th. Multiply 1 £. 11 sh. $7\frac{1}{2}$ d. by 6·125?
Ans. 9 £. 13 sh. 8 d. $1\frac{3}{4}$ qrs.

SECTION XI.

PROPORTION, INTEREST, DISCOUNT, PARTNERSHIP.

329. The result of the comparison of a magnitude with its measuring unit is expressed by a number. The number is integral or fractional (including under the latter description the subordinate denominations of compound numbers), according as the measure is or is not contained a certain number of times exactly in the magnitude which is measured.

Numbers which are derived from the comparison of magnitudes of the same kind with the same unit can be compared with each other in respect of their quantity.

And since a number contains the unit of number, or some part of it, as often as the magnitude which it represents contains the unit or a corresponding part of the unit or measure of magnitude, it follows that the relations of magnitudes in respect of their quantities are the same with the relations of the numbers by which their quantities are expressed.

The comparison of two numbers, for the purpose of determining their mutual relation in respect of quantity, is made by decomposing the compared number into parts each equal to the whole or to some part of the number with which it is compared, reckoning *one* for each whole of the second number contained in the first and the proper fractional expression for each part of the second number contained in the first, and combining these *ones* and fractions into a single result. The relation of the compared numbers, in respect of quantity, is expressed by this result.

The decomposition is made by division, either actually performed, or, more commonly, indicated by writing the compared number as the numerator, and that with which it is compared as the denominator, of a vulgar fraction.

The quotient, or the fractional indication of the quotient, expresses what is termed the ratio of the two numbers.*

The number (integral or fractional) by which the ratio of two quantities is expressed is, like the quotient of every division, an abstract number.

The same value is expressed by all fractions whose terms are equimultiples of the terms of the same fraction. Likewise the ratio obtained by the comparison of two quantities of the same kind, being an abstract number, and the ratio obtained by the comparison of two like quantities of another kind, being also an abstract number, if these abstract numbers are equal the ratios are equal. Whence it follows that one ratio may be exactly equal to another ratio.

330. When two ratios are equal, the four numbers which compose these ratios form a proportion.

Assuming the numbers 12, 4, 30, 10, the ratio of 12 to 4 or the quotient of 12 by 4 being 3, and the ratio of 30 to 10 or the quotient of 30 by 10 being also 3, the numbers 12, 4, 30, 10 form a proportion.

$$\text{Since } \tfrac{12}{4}=3 \text{ and } \tfrac{30}{10}=3,$$
$$\therefore\ \tfrac{12}{4}=\tfrac{30}{10}.$$

A proportion is sometimes written thus, but more frequently in the manner following,

$$12 : 4 :: 30 : 10;$$

that is, by placing two points between the first number and the second, four between the second and third, and two between the third and fourth.

The expression in words of the preceding numbers and symbols is, As twelve are to four, so are thirty to ten.

A proportion is sometimes also written thus,

$$12 : 4 = 30 : 10;$$

expressed in words, the ratio of twelve to four is equal to the ratio of thirty to ten.

The four numbers composing a proportion are called the terms of the proportion.

The first and third terms of a proportion are named antecedents; and the second and fourth terms, consequents.

The first and fourth terms are also called extremes, and the second and third terms, means.

331. In a proportion, the constant ratio existing between each antecedent and its consequent may be expressed by either a whole or a fractional number.

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* Any number may be considered as the expression of the ratio of the quantity of that number to unity.

For example, in the following proportions,

$$9 : 3 :: 12 : 4$$
$$24 : 18 :: 36 : 27$$
$$5 : 12 :: 15 : 36,$$

$$\text{the ratio of } 9 \text{ to } 3=\frac{9}{3}=3,$$
$$12 \text{ to } 4=\frac{12}{4}=3,$$

that is, in the first proportion the constant ratio is 3, a whole number.

In the second proportion $\frac{24}{18}=\frac{4}{3}$, and $\frac{36}{27}=\frac{4}{3}$, or the constant ratio is $\frac{4}{3}$, a fractional expression.

In the third proportion $\frac{5}{12}=\frac{5}{12}$ and $\frac{15}{36}=\frac{5}{12}$, or the constant ratio is $\frac{5}{12}$, a proper fraction.

332. Let a, b, c, d represent four numbers which are proportional, so that $a : b :: c : d$,

$$\text{or, } \frac{a}{b}=\frac{c}{d};$$

then, if the equal fractions $\frac{a}{b}$, $\frac{c}{d}$ are both multiplied by bd, the products are equal.

$$\text{But } \frac{a}{b}\times bd=\frac{abd}{b}=ad,$$
$$\text{and } \frac{c}{d}\times bd=\frac{bcd}{d}=bc.$$
$$\text{Whence } ad=bc.$$

A conclusion which, expressed in words, is, if four numbers are proportional, the product of the extremes is equal to the product of the means.

Next, let the product of two factors, a, d, be equal to the product of two factors, b, c; then, if the equal products, ad, bc, are both divided by bd, the quotients are equal.

$$\text{But } \frac{ad}{bd}=\frac{a}{b},$$
$$\text{and } \frac{bc}{bd}=\frac{c}{d}.$$
$$\text{Whence, if } ad=bc,\ \frac{a}{b}=\frac{c}{d}, \text{ or } a : b :: c : d.$$

In words,—if the product of two factors is equal to the product of two other factors, the factors of these equal products form a proportion of which the factors of the one product are the extreme, and the factors of the other product the mean terms.

333. If $b=c$, the proportion $a : b :: c : d$ becomes

$$a : b :: b : d,$$
$$\text{and } \therefore ad=b^2.$$

In this case the numbers a, b, d are said to be continual proportionals, or to be in continual proportion; d is a third proportional to a, b, and b is a mean proportional between a, d.

334. Let $ad=bc$, and therefore $ad=bc=cb=da$ (Art. 56).

Then since $ad=bc$, $a : b :: c : d$. - - 1.
$bc=ad$, $b : a :: d : c$. - - 2.
$cb=da$, $c : d :: a : b$. - - 3.
$da=cb$, $d : c :: b : a$. - - 4.

Four numbers which are proportional can hence be formed into four different arrangements, in each of which they compose a proportion.

In each of these proportions the number a occupies a different place; so that among the four it is the 1st, the 2d, the 3d, and the 4th term. The numbers b, c, d, are in like manner made to occupy the place of every term of the proportion. The arrangement of the numbers which compose a proportion can consequently be ordered in such a manner that a given number shall occupy any place which may be required.

Although the proportionality of four numbers which are proportional is not destroyed by these changes of arrangement, the value of the constant ratio is altered by them.

The ratio $\frac{a}{b}$, which stands first in proportion 1, is placed last in proportion 3, and conversely. The same transposition of ratios is made in proportions 2 and 4.

The constant ratio $\frac{a}{b}$ is therefore the same in proportions 1 and 3; the constant ratio $\frac{b}{a}$ is also the same in proportions 2 and 4, but in 2 and 4 it differs from the constant ratio of proportions 1 and 3, unless a, b are equal; for that $\frac{a}{b}$ may be equal to $\frac{b}{a}$, it is necessary that $a=b$, in which case also c must be equal to d. Unless, therefore, $a=b$ and $c=d$, the constant ratio of proportions 1 and 3 is different from that of proportions 2 and 4.

335. Again, let $ad=bc$, and therefore $a\times d=b\times c=c\times b=d\times a$.

Then, since the quotients obtained by dividing equal dividends by the same divisor are equal,

$$\frac{a\times d}{d}=\frac{b\times c}{d},\ \text{or}\ a=\frac{b\times c}{d},\ \text{or}\ a=\frac{c}{d}\times b,$$

$$\frac{b\times c}{c}=\frac{a\times d}{c},\ \text{or}\ b=\frac{a\times d}{c},\ \text{or}\ b=\frac{d}{c}\times a,$$

$$\frac{c\times b}{b}=\frac{d\times a}{b},\ \text{or}\ c=\frac{d\times a}{b},\ \text{or}\ c=\frac{a}{b}\times d,$$

$$\frac{d\times a}{a}=\frac{c\times b}{a},\ \text{or}\ d=\frac{c\times b}{a},\ \text{or}\ d=\frac{b}{a}\times c.$$

But since $ad=bc$, $a : b :: c : d$.

It follows from the preceding formulæ that the numerical value of each of the four terms of a proportion is expressible by a suitable combination of the other three terms, and that, consequently, if any three terms of a proportion are given the fourth term can be found.

The process indicated by these formulæ for deducing a fourth term from the other three is, to divide the product of the mean terms by the given extreme, or the product of the extreme terms by the given mean, according as the term sought is an extreme or a mean. Otherwise,

If the term sought is an antecedent, multiply its consequent by the ratio of the other antecedent to its consequent; and if the term sought is a consequent, multiply its antecedent by the ratio of the other consequent to its antecedent; the result is the antecedent or consequent required.

336. The formulæ of the two preceding articles show that when, of the four numbers composing a proportion, three are given and one is unknown, this, which may be represented by the symbol x, can be deduced from the others, and that the three given terms can be always so arranged that x shall fall in the fourth place; or, $a : b :: c : x$.

Whence, by Article 335, $x=\frac{bc}{a}=\frac{b}{a}\times c$,

That is, the unknown quantity is equal to the product of the second and third terms divided by the first; or it is equal to the third term multiplied by the ratio which the second term of the proportion has to the first term.

In the case of continual proportionals, let

$$a : b :: b : c;$$

$$\text{Then } a=\frac{b^2}{c},\ b^2=ac \text{ and } b=\sqrt{ac},\ c=\frac{b^2}{a}.$$

337. Let $a : b :: c : d$, or, which is the same thing, $\frac{a}{b}=\frac{c}{d}$.

$$\text{Then } \frac{a}{b}=\frac{a\times m}{b\times m}=\frac{a\div m}{b\div m}=\frac{c}{d}.$$

$$\therefore a\times m : b\times m :: c : d, \text{ and } a\div m : b\div m :: c : d.$$

Similarly, $a : b :: c\times n : d\times n$, and $a : b :: c\div n : d\div n$.

$$\therefore a\times m : b\times m :: c\times n : d\times n, \text{ and } a\div m : b\div m :: c\div n : d\div n.$$

In these proportions m, n denote any numbers. If $m=n$,

$$ma : mb :: mc : md.$$

$$a\div m : b\div m :: c\div m : d\div m.$$

Whence, in any proportion,

If the first and second terms are multiplied or divided by the same number, the third and fourth terms remaining unchanged;

Or, if the third and fourth terms are multiplied or divided by the same number, the first and second terms remaining unchanged;

Or, if the first and second terms are both multiplied or both divided by one number, and the third and fourth terms are either both multiplied or both divided by that or by some other number;

In each of these cases the results are proportional.

Again, let $\frac{a}{b}=\frac{c}{d}$.

$$\text{Then } \frac{a}{b}\times m=\frac{c}{d}\times m, \text{ or } \frac{a\times m}{b}=\frac{c\times m}{d}.$$

$$\text{Also, } \frac{a}{b}\div m=\frac{c}{d}\div m, \text{ or } \frac{a}{b\times m}=\frac{c}{d\times m}.$$

Therefore, if $a : b :: c : d$,

$$a\times m : b :: c\times m : d,$$

$$a : b\times m :: c : d\times m.$$

Similarly, $a\div m : b :: c\div m : d$,

$$a : b\div m :: c : d\div m.$$

Whence, if the first and third terms of a proportion are both multiplied or both divided by the same number, the multiple or part of the first term is to the second term as the multiple or part of the third term is to the fourth term.

And if the second and fourth terms are both multiplied or both divided by the same number, the first term is to the multiple or part of the second term as the third term is to the multiple or part of the fourth term.

338. It is proved (Art. 335) that if any three terms of a proportion are given, the fourth term can be found. Whenever, therefore, the conditions of a question are such that the data can be converted into three of the terms of a propertion, of which proportion the unknown quantity is the remaining term, it is evident that the question can be resolved by proportion.

The terms of any proportion, derived from a practical question whose solution involves the finding of a fourth proportional to three given numbers, are usually arranged in such a manner that the unknown term (x) falls in the fourth place. By this arrangement x is given by the formula

$$x=\frac{\text{third term}\times\text{second term}}{\text{first term}}=\text{third term}\times\frac{\text{second term}}{\text{first term}}.$$

When a question in proportion involves only four terms, three given and one unknown, it seems the more direct course to give x, the representative of the unknown quantity, its proper place, whether of antecedent or consequent, and to deduce its value by the formulæ of Article 335.

But (as will hereafter appear) in questions involving more than four proportional numbers, it is of advantage to have the unknown term in the last place; the practice of carrying the unknown term, whether of four or a greater number of proportional quantities, to the last place is therefore followed, for the sake of uniformity, in all cases.

339. The manner of deducing the terms from the conditions of a question and of arranging them in proper order remains to be considered. This can be best done by taking examples.

Let it be required therefore (as a first example) to find the value of 384 yards of cloth, at the rate of 32 £. 10 sh. for 25 yards.

In this question the data are

1st. Two quantities of one kind, 25 yds. and 384 yds.;

2d. One quantity of another kind, 32 £. 10 sh.=650 sh.*;

3d. A relation between one of the quantities of the first kind and the single quantity of the second kind; namely, that the money value of 25 yds. is 32 £. 10 sh.

And the problem is, to find the money value of 384 yds., the other given quantity of the first kind.

Let the price of 384 yds. be represented by x.

Then, since the money values of different quantities of the same commodity have a constant relation to the quantities (Art. 327), it is evident that 384 yds. must contain the whole or some part of 25 yds. as often as x sh., the price of 384 yds., contains the whole or the same part of 650 sh., which is the price of 25 yds.

Or that 384 yards must be the same *multiple part* of 25 yds. that x sh. is of 650 sh.

Now the fraction $\frac{384}{25}$ expresses the repetitions of 25 yds. or of some part of 25 yds., which are contained in 384 yds., that is, $\frac{384}{25}$ expresses the ratio of 384 yds. to 25 yds.

And $\frac{x}{650}$ expresses, in like manner, the repetitions of 650 sh., or of some part of 650 sh., contained in x sh., or the ratio of x sh. to 650 sh.

These fractional expressions or ratios being equal,

$$\frac{384}{25}=\frac{x}{650}$$

$$\text{or } \overset{\text{yds.}}{384} : \overset{\text{yds.}}{25} :: \overset{\text{sh.}}{x} : \overset{\text{sh.}}{650};$$

$$\text{Whence (Art. 335) } x=\frac{384}{\underset{1}{\cancel{25}}}\times\overset{26}{\cancel{650}}\,\text{sh.}=9984\,\text{sh.}=498\,£.\ 4\,\text{sh.}$$

The price of 384 yds. at the given rate is therefore 498 £. 4 sh.

In the proportion deduced from the conditions of the preceding question, the unknown term is not in the fourth place. Taking the reciprocals of the ratios, which are $\frac{25}{384}$ and $\frac{650}{x}$, the proportion $25 : 384 :: 650 : x$ is obtained, in which x is the last term.

This position of x has been obtained by changing the arrangement of the terms of a proportion already formed out of the same quantities. The forming of a proportion in order that another, with x in the last place, may be deduced from it is, however, not necessary.

340. In every proportion deduced from a practical question and consisting of four terms, two of the terms are quantities of one kind, and the other two, either quantities of a different kind, or, if of the same kind, distinguished from the first and second terms by the meaning of the question.

Three of these terms, moreover, are given, and the remaining term, x, is to be found.

Now, if x is in the fourth place, the term which is of the same kind with x must be in the third, otherwise it will be required to find the ratio of two unlike quantities. The term which is of the same kind as the unknown quantity must therefore be written in the third place.

The two remaining terms of the same kind must consequently fall, one in the first, the other in the second place.

To determine the absolute place of each in a given case, it is necessary to resort to the meaning of the question by which the terms are given, and to the formula,

$$x = \text{third term} \times \frac{\text{second term}}{\text{first term}}.$$

The terms whose places are to be fixed are unequal in respect of quantity. If in this formula *first* is greater than *second*, the value of the ratio $\frac{\text{second term}}{\text{first term}}$ is expressed by a proper fraction, and x is less than the third term; and if the *first* is less than the second, the ratio $\frac{\text{second term}}{\text{first term}}$ is expressed by an improper fraction, and x is greater than the third term.

Now, in every question which can be resolved by proportion, the conditions must indicate whether the fourth term or unknown quantity, x, ought to be less or greater than the single term which is of the same kind as x, and which is made the third term of the proportion.

If x ought to be less than the third term, the greater of the two remaining terms is written in the first place of the proportion, and the less in the second; but if x ought to be greater than the third term, the less of the two remaining terms is written in the first place and the greater in the second.

The formula $x = \text{third term} \times \frac{\text{second term}}{\text{first term}}$, then gives the value of the unknown quantity.

Applying these principles to the preceding question: the quantity which is to be found (x) is a sum of money; therefore 650 sh., the quantity of the same kind, is the third term of the proportion.

Again, by the question, 25 yds. cost 650 sh.

Therefore 384 yds. must cost more than 650 sh.

Therefore x is greater than the third term.

Whence 25 yds. is the first and 384 yds. the second term of the proportion, which, as already found, is

$$\overset{\text{yds.}}{25} : \overset{\text{yds.}}{384} :: \overset{\text{sh.}}{650} : \overset{\text{sh.}}{x}.$$

2d Example. The price of 17 cwt. 3 qrs. 19 lb. of coffee being 169 £. 6 sh. 9¾ d., what quantity can be bought with 31 £. 14 sh. 6 d. at the same rate?

The unknown quantity being that weight of coffee whose money value is 31 £. 14 sh. 6 d., that given quantity which is of the same kind, viz. 17 cwt. 3 qrs. 19 lb., is the third term of the proportion.

Next, 169 £. 6 sh. 9¾ d. is the money value of 17 cwt. 3 qrs. 19 lb., and 31 £. 14 sh. 6 d. that of x cwt.

The money value of x cwt. being less than that of 17 cwt. 3 qrs. 19 lb., and commodities of the same kind having to each other the same relation in respect of quantity as their money values, it follows that x cwt. is less than 17 cwt. 3 qrs. 19 lb.; and therefore (Art. 340) that 119 £. 6 sh. 9¾ d. is the first term of the proportion, and 31 £. 14 sh. 6 d. the second:

$$\text{Or that, } \overset{\text{£.}}{169}\ \overset{\text{sh.}}{6}\ \overset{\text{d.}}{9\tfrac{3}{4}} : \overset{\text{£.}}{31}\ \overset{\text{sh.}}{14}\ \overset{\text{d.}}{6} :: \overset{\text{cwt.}}{17}\ \overset{\text{qrs.}}{3}\ \overset{\text{lb.}}{19} : \overset{\text{cwt.}}{x}.$$

$$\therefore x \text{ cwt.} = 17 \text{ cwt. } 3 \text{ qrs. } 19 \text{ lb.} \times \frac{31\ \text{£. } 14\ \text{sh. } 6\ \text{d.}}{119\ \text{£. } 6\ \text{sh. } 9\tfrac{3}{4}\text{d.}}$$

The quotient of 31 £. 14 sh. 6 d. by 119 £. 6 sh. 9¾ d. being obtained by the method of Article 326, and the given weight 17 cwt. 3 qrs. 19 lb. multiplied by this quotient, the result is the value of x.

Now 31 £. 14 sh. 6 d.=30456 qrs., and 169 £. 6 sh. 9¾d.=162567 qrs.

$$\therefore \frac{31\text{ £. } 14\text{ sh. } 6\text{ d.}}{169\text{ £. } 6\text{ sh. } 9\tfrac{3}{4}\text{ d.}}=\frac{30456}{162567}=\frac{3^4\times 376}{3^4\times 2007}=\frac{376}{2007}.$$

The compound quantity 17 cwt. 3 qrs. 19 lb. may be multiplied by the numerator and divided by the denominator of this fraction; in which case the value of x will be found in terms of cwts. qrs. and lb., or of a compound number expressed in terms of one or more of these denominations. But in this instance the process by reduction of the compound multiplicand to lb., and the multiplication of the number of lb. by $\frac{376}{2007}$ is shorter.

$$17\text{ cwt. } 3\text{ qrs. } 19\text{ lb.}=2007\text{ lb.}$$

$$\therefore x = 2007\text{ lb.}\times\frac{376}{2007}=376\text{ lb.}$$

or, since 376 lb.=3 cwt. 1 qr. 12 lb.

$$x=3\text{ cwt. } 1\text{ qr. } 12\text{ lb.}$$

In practice, the calculation is made by reducing the first and second terms, if compound numbers, to the lowest denomination contained in either of them; and the third term, if also a compound number, to the lowest denomination which is contained in it; multiplying the third term by the second, and dividing the product by the first; the result is expressed in units of that denomination to which the third term was reduced.

And since this process is equivalent to the multiplication of the third term by a fractional expression, whose numerator is the second and its denominator the first term of the proportion (Art. 188), the result which it affords is evidently the fourth term of the proportion or quantity required.

The expressions $x = 32\text{ £. } 10\text{ sh.}\times\frac{384}{25}$,

$$x=17\text{ cwt. } 3\text{ qrs. } 19\text{ lb.}\times\frac{31\text{ £. } 14\text{ sh. } 6\text{ d.}}{119\text{ £. } 6\text{ sh. } 9\tfrac{3}{4}\text{ d.}},$$

which involve the answers to questions 1 and 2, are identical with the results given in the same examples by the method of Article 327.

3d Example: If 135 men excavate a canal in 20 days, how long will it take 300 men to accomplish the same work, supposing that the part done daily by each man of the 300 and of the 135 is the same.

The unknown quantity, or answer to the question, is days; 20 days is therefore the third term of the proportion.

135 men taking 20 days to produce a certain effect, the problem is to find the number of days in which 300 men can produce the same effect.

Now, if 135 men produce a given effect in 20 days, twice 135 or 270 men produce twice that effect in 20 days, or once that effect in 10 days; three times 135 or 405 men produce three times that effect in 20 days, or once that effect in 6⅔ days.

Whence, since twice the number of men take half the time, and three times the number of men take one third of the time, which once the number take, to produce a given effect; it follows that the greater the number of men employed, the less is the time taken to execute the same amount of work. Therefore 300 men excavate the canal in less than 20 days.

The fourth term is therefore less than the third; 300 is therefore the first and 135 the second term of the proportion; that is,

$$300\text{ men} : 135\text{ men} :: 20\text{ days} : x\text{ days},$$

$$\text{and } x = 20\times\frac{135}{300}=20\times\frac{27}{60}=\frac{540}{60}=9\text{ days.}$$

4th Example. If when the price of wheat is 5 sh. per bushel, the weight of one pennyworth of bread is 16 oz.; what ought the penny loaf to weigh when the bushel of wheat of the same quality costs 8 sh.?

The third term of the proportion is 16 oz. and x oz. the fourth.

To determine the first and second terms; when wheat is at 5 sh. or 60 d. per bushel, 16 oz. is the 60th part of the whole weight of bread made from the bushel; when at 8 sh. or 96 d. per bushel, x oz. is the 96th of the whole weight made from the bushel. The whole weights being by hypothesis equal, and the 96th part, therefore, less than the 60th, it follows that x is less than 16 oz.

The third term being greater than the fourth, the first must be greater than the second.

$$\therefore 8 \text{ sh.} : 5 \text{ sh.} :: 16 \text{ oz.} : x \text{ oz.}$$

$$\text{And } x=16\times\frac{5}{8}=\frac{\overset{2}{\cancel{16}}\times 5}{\underset{1}{\cancel{8}}}=10 \text{ oz.}$$

341. Proportions involving the solutions of the preceding questions have been deduced from the conditions of the several questions by the method of Article 340. But although obtained by the application of the same general method, the proportions deduced from Examples 1 and 2 differ from those deduced from Examples 3 and 4.

Four numbers or terms are involved in each of these questions. Of these four terms two are of one kind; the other two (one of which is x) of a different kind. Each term also of the second kind is connected by the conditions of the question with one of the terms of the first kind. Thus, in the first Example, two of the four numbers express lengths; and two the respective money values of these lengths. The money value of the first length is connected by the question with the first length, and the money value of the second length with the second length. One length and its money value are the antecedents of the proportion; the other length and its money value the consequents.

The second Example is like the first; for two of the terms express weights, and the other two the prices of these weights respectively. One weight and its price form the antecedents, the other weight and its price the consequents of the proportion.

If in a proportion the numbers or quantities which have to each other a relation similar to that between quantity and price are named *related terms*, the antecedents of the proportions deduced from Examples 1 and 2 are related terms, and the consequents are related terms. Likewise, the first number of the first kind and its related term form an extreme and a mean; the second number of the first kind and its related term a mean and an extreme.

In the third Example two of the four terms express the numbers of men, in two different parties, employed to execute the same quantity of work; and the two other terms the times or numbers of days taken by these parties respectively to execute that work.

The first term of the first kind, expressing the number of men in the first party, is connected by the question with the second term of the second kind, that, namely, expressing the time taken by the first party to execute the work; and the second term of the first kind, which expresses the number of men in the second party, is connected with the first term of the second kind, which expresses the time taken by the second party to execute the same work. One number of men and the related time form an antecedent and a consequent of the proportion, the other number of men and time a consequent and an antecedent.

The fourth Example is like the third. Two of the terms are sums of money, the prices, at different times, of the same quantity of a commodity whose money value is subject to considerable variation; the two others the weights of the different portions of that commodity which correspond to the same unit of value at these different prices.

One sum of money and the related weight form an antecedent and consequent of the proportion, the other sum of money and its related weight a consequent and an antecedent.

Otherwise, in Examples 3 and 4, one term of the first kind and its related term form the extremes of the proportion, the other term of the first kind and its related term the means.

When two related terms form the antecedents and the other two the consequents of a proportion, or when two related terms form an extreme and a mean, and the other two a mean and extreme, the relation of the numbers is termed *direct*, and the two numbers of the first kind are said to be *directly proportional* to the related numbers of the second kind.

And when two related terms form an antecedent and a consequent, and the other two a consequent and an antecedent, or when two related terms form the extremes and the other two the means of a proportion, the relation of the numbers is termed *indirect*, and the two numbers of the first kind are said to be *inversely* or *reciprocally proportional* to the related numbers of the second kind.

In No. 1 and 2 of the preceding Examples the proportion of the four quantities to each other is direct, and in No. 3 and 4 it is inverse.

In direct proportion each term of the first kind is said to be in the direct ratio of the related term of the second kind, and in inverse proportion each term of the first kind is said to be in the inverse ratio of the related term of the second kind.

Thus, the price of a quantity of cloth, for example, is in the direct ratio of the number of yards of the quantity, for the greater the number of yards the greater the sum of money which must be paid for them.

But the time required by a party of men for the execution of a given work is in the inverse ratio of the number of men forming the party, for the greater the number of men employed the less is the time required to complete the same amount of work.

342. This distinction of proportion into direct and inverse is not made because of any difference in the manner of deducing either from the conditions of a question, but because the terms *direct ratio* and *inverse ratio* are of frequent occurrence in mathematical treatises.

Two fractions, for instance, which have the same denominator are said to be in the direct ratio of their numerators, and two fractions which have the same numerator are said to be in the inverse ratio of their denominators.

To explain these expressions, assume two fractions, $\frac{5}{14}$ and $\frac{9}{14}$, which have the same denominator, multiplying each of these fractions by 14,

$$\frac{5}{14} : \frac{9}{14} :: \frac{5}{14}\times 14 : \frac{9}{14}\times 14 \text{ (Art. 337); that is, } \frac{5}{14} : \frac{9}{14} :: 5 : 9.$$

Now the fraction $\frac{5}{14}$, and 5 the numerator of this fraction, are the two antecedents of the proportion.

And the fraction $\frac{9}{14}$, and 9 the numerator of this fraction, are the two consequents.

Therefore the two fractions are directly proportional to their numerators, or are in the direct ratio of their numerators.

Assume next the fractions $\frac{7}{11}$ and $\frac{7}{13}$, which have the same numerator.

Multiplying each of these fractions by the expression $\frac{11\times 13}{7}$;

$$\frac{7}{11} : \frac{7}{13} :: \frac{7}{11}\times\frac{11\times 13}{7} : \frac{7}{13}\times\frac{11\times 13}{7} \text{ (Art. 337);}$$

$$\text{or } \frac{7}{11} : \frac{7}{13} :: 13 : 11.$$

In this proportion the first fraction, $\frac{7}{11}$, and its denominator, 11; are the extreme terms, the other fraction, $\frac{7}{13}$, and its denominator, 13, the means; therefore two fractions which have the same numerator are reciprocally proportional to their denominators, or are to each other in the inverse ratio of their denominators.

343. The relation between the unknown quantity or last term of a proportion, and the given term of the same kind, may depend on many conditions which it may be necessary to combine in order to determine the unknown quantity.

If it be supposed, for instance, that 20 workmen in 18 days execute 500 yards of a certain work, and from this it be required to determine the number of days in which 76 workmen can execute 1265 yards of the same work, the number of days must evidently depend upon the number of workmen employed and the number of yards to be executed.

The data in this question (Ex. 5) are,

The number of workmen in each of two parties, 20 and 76:

Two quantities of work, 500 yards and 1265 yards:

The time in which the first party of 20 workmen can execute 500 yards of the work; viz. 18 days:

And the problem is, to find the time taken by 76 men, working at the same rate, to execute 1265 yards of the same work.

The answer to the question being *time* the term 18 days is the third in the proportion.

If the time taken by 76 men to execute 500 yards of the work were known, the time required for the execution of 1265 yards at the same rate could be found; let the first be represented by x, the second by x'.

The first step, therefore, is to find x, the time taken by 76 men to execute that amount of work (500 yards) which is done by 20 men in 18 days.

Now, if 20 men execute a work in 18 days, 76 men execute it in less than 18 days.

Therefore $76 \text{ men} : 20 \text{ men} :: 18 \text{ days} : x \text{ days} = 18 \times \frac{20}{76} \text{ days}$.

The second step is to find x', the time taken by 76 men to execute 1265 yards of work at the rate of 500 yards in x days ($= 18 \times \frac{20}{76}$ days).

500 yards, the term related to the time x days, being less than 1265 yards, the term related to x' days, the third term is less than the fourth, which is therefore given by the proportion.

$$500 \text{ yds.} : 1265 \text{ yds.} :: x \text{ days} : x' \text{ days} = x \times \frac{1265}{500}.$$

Restoring the value of x from the first proportion,

$$x' = 18 \times \frac{20}{76} \times \frac{1265}{500} = 11\frac{187}{190} \text{ days}.$$

This result is composed of the continual product of three factors; of these the first is that quantity which is of the same kind as the answer to the question.

The second is a fraction having for denominator the antecedent and for numerator the consequent of the first ratio of a proportion of which the first pair of quantities of the same kind and that quantity which is of the same kind as the answer form three terms.

The third is a fraction having for denominator the antecedent and for numerator the consequent of the first ratio of a second proportion of which the second pair of quantities of the same kind and the fourth term of the first proportion form three terms.

The numerical value of the result is obtained by multiplying the third term of the first proportion by the product of the numerators of these fractional factors, and dividing the result by the product of their denominators (Art. 194), that is, by multiplying the number which is of the same kind as the answer by the product of the consequents of the first ratios of all the proportions, and dividing the result by the product of the antecedents of the first ratios of the same proportions.

344. The antecedent and consequent of each pair of terms of the same kind in these proportions can be determined without forming the fourth terms.

Let a be the antecedent, c the consequent of the first pair of terms of the same kind.
a' - - c' - - second - do.
a'' - - c'' - - third - do.
. . . .
t the third term of the first proportion.

Then $a : c :: t : t\times\frac{c}{a}$=4th term of 1st proportion.

$a' : c' :: t\times\frac{c}{a} : t\times\frac{c}{a}\times\frac{c'}{a'}$=4th term of 2d proportion.

But $t\times\frac{c}{a} : t\times\frac{c}{a}\times\frac{c'}{a'} :: t : t\times\frac{c'}{a'}$ (Art. 337).

$\therefore a' : c' :: t : t\times\frac{c'}{a'}$.

Whence the arrangement of the terms a', c', determined by comparison with t, the third term of the 1st proportion, must agree with the arrangement of a', c', determined by comparison with $t\times\frac{c}{a}$, the 4th term of the 1st proportion.

Again, $a'' : c'' :: t\times\frac{c}{a}\times\frac{c'}{a'} : t\times\frac{c}{a}\times\frac{c'}{a'}\times\frac{c''}{a''}$=4th term of 3d proportion.

But $t\times\frac{c}{a}\times\frac{c'}{a'} : t\times\frac{c}{a}\times\frac{c'}{a'}\times\frac{c''}{a''} :: t : t\times\frac{c''}{a''}$.

$\therefore a'' : c'' :: t : t\times\frac{c''}{a''}$.

Whence the arrangement of the terms a'', c'', determined by comparison with t, must also agree with the arrangement of a'', c'', determined by comparison with $t\times\frac{c}{a}\times\frac{c'}{a'}$, the 4th term of the second proportion.....

The calculation of the fourth terms of these proportions is, therefore, unnecessary.

All that is requisite is, to determine the antecedent and consequent of each pair of terms of the same kind by comparison with the number which is of the same kind as the answer to the question, to multiply this number by the product of all the consequents, and to divide the result by the product of all the antecedents.

6th Example. If 15 men working 10 hours a day take 18 days to execute 450 yds. of a certain work, how many men of equal strength, who work 12 hours a day, are required to complete 480 yds. of the same work in 8 days?

The answer to the question being the number (x) of a party of men able to execute a given work under the conditions of the question, 15 (men) is the third term of the proportion.

The pairs of numbers of the same kind, from the comparison of which with 15 the antecedent and consequent of each pair are to be determined, are,

10 hours term related to 15 men; 12 hours term related to x men.
18 days - - - do. 8 days - - - do.
450 yds. - - do. 480 yds. - - do.

Let f', f'', f''', respectively denote the fourth terms of the three proportions which must be employed to determine which of these quantities are antecedents and which consequents; then,

1st. In 10 hours a certain work is executed by 15 men;
in 12 hours the same work is done by fewer than 15 men;
$\therefore$ 12 hrs : 10 hrs :: 15 men : f' men, or 6 : 5 :: 15 : f' (Art. 337).

2d. To execute this work in 18 days 15 men are required;
to execute it in 8 days more than 15 men are required;
$\therefore$ 8 days : 18 days :: 15 men : f'' men, or 4 : 9 :: 15 : f''.

3d. To execute 450 yds., under given conditions, 15 men are required;
to execute 480 yds., under the same conditions, more than 15 are required;
$\therefore$ 450 yds. : 480 yds. :: 15 men : f''' men, or 15 : 16 :: 15 : f'''.

Collecting and arranging the first ratios of these proportions, multiplying the general third term by the product of all the consequents of these ratios, and dividing the result by the product of all the antecedents, as under:

$$\left.\begin{array}{lll} 12\text{ hrs.} & : 10\text{ hrs.} & = 6 : 5 \\ 8\text{ days} & : 18\text{ days} & = 4 : 9 \\ 450\text{ yds.} & : 480\text{ yds.} & =15 : 16 \end{array}\right\} :: 15\text{ men} : x\text{ men} = \frac{\overset{1}{\cancel{15}}\times 5\times\overset{3}{\cancel{9}}\times\overset{\overset{2}{\cancel{4}}}{\cancel{16}}}{\underset{\underset{1}{\cancel{2}}}{\cancel{6}}\times\underset{1}{\cancel{4}}\times\underset{1}{\cancel{15}}} = \frac{30}{1} = 30;$$

x, the number of men required, is therefore 30.

The reduction in the ratios of the quantities of the same kind is made by Article 337; and the omission of factors common to the numerator and denominator of the value of x by Article 192.

When three terms of a proportion are given to find the fourth the proportion is simple.

But when five terms are given to find the sixth, seven to find the eighth, &c., the proportion is said to be compound.

Examples 5 and 6 are instances of compound proportion.

7th Example. Find a third proportional to the numbers 4, 6; and a mean proportional between the numbers 3, 27.

Let x be a third proportional to the numbers 4, 6;

And x' a mean proportional between the numbers 3, 27;

Then, by Article 336,

$$4 : 6 :: 6 : x \therefore x = \frac{6\times 6}{4} = \frac{36}{4} = 9;$$

$$\text{and } 3 : x' :: x' : 27 \therefore x'^2 = 3\times 27 = 81 \text{ and } x' = \sqrt{81} = 9.$$

Hence, a third proportional to two given numbers is obtained by dividing the square of the mean term by the given extreme, and a mean proportional between two given extremes is obtained by extracting the square root of the product of the given extremes.

345. General rule for proportion.

When three terms of a proportion are given to find the fourth, write as the third term that number which is of the same kind as the answer to the question; then, if it appears from the meaning of the question that the answer ought to be less than this third term, make the greater of the two remaining numbers the antecedent and the less the consequent of the first ratio; but if it appears that the answer ought to be greater than this third term, make the less of the two remaining numbers the antecedent and the greater the consequent of the first ratio.

When five, seven, nine, ... terms of a proportion are given to find a sixth, an eighth, a tenth, ... term, distinguish the number which is of the same kind as the answer, and separate the other given numbers into two, three, four, ... pairs of quantities, each pair consisting of like numbers.

Compare the first pair of numbers with the number like the answer, and determine the antecedent and consequent of a proportion of which the two numbers of the first pair are the first and second terms and the number like the answer the third term.

Determine, in the same manner, the antecedent and consequent of each pair of terms of the same kind.

Arrange these ratios under each other in two vertical columns, the first containing all the antecedents, the second all the consequents, and write the number, which is like the answer, as their general third term.

Reduce the single antecedent and consequent in simple proportion, and the antecedent and consequent of each pair of terms of the same kind in compound proportion, to the same denomination; reduce also the third term to the lowest denomination contained in it.

Then multiply the third term by the consequent or product of the consequents, and divide the result by the antecedent or product of the antecedents; the number thus obtained is the fourth term of the proportion or unknown quantity.

a. To find a third proportional to two given numbers, divide the square of the second number by the first number, the quotient is the third proportional required.

And to find a mean proportional between two given numbers, multiply one given number by the other, and extract the square root of the product; this square root is the mean proportional required.

346. Exercises in the rules of proportion.

1st. Find a third proportional to the numbers 24 and 60 ? *Ans.* 150.

2d. - - - - - - $\frac{3}{8}$ and $\frac{5}{12}$? *Ans.* $\frac{25}{54}$.

3d. - - - - - - ·06 and ·099 ? *Ans.* ·16335.

4th. - - - - - - ·3 and ·027 ? *Ans.* ·00243.

5th. - - - - - - $\frac{2}{5}$ and 20 ? *Ans.* 1000.

6th. Find a fourth proportional to the numbers 12, 28, and 42 ? *Ans.* 98.

7th. - - - - - - $\frac{5}{7}$, $\frac{8}{9}$, and $\frac{10}{21}$? *Ans.* $\frac{16}{27}$.

8th. - - - - - - ·05, ·8, and ·79 ? *Ans.* 12·64.

9th. - - - - - - ·01, 7, and $8\frac{1}{4}$? *Ans.* 5775.

10th. Find a mean proportional between 9 and 225 ?.....*Ans.* 45.

11th. - - - - - $\frac{4}{7}$ and $\frac{25}{28}$?.......*Ans.* $\frac{5}{7}$.

12th. - - - - - ·015 and ·375 ? *Ans.* ·075.

13th. What is the price of 96 yds. of cloth at the rate of 8 yds. for 1 £. 4 sh. ?..................................*Ans.* 14 £. 8 sh.

14th. A person's annual income being 146 £., how much is that per day ?..*Ans.* 8 sh.

15th. If the rent of 9 acres of land is 5 £. 12 sh., what is the rent of 72 acres at the same rate?...... *Ans.* 44 £. 16 sh.

16th. If 7 cwt. 1 qr. of sugar cost 26 £. 10 sh. 6 d., what is the price of 43 cwt. 2 qrs.?.................... *Ans.* 159 £. 2 sh.

17th. How many workmen are required to execute in 15 days as much work as is done by 5 men of equal strength in 24 days?.. *Ans.* 8 men.

18th. If 3 paces make 2 yds, how many yards do 160 paces make?.. *Ans.* 106⅔ yards.

19th. What is the price of 6 bushels of oats at the rate of 1 £. 14 sh. 6 d. per chaldron of 36 bushels? *Ans.* 5 sh. 9 d.

20th. What is the value of a bar of pure silver weighing 73 lb. 5 oz. 15 dwt., at the rate of 5 sh. 9 d. per ounce?
Ans. 253 £. 10 sh. 0¾ d.

21st. A garrison of 536 men have provisions for a year (365 days); how long will these provisions last, at the same rate for each man, if the garrison is increased to 1124 men?.............................. *Ans.* $174\frac{16}{281}$ days.

22d. What is the amount of tax upon 763 £. 15 sh. at the rate of 3 sh. 6 d. per pound sterling? *Ans.* 133 £. 13 sh. 1½ d.

23d. What quantity of corn can be bought for 42 £. at the rate of 6 sh. per bushel?.................... *Ans.* 39 qrs. 3 bush.

24th. A party of labourers working 4 hours daily complete a certain work in 12 days, what time ought the same party to take if they work 6 hours a day?.......... *Ans.* 8 days.

25th. What cost 20 pieces of lead, each weighing 1 cwt. 12 lb., at the rate of 16 sh. 4 d. per cwt.?...... *Ans.* 18 £. 1 sh. 8 d.

26th. A plain of a certain extent supplied 3000 horses with forage for 20 days; for how many days ought the same plain to have supplied 2000 horses?......... *Ans.* 27 days.

27th. The debts of a bankrupt amount to 977 £., and his property to 420 £. 6 sh. 3¼ d.; how much per pound can he pay to his creditors?.............................. *Ans.* 8 sh. 7¼ d.

28th. Suppose that a gentleman's income is 525 £. a year, and that his average daily expenditure is 19 sh. 7 d.; how much does he save yearly?......... *Ans.* 167 £. 12 sh. 1 d.

29th. The governor of a besieged fortress has bread for 54 days, at the rate of 1½ lbs. per day for each soldier; at what rate must he fix the daily allowance if he intends to hold out 81 days?.. *Ans.* 1 lb.

30th. If the penny loaf weighs 9 ounces when the bushel of wheat costs 6 sh. 3 d., what ought it to weigh when the price of wheat is 8 sh. 2½ d. per bushel?
Ans. 6 oz. $13\frac{127}{197}$ dr.

31st. At 1 £. 7 sh. 8 d. the acre; what is the rent of 173 ac. 2 ro. 14 poles of land?............ *Ans.* 240 £. 2 sh. $7\frac{1}{20}$ d.

32d. If 5 yds. of cloth cost 14 sh. 2 d., what must be given for 9 pieces, each measuring 21 yds. 1 qr.? *Ans.* 27 £. 1 sh. 10½ d.

33d. If a gentleman's estate is worth 2107 £. 12 sh. a year, what may he spend per day in order to save 500 £. of his yearly income?.................. *Ans.* 4 £. 8 sh. $1\frac{19}{365}$ d.

34th. If for the carriage of 5 cwt. 14 lbs. a distance of 96 miles the charge is 1 £. 12 sh. 6 d., how far ought 3 cwt. 1 qr. to be carried for the same money?

Ans. 151 mi. 3 fur. $3\frac{1}{13}$ po.

35th. What is the expense of keeping 11 horses for a year, at the rate of $11\frac{1}{2}$ d. per day for each horse?

Ans. 192 £. 7 sh. $8\frac{1}{2}$ d.

36th. What cost 43 qrs. 5 bush. of corn at 1 £. 8 sh. 6 d. per quarter?..............................*Ans.* 62 £. 3 sh. $3\frac{3}{4}$ d.

37th. If 3 cwt. of tea cost 40 £. 12 sh., at how much per lb. must it be retailed in order that 10 £. may be gained by the sale of the whole quantity?..............*Ans.* $3\frac{1}{84}$ sh.

38th. How many men are required to cut a trench of 135 yds. in length in 8 days, when 16 men complete 54 yds. of the same trench in 6 days?..............*Ans.* 30 men.

39th. A family of 9 persons expends 120 £. in 8 months; what sum ought to serve a family of 24 persons during 16 months, at the same rate of expenditure?...*Ans.* 640 £.

40th. 27 shillings being the wages of 4 men for 7 days; required the wages of 14 men for 10 days at the same rate?

Ans. 6 £. 15 sh.

41st. A courier travels 130 miles in 3 days of 12 hours each; in how many days of 10 hours each can he travel 360 miles at the same hourly rate?..............*Ans.* $9\frac{63}{65}$ days.

42d. If 120 bush. of corn serve 14 horses 56 days, how many days will 94 bush. serve 6 horses?......*Ans.* $102\frac{16}{45}$ days.

43d. If $\frac{3}{8}$ yds. cost $\frac{3}{5}$ £., what ought $\frac{5}{16}$ yds. to cost?..*Ans.* 10 sh.

44th. If $\frac{3}{16}$ of a lottery ticket is worth 273 £. 2 sh. 6 d., what is $\frac{5}{32}$ of the same ticket worth?...*Ans.* 227 £. 12 sh. 1 d.

45th. If $\frac{1}{8}$ of a ship is worth 73 £. 1 sh. 3 d., what part of her is worth 250 £. 10 sh.?...............................*Ans.* $\frac{3}{7}$.

46th. If the penny loaf weigh 7 oz. when wheat is at 5 sh. 6 d. the bushel, what is the bushel worth when the penny loaf weighs $2\frac{1}{2}$ oz.?........................*Ans.* 15 sh. $4\frac{4}{5}$ d.

47th. If $2\frac{1}{4}$ lbs. of bread are bought for $2\frac{1}{2}$ d. when the price of wheat is 4 sh. per bushel, what ought the 20-penny loaf to weigh when wheat is 6 sh. per bush.?......*Ans.* 12 lbs.

48th. The expense of building a wall 6 feet thick, 9 feet high, and 432 feet long, being 720 £., what must be the cost of building another wall 12 feet thick, 18 feet high, and 576 feet long, at the same rate?..............*Ans.* 3840 £.

49th. If 240 men in 5 days of 11 hours each dig a trench 230 yds. long, 3 yds. wide, and 2 yds. deep, in how many days of 9 hours long can 24 men dig a trench 420 yds. long, 5 yds. wide, and 3 yds. deep?

Ans. $278\frac{68}{69}$ days.

50th. If 24 pioneers in $2\frac{1}{3}$ days of $12\frac{1}{2}$ hours long can make a trench 144·75 yds. long, $4\frac{1}{2}$ yds. wide, and $2\frac{1}{3}$ yds. deep, what length of trench can 90 pioneers make in $4\frac{1}{5}$ days of $9\frac{2}{3}$ hours long, the trench being $4\frac{7}{8}$ yds. wide and $3\frac{1}{5}$ yds. deep?.............................*Ans.* $508\frac{1193}{2080}$ yds.

INTEREST.

347. If the sum of 5£. is paid for the use of 100£. during 1 year, how much must be paid for the use of 1486£. during 3 years, at the same rate.

In this example of compound proportion,

5£., the quantity of the same kind as the answer, is the third term.

100£. and 1486£. are the terms of the first ratio.

1 year and 3 years are the terms of the second ratio.

100£. produce 5£., 1486£. must produce a greater sum; and 1 year giving 5£., 3 years must give a greater sum.

Whence,

$$\left.\begin{array}{r}100\pounds. : 1486\pounds.\\ 1\text{ year} : 3\text{ years}\end{array}\right\} :: 5\pounds. : x\pounds. = \frac{5\times1486\times3}{100} = 222\cdot7\pounds. = 222\pounds.\ 14\text{sh}.$$

Questions, in which the compensation made for the loan of 100£. for 1 year is given to find what must be paid for the use of any sum greater or less than 100£. during any given period of time, depend for their solutions on the rules of proportion.

Some particular names are employed to describe the data in such questions, and the rule expressed in terms of these names is called *the rule of Interest.*

The compensation paid yearly by the borrower to the lender for the use of 100£. is the *interest* of 100£. for 1 year. This (which in the preceding example is 5£.) is described as *the rate per cent. per annum.**

The rate per cent. per annum is a matter of agreement between borrower and lender. See the acts of parliament 7 Will. 4. & 1 Vict. cap. 80., 2 & 3 Vict. cap. 37., 3 & 4 Vict. cap. 83., and 4 & 5 Vict. cap. 54.

The money lent is named the *principal.*

The compensation for the loan is the *interest.*

The sum of the principal and interest is the *amount.*

Technically expressed, the preceding question runs thus: What is the interest of 1486£. for 3 years at 5 per cent. per annum?

The result already found is $x = \frac{5\times3\times1486}{100} = 222\pounds.\ 14\text{sh}.$

If p is put for the principal or sum lent, r for the rate per cent. per annum, and t for the time, it is found, as above, that

$$\left.\begin{array}{r}100 : p\\ 1 : t\end{array}\right\} :: r : x = \frac{prt}{100}.$$

In words; to find the interest of any sum of money for any given rate and time, multiply the principal by the rate per cent. per annum, this product by the time during which the money is lent, and divide the result by 100.

If the time is any number of years, t is a whole number; if months, or months and days (it being the custom in these calculations to reckon 12 months to the year, and 30 days to the month) the numerical value of t is fractional; if days only, the number of days may be made the numerator and 365 the denominator of the numerical value of t.

The interest of 100£. for 1 year being denoted by r, the interest of p£. for 1 year at rate r by i, and any period of time less or greater than a year by t;

$$100 : p :: r : i \text{ (Art. 347)} \therefore \frac{p}{100} = \frac{i}{r} \text{ (Art. 330).}$$

$$\text{But } \frac{i}{r} = \frac{i\times t}{r\times t} \therefore \frac{p}{100} = \frac{it}{rt},$$

$$\text{or } 100 : p :: rt : it \left(=\frac{prt}{100}\right).$$

---

* Cent. an abbreviation of centum, the Latin word for a hundred; per annum, or per an., also Latin for every year, yearly.

a. The value of each of the quantities i, p, r, can be expressed in terms of the other quantities involved in the proportion

$$100 : p :: rt : it.$$

For, by Article 335, $it=\frac{prt}{100}$; $i=\frac{pr}{100}$ - (A),

$$rt=\frac{100\times it}{p};\ r=\frac{100\times i}{p} \quad - \quad (B),$$

$$p=\frac{100\times it}{rt};\ p=\frac{100\times i}{r} \quad - \quad (C);$$

Whence,

348. Case A. The principal, rate per cent., and time being given, to find the interest,

Form the continual product of the numbers expressing the principal, rate per cent., and time, and divide this product by 100; the result is the interest required.

Example. What is the interest of 170£. for 1½ years at 5 per cent. per annum?

$$p=170\text{£.},\ r=5,\ t=1\tfrac{1}{2}=1{\cdot}5.$$

Substituting these values of p, r, t in formula A,

$$it=\frac{170\times 5\times 1{\cdot}5}{100}=\frac{1275}{100}=12{\cdot}75\text{£.}=12\text{£. } 15\text{ sh.}$$

349. Case B. The principal and its interest for any specified number of years being given, to find the rate per cent. per annum,

Multiply 100£. by the interest for the given time; divide the product by the principal, and this quotient by the number of years; or multiply 100£. by the interest for 1 year, and divide the product by the principal; the result in either case is the rate per cent. per annum.

Example. The interest of 25£. for 3½ years, being 3£. 18 sh. 9 d.; what is the rate per cent. per annum?

$$p=25,\ t=3\tfrac{1}{2}=3{\cdot}5,\ it=3\text{£. } 18\text{ sh. } 9\text{ d.}=3{\cdot}9375\text{£.}$$

$$i=\frac{it}{t}=\frac{3{\cdot}9375}{3{\cdot}5}=1{\cdot}125.$$

Substituting these values of p, t, and it in formula B,

$$rt=\frac{100\times 3{\cdot}9375}{25}=15{\cdot}75,$$

$$\text{and } r=\frac{15{\cdot}75}{3{\cdot}5}=4{\cdot}5=\text{rate per cent. per annum,}$$

$$\text{or } r=\frac{100\times 1{\cdot}125}{25}=4{\cdot}5.$$

350. Case C. The rate per cent., time, and interest for that time being given, to find the principal.

Multiply 100£. by the interest for the given time, and divide the result by the product of the rate per cent. and time; or multiply 100£. by the interest for 1 year, and divide the result by the rate per cent.; the quotient is the principal.

Example. What principal, being put to interest at 5 per cent. per annum, will in 7 years produce 86£. 16 sh. of interest?

$$it=86\text{£. } 16\text{ sh.}=86{\cdot}8,\ r=5,\ t=7.$$

Substituting these values of it, r, and t in formula C,

$$p=\frac{100\times 86{\cdot}8}{5\times 7}=\frac{8680}{35}=248\text{£.}$$

351. The principal rate per cent. per annum and interest for an unknown time may be given to find the time.

In this case let the interest of p for the unknown time be denoted by i'.

$$\text{Then } 100 : p :: rt : i'$$

$$rt = \frac{100 \times i'}{p} \text{ and } t = \frac{100 \times i'}{pr} \quad - \quad \text{(D)};$$

Whence,

Case D. The principal and its interest for an unknown time, at a given rate per cent., being given to find the time;

Form first the product of 100£. by the given interest; second, the product of the principal by the rate per cent. per annum; divide the first by the second product; the result is the time required.

Example. In what time will the interest of 547£. 15 sh. amount to 82£. 3 sh. 3 d. at 5 per cent. per annum?

$p = 547$£. 15 sh. $= 547{\cdot}75$£., $r = 5$, $i' = 82$£. 3 sh. 3 d. $= 82{\cdot}1625$£.

Substituting these values of p, r, i' in formula D,

$$t = \frac{100 \times 82{\cdot}1625}{547{\cdot}75 \times 5} = \frac{8216{\cdot}25}{2738{\cdot}75} = 3 \text{ years.}$$

$$\text{Since } 100 : p :: rt : it \text{ (Art. 347)},$$

$$\text{therefore } 100 : rt :: p : it \text{ (Art. 332)},$$

$$\text{or } \frac{rt}{100} = \frac{it}{p}.$$

Now, if equal numbers are added to the same number, the sums are equal; therefore,

$$1 + \frac{rt}{100} = 1 + \frac{it}{p}, \text{ or } \frac{100}{100} + \frac{rt}{100} = \frac{p}{p} + \frac{it}{p},$$

$$\text{or } \frac{100 + rt}{100} = \frac{p + it}{p} \text{ (Art. 171)},$$

$$\text{or } 100 : 100 + rt :: p : p + it.$$

In this proportion rt being the interest of 100£. for t years at r per cent. per annum, $100 + rt$ is the amount of 100£. for t years at rate r (Art. 347); and it being the interest of p£. for t years at rate r, $p + it$ is the amount of p£. for time t at rate r.

Denoting $p + it$ by a (the first letter of the word Amount),

$$a = \frac{(100 + rt) \times p}{100} \text{ (Art. 335)} \quad - \quad - \text{ (E)};$$

Whence,

352. Case E. The principal, rate per cent., and time being given, to find the amount,

Multiply the amount of 100£. for the given rate and time by the principal, and divide this product by 100; the result is the amount required.

Example. Find the amount of 450£. in $2\frac{1}{2}$ years, at four per cent. per annum?

$$r = 4,\ t = 2\tfrac{1}{2} = 2{\cdot}5,\ rt = 4 \times 2{\cdot}5 = 10,\ 100 + rt = 110,\ p = 450.$$

Substituting these values of rt and p in formula E,

$$a = \frac{110 \times 450}{100} = \frac{49500}{100} = 495 \text{£}.$$

The same result may be obtained by computing the interest of 450£. for $2\frac{1}{2}$ years at four per cent. per annum, and adding that interest to 450£. (Art. 347).

353. Numbers expressing three of the four quantities a, p, r, t, being given, the fourth can be found as in Article 347 a. It seems unnecessary to

repeat the detail of that article. Therefore let it further be only required to find p, having given r, t, a.

To determine p, the proportion

$$100 : 100+rt :: p : a$$

$$\text{gives } p=\frac{100\times a}{100+rt} \text{ (Art. 335) - - (F);}$$

Therefore,

Case F. Having given the rate per cent., time, and amount, to find the principal.

Multiply the given amount by 100, and divide the product by the amount of 100£. for the given rate and time; the result is the principal required.

Example. The amount of an unknown principal in 3½ years at 4½ per cent. per annum is 289£. 7 sh. 6 d. Required the principal?

$a=$289£. 7 sh. 6 d. $=289\cdot375$£.; $r=4\frac{1}{2}=4\cdot5$; $t=3\frac{1}{2}=3\cdot5$; $rt=4\cdot5\times 3\cdot5=15\cdot75$; $100+rt=100+15\cdot75=115\cdot75$.

Substituting these values of a, rt, in formula F,

$$p=\frac{289\cdot375\times100}{115\cdot75}=\frac{28937\cdot5}{115\cdot75}=250\text{£}.$$

354. If a sum of money, a, due by one person to another, is not payable until after the expiration of a certain time, t, the expression $\frac{100\times a}{100+rt}$ gives that other sum of money, p, which, placed at interest for the time t, and at a certain rate of interest, r, will produce the sum a.

When the rate per cent. per annum, time to come, and amount at the expiration of the given future time of an unknown principal are given to find that principal, the sum of money, a, is described as a£. due t years hence; p is termed the present worth, r the rate per cent. of discount; and the rule of interest, with these changes of terms and object, is called

The Rule of Discount.

The last example, expressed as a question of discount, is, What is the present worth of 289£. 7 sh. 6 d. due 3½ years hence, discount being allowed at the rate of 4½ per cent. per annum?

As already found, the present worth is,

$$p=\frac{289\cdot375\times100}{115\cdot75}=\frac{28937\cdot5}{115\cdot75}=250\text{£}.$$

The difference between a and p, which is evidently the interest of p for the given time and rate, is termed in this rule *the discount.*

Discount, then, is the difference between a sum of money payable at a future time and the value of that sum in ready money.

Thus, in the last example,

> 289£. 7 sh. 6 d. due 3½ years hence at 4½ per cent. discount is worth 250£. payable now.
>
> And 289£. 7 sh. 6 d.—250£.=39£. 7 sh. 6 d. is the discount or compensation allowed for immediate payment.

355. The discount can be obtained otherwise than by subtraction of p from a.

Since the interest of different principal sums at the same rate and for the same time are proportional to the principals, it follows that

$$100 : rt :: p : it, \text{ or } \frac{100}{rt}=\frac{p}{it}.$$

$$\therefore\ 1+\frac{100}{rt}=1+\frac{p}{it}$$

$$\text{Or } \frac{100+rt}{rt}=\frac{p+it}{it}=\frac{a}{it}.$$

$$\therefore\ 100+rt : rt :: a : it,$$

$$\text{And } it \text{ or } d=\frac{a\times rt}{100+rt}\ \text{ - - } \quad (G);$$

Whence,

Case G. A sum of money payable at a given future time being given, to find the discount which must be allowed, at a given rate per cent. per annum, for prompt payment.

Multiply the given sum by the product of the given rate per cent. and time, and divide the result by the amount of 100£. for the given rate and time; the quotient is the discount required.

Example. What discount must be allowed for prompt payment of 1000£. payable in two years, allowing discount at the rate of five per cent. per annum?

$a=1000$, $r=5$, $t=2$, $rt=5\times 2=10$; and $100+rt=100+10=110$.

Substituting these values of a, rt, and $100+rt$, in formula G,

$$d=\frac{1000\times 10}{110}=\frac{1000}{11}=90\tfrac{10}{11}\text{£.}=90\text{£. } 18\text{ sh. } 2\tfrac{2}{11}\text{ d.}$$

As the discount can be found by subtraction of the present worth from the amount or sum of money due at time t, so the present worth can be found by subtraction of the discount from the sum due.

356. This is the discount of treatises of commercial arithmetic, but it is not that of dealers in money. The discount of the latter is the interest of the sum at a certain rate per cent. for a time equal to the interval between the day of prompt payment and the day when the money falls due.

The discount of 1000£. payable in two years at five per cent. per annum, by this mode of computation, is the interest of 1000£. for two years at five per cent. per annum;

$$\text{Or } \frac{1000\times 10}{100}=100\text{£.}$$

This manner of computing discount is less favourable than the other to the person receiving, and consequently more favourable than the other to the person giving, prompt instead of future payment.

By the first method the discount from 1000£. is 90£. 18 sh. $2\frac{2}{11}$ d.,

And the owner of the 1000£. receives 909£. 1 sh. $9\frac{9}{11}$ d.

By the second the discount is 100£.,

And the owner receives 900£.

The difference, or 9£. 1 sh. $9\frac{9}{11}$ d., is the immediate loss to the owner of the 1000£. from the calculation of the discount by the second method.

If $909\frac{1}{11}$£. are placed at interest for 2 years, at 5 per cent. per annum, the amount is $\frac{909\frac{1}{11}\times 110}{100}=\frac{100000}{100}=1000$£.; consequently, by placing $909\frac{1}{11}$£. at interest for 2 years at 5 per cent. per annum, the owner receives the full amount of his bill.

But if 900£. are placed at interest for 2 years at 5 per cent. per annum, the amount is only $\frac{900\times 110}{100}=\frac{99000}{100}=990$£.

Whence, placing the money received, in the second instance, at interest for 2 years at 5 per cent., the owner receives 10£. less than the full amount of his bill; that is, by getting his bill discounted in this way the owner is a loser, when the bill becomes due, to the extent of 10£.

The difference between what was termed the immediate loss, namely, 9£. 1 sh. $9\frac{2}{11}$ d., and the loss when the money falls due, is equal to the interest of $9\frac{1}{11}$£. during the interval.

$$\text{For } \frac{9\frac{1}{11}\times 10}{100}=\frac{9\frac{1}{11}}{10}=\frac{100}{110}=\frac{10}{11}\text{£.}=18\text{ sh. } 2\tfrac{2}{11}\text{ d.}$$

$$\text{and } 9\text{£. } 1\text{ sh. } 9\tfrac{9}{11}\text{ d.}+18\text{ sh. } 2\tfrac{2}{11}\text{ d.}=10\text{£.}$$

357. If a given sum of money is placed at interest for a certain time at a fixed rate of interest, and the amount of this sum, at the expiration of the first year or of the time fixed for the first payment of interest, is made the principal for the second year or term fixed for the second payment of interest; the amount at the expiration of the second year or term is made the principal for the third year, &c.; in this case the result, at the expiration of the 2d, 3d, 4th . . . t year or fixed period of time, is the amount, at compound interest, of the given principal for that time; and the difference between this amount and the original principal is the compound interest of that principal for the same time.

Let the principal, p, be placed at interest for 2, 3, . . . t years at the rate r for the time of each payment.

Then $100 : 100+r :: p : p\times\frac{100+r}{100}$=amount of p for the time of the first payment at rate r,

Or, since the ratio $\frac{100+r}{100}=1+\frac{r}{100}$,

$p\times\frac{100+r}{100}=p\left(1+\frac{r}{100}\right)$=amount of p for the first payment of interest.

Similarly,

$100 : 100+r : p\left(1+\frac{r}{100}\right) :: p\left(1+\frac{r}{100}\right)\left(1+\frac{r}{100}\right)=p\left(1+\frac{r}{100}\right)^2$=amount of p at the time of the second payment of interest,

$100 : 100+r : p\left(1+\frac{r}{100}\right)^2 : p\left(1+\frac{r}{100}\right)^2\left(1+\frac{r}{100}\right)=p\left(1+\frac{r}{100}\right)^3$=the amount of p at the time of the third payment of interest.

. . . .

The constant ratio $\frac{100+r}{100}$ or $1+\frac{r}{100}$, by which p and the successive amounts are multiplied, expresses the amount of 1 £. for the first year, or for the time fixed for the first payment of interest, at the given rate r.

For $100 : 100+r :: 1$ £. : $\frac{100+r}{100}=1+\frac{r}{100}$=amount of 1 £. at rate r.

The formula $p\left(1+\frac{r}{100}\right)^2$, or $p\left(1+\frac{r}{100}\right)^3$, ... $p\left(1+\frac{r}{100}\right)^t$ which expresses the amount, at compound interest, of p £. for 2 or 3 or . . . t times of payment, at the rate r, indicates that the amount of a sum of money placed at compound interest for a given time and rate is equal to the product of two factors, of which the one is the given principal, and the other the amount of 1 £., calculated at the given rate for the time of the first payment, and raised to a power the exponent of which expresses the number of payments.

1st Ex. Required the amount and interest of 500 £. in 3 years at 4 per cent. per annum, compound interest?

In this example $p=500$, $r=4$, $\frac{r}{100}=\cdot04$, $1+\frac{r}{100}=1\cdot04$, $t=3$,

and $500\times1\cdot04^3$=the amount required.

Now $1\cdot04^3=1\cdot04\times1\cdot04\times1\cdot04=1\cdot124864$,

and $500\times1\cdot124864=562\cdot432$ £.=562 £. 8 sh. 7½ d.

Whence the amount of 500 £. in 3 years at 4 per cent. per annum, compound interest, is 562 £. 8 sh. 7½ d.,

And the interest=562 £. 8 sh. 7½ d.−500 £.=62 £. 8 sh. 7½ d.

2d Ex. Required the compound interest of 56 £. for 2½ years at 4 per cent. per annum, the interest being payable half-yearly?

$$p=56,\ r=\frac{4}{2}=2,\ \frac{r}{100}=\frac{2}{100}=\cdot02,\ 1+\frac{r}{100}=1\cdot02,\ t=2\frac{1}{2}\div\frac{1}{2}=5,$$

$$p\times\left(1+\frac{r}{100}\right)^t=56\times1\cdot02^5=56\times1\cdot10408=61\cdot82848\text{ £.}$$

∴ 61·82848 £.—56 £.=5·82848 £.=5 £. 16 sh. 6 d. 3·3408 d.=the compound interest required.

Whence, to find the amount, at compound interest, of a given principal for a given time and rate,

Rule. Find the amount of 1 £. for the time of the first payment of interest, raise this amount to a power whose exponent is equal to the number of payments of interest, and multiply the result by the given principal; the product is the amount required.

To find the compound interest; calculate the amount and subtract the principal from it; the remainder is the compound interest required.

358. The allowance given by a merchant to an agent who buys or sells goods for him is named commission; it is allowed at so much per cent. on the value of the goods, the rate being fixed by usage, or by agreement between the parties concerned; commission is computed like interest, with this difference, that time is not an element in the calculation.

Ex. An agent buys goods for his employer to the value of 7864 £. 16 sh. 8 d., what is his commission at 2½ per cent.?

$$7864\text{ £. }16\text{ sh. }8\text{ d.}=7864\tfrac{5}{6}\text{ £.}=\frac{47189}{6}\text{ £.}\quad 2\frac{1}{2}=\frac{5}{2}.$$

$$100\text{ £.} : \frac{5}{2}\text{ £.}=40 : 1 :: \frac{47189}{6} : \frac{47189}{40\times6}=\frac{47189}{240}\text{ £.}=196\text{ £. }12\text{ sh. }5\text{ d.}$$

359. When one party engages, in consideration of a sum of money, to make good to another any damage which his property may accidentally sustain from fire, shipwreck, &c., the security which the receiver of the money gives to the owner of the property is termed a Policy of Insurance, and the money paid is the premium, or, simply, the insurance; the premium is so much per cent., according to the hazard, on the value of the property insured.

The premium to be paid for the insurance of a given amount of property at a fixed rate per cent., and for a given time, is calculated in the same manner as interest.

Ex. What must be paid for the insurance of goods valued at 712 £. 6 sh. for 8 months, at the rate of 7½ per cent. per annum?

$$712\text{ £. }6\text{ sh.}=712\cdot3\text{ £., }8\text{ mths.}=\frac{8}{12}\text{ or }\frac{2}{3}\text{ yr., }7\frac{1}{2}=\frac{15}{2}\text{ and }\frac{2}{3}\times\frac{15}{2}=5.$$

$$\therefore\text{ the insurance}=\frac{712\cdot3\times5}{100}=\frac{3561\cdot5}{100}=35\cdot615\text{ £.}=35\text{ £. }12\text{ sh. }3\text{ d. }2\cdot4.$$

360. The national debt of Great Britain, under the denomination of the public funds or stocks, is distinguished by several names, some descriptive of portions of the debt, others of the rate of interest which portions of it bear.

The interest, whether at the rate of 3, 3½, 4, . . . per cent., is computed in the same manner as the interest of any other capital.

The public creditor reckons his funded property by hundreds; for each nominal hundred of, for instance, a three per cent. stock, he is entitled yearly to 3 £. of interest, which is payable in two half-yearly dividends at the Bank of England.

One hundred pounds of stock may be worth less or more than 100£. of money; its value depending on the rate of interest, the means of employing capital, the general opinion of the national good faith and solvency, &c.

The calculation of the sum of money which must be given or received for any amount of stock, is made by proportion.

1st Ex. What sum will purchase 1000£. of a 3 per cent. stock at 93½ per cent.?

$$100 : 93{\cdot}5 :: 1000 : \frac{93{\cdot}5 \times 1000}{100} = 935\text{ £.} = \text{sum required.}$$

2d Ex. What is the value of 875£. 10sh., bank stock, at 131¼ per cent.?

$$875\text{ £. }10\text{sh.} = 875{\cdot}5,\ 131\tfrac{1}{4} = 131{\cdot}25,$$

$$100 : 131{\cdot}25 :: 875{\cdot}5 : \frac{131{\cdot}25 \times 875{\cdot}5}{100} = \frac{114909{\cdot}375}{100} = 1149\text{ £. }1\text{ sh. }10\tfrac{1}{2}\text{ d.}$$

361. Exercises in simple and compound interest, discount, commission, insurance, and sale of stocks:

1st. What is the interest of 547 £. 15 sh. for 3 years at 5 per cent. per annum?......................*Ans.* 82 £. 3 sh. 3 d.

2d. What is the interest of 765 £. 12 sh. 6 d. for 1 year at 4½ per cent. per annum?.................*Ans.* 32 £. 4 sh. 0¾ d.

3d. What is the interest of 317 £. 16 sh. for 5¾ years at 3½ per cent. per annum?..............*Ans.* 63 £. 19 sh. 1·74 d.

4th. What is the interest of 712 £. 6 sh. for 8 months at 7½ per cent. per annum?..............*Ans.* 35 £. 12 sh. 3½ d.

5th. What is the interest of 100 £. for 254 days at 4⅞ per cent. per annum?.......................*Ans.* 3 £. 7 sh. 10·192 d.

6th. What is the interest of 245 £. 13 sh. 4 d. for 246 days at 4¾ per cent. per annum?............*Ans.* 7 £. 17 sh. 3½ d.

7th. The interest of 956 £. for 7½ years is 358 £. 10 sh.; required the rate per cent. per annum?..*Ans.* 5 per cent.

8th. The interest of 217 £. 17 sh. 8 d. for 4¾ years is 36 £. 4 sh. 5 d. 2·18 qrs.; required the rate per cent. per annum?...................................*Ans.* 3½ per cent.

9th. What principal, being put to interest at 4½ per cent. per annum, will in 12 years produce 236 £. 18 sh. 2·76 d. of interest?.............................*Ans.* 438 £. 14 sh. 6d.

10th. What principal, being put to interest at 4⅝ per cent. per annum, will in 3 years 11 mths. produce 15 £. 10 sh. 6 d. 3·168 qrs. of interest?...............*Ans.* 85 £. 14 sh. 6 d.

11th. In what time will the interest of 1 £. come to 15 sh., allowing interest at the rate of 4½ per cent. per annum? *Ans.* 16⅔ years.

12th. In what time will 355 £. 15 sh. produce 56 £. 18 sh. 4¾ d. of interest, allowing interest at the rate of 4 per cent. per annum?.......................................*Ans.* 4 years.

13th. Find the amount of 225 £. in 3 yrs. at 5 per cent. per annum?...................................*Ans.* 258 £. 15 sh.

14th. Find the amount of 79 £. 16 sh. in 5½ years at 4⅜ per cent. per annum?...................*Ans.* 99 £. 0 sh. 0·45 d.

15th. The amount of a certain principal in 10 months at 5 per cent. per annum is 500 £.; required the principal? *Ans.* 480.

16th. What is the present worth of 430 £. due 9 months hence, discount being allowed at the rate of $4\frac{1}{2}$ per cent. per annum ?..............................*Ans.* 415 £. 19 sh. $2\frac{1}{2}$ d.

17th. What is the present worth of a bill for 682 £. 18 sh. $9\frac{1}{2}$ d. due in $3\frac{3}{4}$ years, discount being allowed at the rate of $4\frac{1}{2}$ per cent. per annum ?.............*Ans.* 584 £. 6 sh. 8 d.

18th. What must be allowed for discounting a bill of 85 £. 10 sh. due in 66 days, at 5 per cent. per annum ? *Ans.* 15 sh. $3\frac{3}{4}$ d.

19th. What discount should be allowed for prompt payment of 560 £. due in 2 years, discount being allowed at the rate of $4\frac{1}{2}$ per cent. per annum ?..*Ans.* 46 £. 4 sh. $9\frac{27}{109}$ d.

20th. What are the discount and present worth of a bill of 546 £. 16 sh. due in 146 days, allowing discount at the rate of $4\frac{3}{4}$ per cent. per annum ? *Ans.* 10 £. 3 sh. $10\frac{3}{4}$ d. and 536 £. 12 sh. $1\frac{1}{4}$ d.

21st. What is the amount of 720 £. for 4 years at 5 per cent. per annum compound interest ?...*Ans.* 875 £. 3 sh. $3\frac{1}{4}$ d.

22d. What is the compound interest of 370 £. for 6 years at 4 per cent. per annum ?..............*Ans.* 98 £. 3 sh. $4\frac{1}{2}$ d.

23d. What is the amount of 50 £. in 5 years or 10 half years at 5 per cent. per annum compound interest, the interest payable half-yearly ?..................*Ans.* 64 £. 0 sh. 1 d.

24th. What is the compound interest of 410 £. for $2\frac{1}{2}$ years at $4\frac{1}{2}$ per cent. per annum, the interest payable half-yearly ?...............................*Ans.* 48 £. 4 sh. $11\frac{1}{4}$ d.

25th. What is the commission on 4835 £. at $1\frac{3}{8}$ per cent. ? *Ans.* 66 £. 9 sh. $7\frac{1}{2}$ d.

26th. What is the commission on 476 £. 12 sh. 9 d. at 6 sh. 8 d. or $\frac{1}{3}$ £. per cent. ?.....................*Ans.* 1 £. 11 sh. $9\frac{1}{4}$ d.

27th. What premium must be paid for the insurance of a ship valued at 2860 £. at $13\frac{1}{2}$ per cent. ?......*Ans.* 386 £. 2 sh.

28th. What is the insurance on 107 £. for 117 days at $4\frac{3}{4}$ per cent. per annum ?......................*Ans.* 1 £. 12 sh. 7 d.

29th. At $110\frac{1}{4}$ per cent., what is the purchase of 2054 £. 16 sh. 4 per cent. stock ?..................*Ans.* 2265 £. 8 sh. 4 d.

30th. What sum will purchase 432 £. 15 sh. South Sea stock at $113\frac{5}{8}$ per cent. ?.....................*Ans.* 491 £. 14 sh. $2\frac{1}{2}$ d.

362. Let A, C, be two given numbers, of which A is divided into the parts $a', a'', a''' \ldots.$; it is required to divide C into parts $c', c'', c''' \ldots.$ proportional respectively to $a', a'', a''' \ldots.$ the parts of A in such manner that

$$a' : c' :: a'' : c'' :: a''' : c''' \ldots..$$

Or, what is the same thing, that

$$a' : c' :: a'' : c''; \; a' : c' :: a''' : c'''; \ldots..$$

$$\text{Since } a' : c' :: a' : c'; \; c'a' = a'c' \text{ (Art. 332)};$$

$$a' : c' :: a'' : c''; \; c'a'' = a'c'';$$

$$a' : c' :: a''' : c'''; \; c'a''' = a'c'''.$$

Whence by addition, $c'a' + c'a'' + c'a''' = a'c' + a'c'' + a'c'''$;

$$\text{or } (a' + a'' + a''')c' = (c' + c'' + c''')a' \text{ (Art. 70)};$$

$$\therefore a' + a'' + a''' \ldots : c' + c'' + c''' \ldots :: a' : c' \text{ (Art. 332)}.$$

But $a' + a'' + a''' \ldots = \text{A}$; $c' + c'' + c''' = \text{C}$.

$$\therefore \text{A} : \text{C} :: a' : c'.$$

And since $a' : c' :: a'' : c'' :: a''' : c''' \ldots.$ it follows that

$$A : C :: \begin{cases} a' : c' ; \therefore c' = a' \times \frac{C}{A}. \\ a'' : c'' ; \therefore c'' = a'' \times \frac{C}{A}. \\ a''' : c''' ; \therefore c''' = a''' \times \frac{C}{A}. \\ \ldots\ldots \quad \ldots\ldots\ldots \end{cases}$$

The required division is made by means of these proportions; for the divided number, A, is to the number which is to be divided, C, as any part of the divided number is to a fourth proportional, which is the corresponding part of the number which is to be divided.

In practice, it is not necessary to form the proportions, but merely to multiply successively the parts of the divided number by the ratio of the undivided to the divided number; the results are the corresponding parts of the undivided number.

1st Example. Given $A=12$; $C=20$; $a'=2$; $a''=3$; $a'''=7$; divide 20 into parts proportional to 2, 3, 7.

The ratio $\frac{C}{A}=\frac{20}{12}=\frac{5}{3}$.

$$\therefore c'=2\times\frac{5}{3}=\frac{10}{3}=3\tfrac{1}{3};$$

$$c''=3\times\frac{5}{3}=\frac{5}{1}=5;$$

$$c'''=7\times\frac{5}{3}=\frac{35}{3}=11\tfrac{2}{3};$$

Whence the parts required are $3\frac{1}{3}$, 5, $11\frac{2}{3}$.

The sum of these parts ought to be equal to C.

In this example $3\frac{1}{3}+5+11\frac{2}{3}=20=C$.

When the sum of the parts is not equal to C, the calculation must be revised.

2d Example. Divide 100 into parts which shall be in the proportion $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$.

$$a'=\frac{1}{2}=\frac{30}{60};\ a''=\frac{1}{3}=\frac{20}{60};\ a'''=\frac{1}{4}=\frac{15}{60};\ a^{\text{IV}}=\frac{1}{5}=\frac{12}{60}.$$

$$A=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}=\frac{30}{60}+\frac{20}{60}+\frac{15}{60}+\frac{12}{60}=\frac{77}{60}.$$

$$C=100;\ \frac{C}{A}=100\div\frac{77}{60}=\frac{100\times60}{77}.$$

$$\therefore c'=\frac{30}{60}\times\frac{100\times60}{77}=\frac{30\times100}{77}=\frac{3000}{77}=38\tfrac{74}{77}.$$

$$c''=\frac{20}{60}\times\frac{100\times60}{77}=\frac{20\times100}{77}=\frac{2000}{77}=25\tfrac{75}{77}.$$

$$c'''=\frac{15}{60}\times\frac{100\times60}{77}=\frac{15\times100}{77}=\frac{1500}{77}=19\tfrac{37}{77}.$$

$$c^{\text{IV}}=\frac{12}{60}\times\frac{100\times60}{77}=\frac{12\times100}{77}=\frac{1200}{77}=15\tfrac{45}{77}.$$

When a', a'', a''', ... are given, and not A, it is necessary to add together the proportional parts a', a'', a''' ...; their sum is A.

When the given proportional parts a', a'', a''', ... are fractions, the common denominators of these fractions reduced for addition, and the denominator of their sum, A, must be the same. The ratios of fractions which have the same denominator being equal to the ratios of the numerators, the denominators may be suppressed; in the preceding calculation they are cancelled against each other.

363. Proportion, when employed to divide a number into parts proportional to other given numbers, is named Distributive Proportion, Partnership, Fellowship, &c. The terms partnership and fellowship are used when the rule is employed to determine the particular gain or loss of the individuals engaged in any joint undertaking.

The principle upon which the apportionment of gain or loss is made in such cases is that the gain or loss of each individual is proportional to that individual's share of the joint undertaking, or of the money expended upon it, when all the shares have been the same time embarked in the undertaking: and to the product of the share by the time during which it has been so embarked when the shares have been acquired or their values contributed at different times.

1st Example. Three merchants engage in a joint speculation, towards which the first contributes 750£., the second 1127£., and the third 1280£.; the gain being 600£. it is required to find the share of each partner?

$$A=750+1127+1280=3157\ \text{£.};\ C=600\ \therefore \frac{C}{A}=\frac{600}{3157}.$$

$$a'=750;\ a''=1127;\ a'''=1280;$$

$$\therefore c'=750\times\frac{600}{3157}=\frac{450000}{3157}=142\ \text{£. 10 sh. 9 d. 2·88 qrs.}=\text{1st share.}$$

$$c''=1127\times\frac{600}{3157}=\frac{676200}{3157}=214\ \text{£. 3 sh. 9 d. 2·88 qr.}=\text{2d share.}$$

$$c'''=1280\times\frac{600}{3157}=\frac{768000}{3157}=243\ \text{£. 5 sh. 4 d. 2·24 qr.}=\text{3d share.}$$

2d Example. A tradesman begins business with a capital of 1250£.; 5 months afterwards he borrows 2000£. from one person, and 6 months after obtaining the first loan he borrows 3000£. from another; at the end of two years the gain on the business is 4000£. If the tradesman is allowed 5 per cent. upon the whole gain for conducting the business, and the balance is divided between him and his creditors in proportion to the value of the contribution of each to the common stock, what is the share which each receives of the gain?

The allowance for management being $4000\text{£.}\times\frac{5}{100}=200\text{£.}$, $4000\text{£.}-200\text{£.}$ or 3800£. is the sum which is to be shared among the partners in proportion to the value of their contributions, that is, in proportion to the product of the contribution of each by the time during which it has been embarked in the business.

Now 1250£. for 24 months$=1250\times24=30000$£. for 1 month;

2000£. for 24−5 or 19 months$=2000\times19=38000$£. for 1 month;

3000£. for 19−6 or 13 months$=3000\times13=39000$£. for 1 month.

Hence the question is reduced to the division of 3800 £. into parts proportional to the three numbers, 30000, 38000, 39000,
or to the three numbers - 30, 38, 39;

$$\therefore a'=30;\ a''=38;\ a'''=39;\ A=30+38+39=107.$$

$$C=3800;\ \frac{C}{A}=\frac{3800}{107}.$$

$$c'=30\times\frac{3800}{107}=\frac{114000}{107}=1065\ \text{£. 8 sh. 4 d. 3·68 qrs.}=\text{1st share.}$$

To this add, for management, 200£. 0 sh. 0 d.

1265£. 8 sh. 4 d. 3·68 qrs.

$$c''=38\times\frac{3800}{107}=\frac{144400}{107}=1349\ \text{£. 10 s. 7 d. 3·2 qrs.}=\text{2d share.}$$

$$c'''=39\times\frac{3800}{107}=\frac{148200}{107}=1385\ \text{£. 0 sh. 11 d. 1·12 qrs.}=\text{3d share.}$$

364. To divide a given number into parts proportional to other given numbers.

Rule. Form the sum of the given proportional numbers, and the ratio of the given undivided number to this sum; multiply the given proportional numbers successively by this ratio; the products are the proportional parts required.

When the proportional numbers (as in the 2d Ex. of Art. 363) are contributions, for different periods of time, to a common stock, multiply each contribution by the time of its continuance; take these products for the proportional numbers, and find the parts of the undivided number as before.

365. Exercises in Distributive Proportion and Partnership.

1st. Divide 240 into parts proportional to the numbers 1, 2, 3? *Ans.* 40, 80, 120.

2d. Divide 1000 into parts proportional to the numbers 3, 5, 8?.................................... *Ans.* 187½, 312½, 500.

3d. Divide 60 into parts proportional to the numbers 1, $\frac{2}{3}$, $\frac{5}{8}$? *Ans.* $26\frac{2}{11}$, $17\frac{5}{11}$, $16\frac{4}{11}$.

4th. Two persons, A and B, made a stock of 120£., of which A contributed 75£. and B the rest; by trading they gain 30£. What is the share of each?
Ans. A's share is 18£. 15 sh.; B's, 11£. 5 sh.

5th. Three persons, A, B, and C, freighted a ship with 340 tuns of wine, of which A supplied 110 tuns, B 97, and C the rest; in a storm the seamen threw overboard 85 tuns. How much must each person sustain of the loss?
Ans. A, 27½ tuns; B, 24¼; and C, 33¼.

6th. A field containing 37 ac. 2 ro. 14 po. is to be divided among three persons, A, B, C, in proportion to the values of their estates; now if A's estate is worth 500£. a year, B's 320£., and C's 75£., what quantity of land must each have?
Ans. A, 20 ac. 3 ro. $39\frac{139}{179}$ po.; B, 13 ac. 1 ro. $30\frac{46}{179}$ po.; C, 3 ac. 0 ro. $23\frac{173}{179}$ po.

7th. A general imposed a contribution of 700£. on four villages, to be paid in proportion to the number of inhabitants contained in each; the first contained 250, the second 350, the third 400, and the fourth 500 persons. What part had each village to pay?
Ans. First, 116£. 13 sh. 4 d.; second, 163£. 6 sh. 8 d.; third, 186£. 13 s. 4 d.; fourth, 233£. 6 sh. 8 d.

8th. Five companies consisting the first of 54 men, the second of 51 men, the third of 48 men, the fourth of 39 men, and the fifth of 36 men, are detached to a post, the duty of which requires 76 men a day; if the companies are required to contribute in proportion to their strength, what number of men ought to be furnished by each company daily?
Ans. First, 18 men; second, 17 men; third, 16 men; fourth, 13 men; fifth, 12 men.

9th. A and B traded in partnership and gained 100£.; A contributed 640£. to the common stock, and B as much as entitled him to receive 60£. for his share of the gain. What was B's contribution?..................... *Ans.* 960 £.

10th. A sum of money is to be divided among A, B, C, D, in such a manner that A is to receive $\frac{1}{3}$ of the whole; B, $\frac{1}{4}$; C, $\frac{1}{6}$; and D the remainder. What is that sum, D's share being 28£.?................................ *Ans.* 112£.

11th. A had, in company, 50£. for 4 months, and B had 60£. for 5 months; the gain on the joint speculation being 24£., how must it be divided between them?
Ans. A's share, 9£. 12 sh.; B's share, 14£. 8 sh.

12th. Two troops of cavalry rent a field in common, for which they are to pay 36£.; one company put into the field 23 horses for 27 days, the other 21 horses for 39 days. How much ought each to pay of the rent?
Ans. The first comp., 15 £. 10 sh. 6 d.; the second, 20£. 9 sh. 6 d.

13th. A began business on the 1st of January with a capital of 1000£.; on the 1st of March following he took B as a partner, with a capital of 1500£.; on the 1st of June A and B admit a third partner, C, who has a capital of 2800£.; after trading together till the end of the year they find that their gain amounts to 1776£. 10 sh. How must this be divided amongst the partners?
Ans. A's share, 457£. 9 sh. $4\frac{1}{4}$ d.; B's, 571£. 16 sh. $8\frac{1}{4}$ d.; C's, 747£. 3 sh. $11\frac{1}{4}$ d.

14th. A and B entered into partnership for a year; A contributed 500£. on the 1st of January, and B on the 1st of May as much as entitled him to half the gain at the end of the year. What was B's contribution?
Ans. 750£.

366. Many treatises of commercial arithmetic contain a number of rules applicable to the business of bankers and merchants.

Generally, each rule embodies instructions for the application of proportion to some particular class of questions; but as the application may be made without the help of a rule by any one who understands the principles of proportion and the meaning of such questions, it seems unnecessary to consider these rules in detail.

A few questions, falling under some of the more important, are subjoined.

1st Example. A silk mercer bought 100 yards of riband at 3 yards for 1 sh., and 100 yards at 2 yards for 1 sh., and sold the whole at the rate of 5 yards for 2 sh. Did he gain or lose? How much in all? And how much per cent.?

3 yds. : 100 yds. :: 1 sh. : $\frac{100}{3}$ sh. $=$ 1 £. 13 sh. 4 d. $=$ price of 100 yds. at 3 yds. for 1 sh.;

2 yds. : 100 yds. :: 1 sh. : $\frac{100}{2}$ sh. $=$ 2 £. 10 sh. 0 d. $=$ price of 100 yds. at 2 yds. for 1 sh.;

$\therefore$ 1 £. 13 sh. 4 d. $+$ 2 £. 10 sh. $=$ 4 £. 3 sh. 4 d. $=$ price paid for the whole.

And 5 yds. : 200 yds. :: 2 sh. : $\frac{\overset{40}{\cancel{200}}\times 2}{\underset{1}{\cancel{5}}}$ sh. $=4$£. $=$ price received for 200 yds. at 5sh. for 2 yds.;

$\therefore$ 4£. 3 sh. 4 d — 4£. $=$ 3 sh. 4 d. $=$ the loss upon the whole.

Again, since 4£. 3 sh. 4d. $=4\frac{1}{6}$ £. $=\frac{25}{6}$ £.; 3 sh. 4 d. $=\frac{1}{6}$ £.;

and that $\frac{25}{6}$ £. : $\frac{1}{6}$ £. $=25:1::100:\frac{100}{25}=4$£.;

it appears that the loss is 4 per cent.

This question is one of a class which, in treatises of commercial arithmetic, is considered under the rule of *Profit and Loss*.

2d Example. How many lbs. of tea at 5 sh. 9 d. per lb. must be given in exchange for 426 yds. of cloth at 13 sh. 4 d. per yard?

The first step towards a solution is, to find the price of 426 yds. at 13 sh. 4 d., or $\frac{2}{3}$£. per yard.

1 yd. : 426 yds. :: $\frac{2}{3}$£. : $\frac{\overset{142}{\cancel{426}}\times 2}{\underset{1}{\cancel{3}}}$ $=284$£. $=$ price of 426 yds. at 13 sh. 4 d. per yard.

The second step is, to find how many lbs. of tea at 5 sh. 9 d. per lb. can be bought for 284£.

$$5\text{ sh. }9\text{ d.}=\frac{69}{240}\text{£.}=\frac{23}{80}\text{£., and}$$

$$\frac{23}{80}\text{£.}:284\text{£.}::1\text{ lb.}:\frac{284\times 80}{23}\text{lb.}=\frac{22720}{23}\text{lb.}=987\tfrac{19}{23}\text{ lbs.}$$

Therefore $987\frac{19}{23}$ lb. of tea at 5 sh. 9 d. per lb., are equivalent to 426 yards of cloth at 13 sh. 4 d. per yard.

This is a simple case of the exchange of commodities by means of their money values.

Such exchanges are described by the term *Barter*.

Any sum of money expressed in terms of the coins or circulating medium of one country may be reduced to an equivalent sum expressed in terms of the coins or circulating medium of another country (if the money of the two countries has a given relation) by a process like that of Example 2, or by the method of Article 316.

The process for the reduction of any sum of the money of one country to an equivalent sum of the money of another country is termed *Exchange*.

3d Example. How many Spanish piastres at 3 sh. $5\frac{1}{2}$ d. per piastre are equivalent to 1000£. 16 sh. 10 d. sterling?

$$3\text{ sh. }5\tfrac{1}{2}\text{d.}=\frac{83}{480}\text{£.}:1000\text{£. }16\text{ sh. }0\text{ d.}=1000\frac{101}{120}\text{£.}=\frac{120101}{120}\text{£.};$$

$$\therefore\frac{83}{480}\text{£.}:\frac{120101}{120}\text{£.}::1\text{ pi.}:\frac{\overset{1447}{\cancel{120101}}\times\overset{4}{\cancel{480}}}{\underset{1}{\cancel{120}}\times\underset{1}{\cancel{83}}}\text{ pi.}=5788\text{ piastres.}$$

The calculation becomes considerably more complex when the money of one country is to be changed into that of another; the relative value of the money of the two countries being given indirectly in terms of the money of a third country, or of a third and fourth country, &c.

4th Example. Suppose that 1£. or 20sh. are equal to 25 francs,
15 francs are equal to 8 German florins,
50 florins are equal to 9 Hamburgh ducats,
19 ducats are equal to 40 Russian roubles,
it is required to find how many Russian roubles are equivalent to 100£. sterling?

Since 20 sh. = 25 francs, 1 sh. = $\frac{25}{20}$ francs = $\frac{5}{4}$ francs;

15 francs = 8 florins, 1 franc = $\frac{8}{15}$ florins;

therefore 1 sh. = $\frac{5}{4}$ of $\frac{8}{15}$ florins.

Again, since 50 florins = 9 ducats, 1 florin = $\frac{9}{50}$ ducats;

therefore 1 sh. = $\frac{5}{4}$ of $\frac{8}{15}$ of $\frac{9}{50}$ ducats.

Lastly, since 19 ducats = 40 roubles, 1 ducat = $\frac{40}{19}$ roubles;

therefore 1 sh. = $\frac{5}{4}$ of $\frac{8}{15}$ of $\frac{9}{50}$ of $\frac{40}{19}$ roubles.

$$\therefore\ 100\,\pounds.\ \text{or}\ 2000\,\text{sh.}=\frac{\overset{40}{\cancel{2000}}}{1}\times\frac{\overset{1}{\cancel{5}}}{\underset{1}{\cancel{4}}}\times\frac{\overset{2}{\cancel{8}}}{\underset{\underset{1}{\cancel{3}}}{\cancel{15}}}\times\frac{\overset{3}{\cancel{9}}}{\underset{1}{\cancel{50}}}\times\frac{40}{19}=\frac{9600}{19}\,\text{ro.}=505\tfrac{5}{19}\,\text{ro.}$$

Questions such as the last can be resolved by different processes; but this method of forming each of the given relations into a ratio expressed by a fraction, and combining the ratios into a complex fraction by means of which the money which is to be exchanged is connected with the money for which it is required to exchange it, seems the most simple.

The relations between the money of England and France, France and Germany, &c. are expressed in round numbers; the object being merely to indicate the manner of resolving questions of this kind.

5th Example. If a grocer mixes 4 cwt. of sugar at 56 sh. per cwt., 7 cwt. at 43 sh. per cwt., and 5 cwt. at 37 sh. per cwt.; what is 1 cwt. of the mixture work?

4 cwt. at 56 sh. per cwt. = 224 sh. = 11 £. 4 sh.
7 cwt. at 43 sh. per cwt. = 301 sh. = 15 £. 1 sh.
5 cwt. at 37 sh. per cwt. = 185 sh. = 9 £. 5 sh.
16 cwt. of the mixture cost - 35 £. 10 sh.

The question now is, if 16 cwt. cost 35 £. 10 sh., what is the price of 1 cwt.? To find this,

$$16\ \text{cwt.} : 1\ \text{cwt.} :: 35\tfrac{1}{2}=\frac{71}{2}\,\pounds. : \frac{71}{2\times 16}\,\pounds.=\frac{71}{32}\,\pounds.=2\,\pounds.\ 4\,\text{sh.}\ 4\tfrac{1}{2}\,\text{d.}$$

Whence the price of 1 cwt. of the mixture is 2 £. 4 sh. 4½ d.

This question is a particular example of the most simple case of *Alligation* or the rule of mixtures. The more difficult cases are most simply resolved with the help of algebra.

6th Example. Suppose that the distance between two fixed points on level ground has been measured four times; that the results of these measurements disagree,

the first being 250·439 yards,
the second being 250·695 yards,
the third being 249·750 yards,
the fourth being 251·158 yards;

and that it is required to assign as nearly as possible the true distance; this is also considered a question falling under the rule of alligation.

The distance being constant and the measurements different, it is certain that some and perhaps the whole of them are inaccurate. To approximate to the true result, it is evident that if the true distance had been obtained each time the sum of the results must be equal to four times the distance; it is also evident that the same thing must happen if some of the results erred in excess and others in defect in such a manner that the errors in excess compensated those in defect.

Whence, in this case also, the true distance must be obtained by dividing the sum of the measurements by their number.

The sum of the four measurements is 1002·042 yards, and 1002·042 ÷ 4 = 250·5105 yards.

250·5105 yards is assumed to be nearly the distance required.

The result obtained by dividing any number of different results obtained by experiment or observation by the number of repetitions is termed a *mean value.*

It is not to be expected that in any case the errors in excess and defect will exactly compensate each other, but the approximation of a mean to a true value may be considered close in proportion as the number of repetitions is great.

7th Example. Suppose that A executes a certain work in 6 days, and that B requires 8 days to execute the same work, in what time can A and B accomplish it if both work together?

Since A does the whole in 6 days he does $\frac{1}{6}$ part in 1 day, and since B does the whole in 8 days he does $\frac{1}{8}$ part in 1 day; therefore A and B working together do $\frac{1}{6}+\frac{1}{8}=\frac{7}{24}$ parts of the whole work in 1 day.

The question is consequently reduced to this: $\frac{7}{24}$ parts of the work are done in 1 day, what time is required to do the whole, or $\frac{24}{24}$ parts.

And the proportion $\frac{7}{24} : \frac{24}{24} = 7 : 24 :: 1$ day : $\frac{24}{7} = 3\frac{3}{7}$ days, gives the time required.

When an arithmetical question involves many or complex conditions the investigation for the discovery of the unknown quantity is simplified by the employment of the resources of algebra.

In Example 7, let the time taken by A be denoted by the general symbol a, the time taken by B by the symbol b, the time required by both, or the unknown quantity, by x.

Then in 1 day A does a part of the work expressed by $\frac{1}{a}$, and in x days a part expressed by $\frac{x}{a}$.

Similarly, in 1 day and x days B does parts expressed by $\frac{1}{b}$ and $\frac{x}{b}$ respectively.

Therefore in x days A and B together do a part of the work expressed by $\frac{x}{a}+\frac{x}{b}$.

But by hypothesis A and B together do the whole work in x days; therefore, denoting the whole work by 1,

$$\frac{x}{a}+\frac{x}{b}=1.$$

Multiplying the equal quantities by ab the products are equal;

$$\text{or } bx+ax=ab,$$

$$\text{or } (a+b)x=ab \text{ (Art. 78).}$$

Dividing the equal quantities by $a+b$, the quotients are equal;

$$\text{therefore } x=\frac{ab}{a+b}.$$

This result involves not alone the answer of question 7, but that of every question in which the times taken by two agents acting separately to produce an effect being given, it is required to find the time taken by the same agents acting together.

If, for example, a is replaced by 6 and b by 8,

$$\frac{ab}{a+b} \text{ becomes } \frac{6\times 8}{6+8}=\frac{48}{14}=3\tfrac{3}{7}, \text{ the answer of question 7.}$$

8th Example. If 2 d. in the shilling is a dealer's profit, how much does he gain per cent.?..*Ans.* 16⅔ per cent.

9th. A draper bought 124 yards of linen for 31£., how must he sell it per yard to gain 10£. 6 sh. 8 d. on the whole?............*Ans.* At 6 sh. 8 d. per yard.

10th. A merchant receives 126 yards of cloth in barter for 189 gallons of wine at 6 s. 8 d. per gallon, what is the cloth valued at per yard?
Ans. At 10 sh. per yard.

11th. How many bushels of wheat at 4 sh. per bushel must be given in exchange for 120 musket barrels at 15 sh. 6 d. each?........*Ans.* 465 bushels.

12th. In Portugal 1000 reas make 1 millrea, and 1 millrea is worth 63⅜ d. sterling; it is hence required to find, in British money, the value of 827 millreas 160 reas?..*Ans.* 218 £. 8 sh. 5¼ d.

13th. If 3 francs are worth 32 pence sterling, 240 pence are worth 408 deniers of Holland, and 50 deniers are worth 190 Spanish maravedis; it is required to find how many maravedis are worth 90 francs?
Ans. 6201⅗ maravedis.

14th. A distiller mixes several sorts of spirits; viz. 36 gallons at 8 sh. per gallon, 2 gallons at 7 sh. per gallon, and 13 gallons at 1 sh. per gallon, with 12 gallons of water; what is a gallon of the mixture worth?.........*Ans.* 5 sh.

PART II.

ELEMENTS OF ALGEBRA.

SECTION I.

DEFINITIONS AND FIRST PRINCIPLES.

1. The result of every arithmetical question is expressed by some number, whole or fractional, in which all distinction of the given quantities, and of the operations by which they are combined to form this number, is lost.

If numbers are represented by general characters, and arithmetical operations are indicated by symbols, the given quantities can be made to preserve their distinction throughout a calculation, and the result to exhibit the arithmetical operations which must be performed with these quantities in order to obtain the numerical value required.

But these operations cannot be effected unless particular values are assigned to the given quantities. Thus in Example 7, Art. 366, the result $\frac{ab}{a+b}$ indicates, that to obtain the time in which two agents acting jointly produce a certain effect, when the times taken by them singly to produce that effect are given, it is necessary to divide the product of the given times by their sum. But to obtain a numerical value from the expression $\frac{ab}{a+b}$, it is necessary to assign particular values to a, b, and to perform with these values the operations indicated in the expression. Thus, if $a=6$ and $b=8$,

$$\frac{ab}{a+b}=\frac{6\times 8}{6+8}=\frac{48}{14}=3\tfrac{3}{7}.$$

The formula $\frac{ab}{a+b}$ involves the answer not to this particular question only, but to every question in which the times, taken by two agents acting singly to produce the same effect, are given to find the time in which that effect can be produced by the same agents acting together.

Making a successively 4, 6, 10
b - 9, 12, 18,

$$\text{the results are } \frac{4\times 9}{4+9}=\frac{36}{13}=2\tfrac{10}{13}$$

$$\frac{6\times 12}{6+12}=\frac{72}{18}=4$$

$$\frac{10\times 18}{10+18}=\frac{180}{28}=6\tfrac{3}{7}.$$

It is by the substitution of general characters for particular numbers, and symbols of operation for the operations themselves, that Algebra has been derived from Arithmetic.

2. Two expressions of equal value, with the symbol = placed between them, form an equation. Thus, if the quantities denoted by a and b are of the same value, the symbolical expression of equality, $a=b$, is termed an Equation.

Similarly, if the sum of a and b is equal to the sum of c and d;

Or if the difference between a and c is equal to the difference between b and d;

the equations $a+b=c+d$,

$a-c=b-d$ respectively express these equalities.

The quantities on each side of the symbol $=$ are named Members of the Equation; and each of the quantities which is connected with the others by the symbols $+$ or $-$ is called a Term.

The changes which may be made among the quantities composing an equation depend chiefly on the following self-evident principles:

1st. That if to equal quantities the same quantity or equal quantities are added the sums are equal.

2d. That if from equal quantities the same quantity or equal quantities are taken away the remainders are equal.

3d. That if two quantities are equal, their doubles, their triples, their quadruples, and, in general, their equimultiples, are equal.

4th. And that if two quantities are equal, their halves, their third parts, and, in general, their like parts or submultiples, are equal.

Whence, if both members of an equation are augmented or diminished, or multiplied or divided by the same quantity, or by equal quantities, the sums, the differences, the products, the quotients form an equation; and if each of the members of one equation is added to the corresponding member of another equation, or subtracted from, or multiplied by, or divided by it, in each case the result forms an equation.

Let $a=b$, and let m be another quantity,

$$\text{then } a+m=b+m$$
$$a-m=b-m, \text{ or } m-a=m-b$$
$$ma=mb$$
$$\frac{a}{m}=\frac{b}{m}.$$

Also, let $c=d$. Then, since $a=b$ and $c=d$,

$$a+c=b+d$$
$$a-c=b-d, \text{ or } c-a=d-b$$
$$ac=bd$$
$$\frac{a}{c}=\frac{b}{d}.$$

3. The expression $a-b$ indicates that the quantity b is to be subtracted from the quantity a.

a and b being given quantities, three cases may occur; 1st. $a>b$, 2d. $a=b$, 3d. $a<b$.

1st. Let a be greater than b, and let the difference be denoted by c.

This is the common case of arithmetical subtraction.

The subtraction and remainder are indicated by the equation,

$$a-b=c.$$

2d. Let a be equal to b.

The numerical subtraction in this case also is possible, and the remainder zero. The expression of the difference is,

$$a-b=0.$$

3d. Let a be less than b by a quantity d.

In this case the numerical subtraction is impossible.

$$\text{Since } a=a$$
$$\text{and } \quad b=a+d$$
$$a-b=a-a-d \text{ (Art. 2)},$$
$$\text{or } a-b=0-d.$$

Therefore, if b exceeds a by a quantity d, when a part of b equal to a has been taken from a, the remainder is zero, and there is still the quantity d to be subtracted.

Although the process of subtraction necessarily supposes two quantities, a minuend and a subtrahend, and that in this instance there is only a subtrahend (the quantity d), d may be distinguished as a subtractive quantity by writing it $-d$. Prefixed thus to a monomial, the symbol $-$ serves to intimate that the quantity d ought to enter subtractively into any combination of which it forms a part.

If, for example, it is required to add the subtractive quantity $(-d)$ to c, the sum $c+(-d)$ is $c-d$.

The result obtained by adding the subtractive quantity $(-d)$ to c is identical with that obtained by subtracting b or $a+d$ from the sum of a and c.

$$\text{For } a+c=a+c$$
$$\text{and } \quad b=a+d$$
$$\therefore a+c-b=c-d \text{ (Art. 2, and Part I. Art. 43).}$$

Since the expression of the difference between c and d is $c-d$, it follows that if $(-d)$ is added to c, or d is subtracted from c, the result is the same.

To illustrate the subject of subtractive monomial quantities, suppose that by one speculation a merchant has gained a £.; that by another he has lost b£., and that it is required to find the change produced in his capital by this gain and loss, it is evident that if $a>b$, the capital is augmented by the quantity $a-b$; and that if $b>a$, the capital is diminished by the quantity $b-a$.

In the case of $b>a$, the expression $a-b$, separately considered, conveys no meaning except that of a subtraction which cannot be effected; yet by the conventions of algebraic notation $a-b$ is still employed to express the change of capital; and since b cannot be subtracted from a, a is subtracted from b, and the symbol or sign $-$ is prefixed to the remainder. By this convention it is indicated that the result is to be regarded as not an augmentation, but a diminution of the capital. If $a=700$£. and $b=500$£., there is a real augmentation of 200£.; if, on the contrary, $a=500$ £. and $b=700$ £., there is a diminution of 200£. But, instead of indicating that the capital suffers a diminution of 200£., this notation expresses, in terms which are equivalent (though at variance with common language), that there is an augmentation or gain of -200 £.

Instances are numerous in which it becomes necessary to consider quantities of the same kind in two contrary relations; additive in the one, subtractive in the other, as expressing the gains and losses of business; income and expenditure; the acceleration or retardation of the time indicated by a watch; the distance, at different times, between a fixed point and the place of a moving body, according as the body moves away from or towards the point; time subsequent or anterior to a given epoch, &c. &c.

a. In the expression $a-b$, let a have a constant value, and suppose that the value of b begins from zero, and increases until it exceeds that of a:

$$\text{When } b=0,\ a-b=a.$$

As b increases the differences diminish.

$$\text{When } b=a,\ a-b=0.$$

When $b>a$, the difference is subtractive; and the greater b becomes the greater are these subtractive quantities, considered with respect to their absolute values.

If $a=4$ and b is made equal, successively, to 0, 1, 2, 3, 4, the corresponding values of $a-b$ are 4, 3, 2, 1, 0. If b, continuing to increase, becomes in succession 5, 6, 7, 8, the corresponding values of $a-b$ are -1, -2, -3, -4. Because these subtractive values form a continuation of the decreasing series of additive numbers 4, 3, 2, 1, 0, they are, by convention, though the practice is objectionable, regarded as less than zero, and each of the terms is considered less than that which precedes it.

In the language of this convention it is said that -1 is less than 0, and -2 less than -1; or that

$$(-1)<0;\ \text{and } (-2)<(-1)$$
$$\text{or otherwise, that } 0>(-1) \text{ and } (-1)>(-2).$$

From this convention, by which a subtractive monomial quantity is regarded as less than zero, probably originated the use of the term *Negative*, which is commonly employed to describe subtractive quantities. The opposed term, *Positive*, is generally employed to distinguish quantities which are additive.

In conclusion, to express in a general manner the opposition of quantities which are subtractive to quantities which are additive, the sign minus (—) is employed in algebra, and the monomial quantity to which this sign is prefixed is called a negative quantity. Negative quantities, then, originate from subtraction in this manner: when the subtrahend is greater than the minuend, instead of taking the subtrahend from the minuend, the latter is taken from the former; and this change of order is indicated by the sign — prefixed to the remainder.

4. Abstractedly considered, a monomial quantity has no sign; but from the conventions established in Article 3, monomials arising from the subtraction of a greater number from a less are considered as subtractive quantities, and distinguished by the prefix —. Monomials which have no sign prefixed are considered to have +.

The signs + and —, by which isolated quantities are preceded, indicate the operations which are to be performed with them, or the relation of these quantities to others with which they are combined.

Whence $+a$, $-a$, are complex algebraic forms which express not a quantity simply, or an operation simply, but the combination of both.

Since it may be required to combine subtractive with additive or subtractive quantities, it becomes necessary to determine rules for such combinations.

a. Let a, a' denote two quantities whose sum is required: if a and a' are both additive, their sum is $a+a'$ (Part I. Art. 25). If a is additive and a' subtractive, the sum is $a+(-a')$ or $a-a'$ (Part II. Art. 3). If a is subtractive and a' additive, the sum is $-a+a'$ or $a'-a$. For if a third quantity, b, is assumed, it is evident that, whether a units are subtracted from b and a' units are added to the remainder, or a' units are added to b and a units subtracted from the sum, the absolute number of units expressed by $b-a+a'$ and by $b+a'-a$ must be the same, and therefore in $-a+a'$ and in $a'-a$.

If a, a' are both subtractive, assuming a third quantity, b, greater than the sum of the absolute numbers a, a', it is evident that the sum of $b+(-a)+(-a')$ can be obtained by adding $-a$ to b, and then to this sum, which is $b-a$, adding $-a'$. The result is $b-a-a'$. The sum can also be obtained by adding at once $-(a+a')$ to b. The sum of two subtractive quantities $-a$ and $-a'$ is therefore $-a-a'$, or $-(a+a')$.

Combining the symbols + and —, thus, ±, to indicate that the quantity to which they are prefixed may be taken additively or subtractively, the result of this discussion is expressible by the formula,

$$\pm a+(\pm a')=\pm a\pm a',$$

or by this proposition:

The sum of two monomial quantities, of whatever signs, is obtained by connecting the quantities together by means of their proper signs.

The result obtained by the addition of quantities, of which some are subtractive, is termed the algebraic sum of these quantities.

Algebraic addition, it ought to be observed, does not necessarily imply augmentation; for the numerical value of the sum $a+(-a')$ is less than that of one of the quantities, a.

b. Let m, s, be two quantities whose difference is required:

m and s being both additive, and $m>s$, the expression of the difference is $m-s$ (Part I. Art. 43); and if $m-s=d$, d is an additive quantity.

If $s>m$, and $s-m=d$, then $m-s=-d$, and the difference is a subtractive quantity (Part II. Art. 3, Case 3).

b′. Next, let the difference between m, an additive, and $(-s)$, a subtractive quantity, be required.

Let the difference$=d$, that is, let

$$m-(-s)=d.$$

Then, since $(-s)=(-s)$,

by addition, $m-(-s)+(-s)=d+(-s)$ (Part II. Art. 2);

but $m-(-s)+(-s)=m$, and $d+(-s)=d-s$.

Therefore $m=d-s$.

Now $m-(-s)=d$, and $m=d-s$.

Hence $m-(-s)$ is greater than m by the additive quantity s.

Therefore $m-(-s)=m+s$.

b$''$. Again, let m, the minuend, be a subtractive, and s, the subtrahend, an additive quantity.

Since m is subtractive, and s becomes, by the operation, also subtractive, the expression of the remainder, $-m-(+s)$ is $-m-s$, or$-(m+s)$. (See the Case b$''''$ of this Article.)

b$'''$. Suppose that a is an additive quantity greater than $m+s$. Then, since $a>m$, $a+(-m)$ or $a-m$ is an additive quantity, let $a-m=a'$.

Since $a>(m+s)$, $a'>s$.

Again, $a'-(+s)=a'-s=a''$, an additive quantity.

Now $a''=a'-s$; and $a'=a-m$.

Therefore $a''=a-m-s=a-(m+s)$; that is, the effect produced on the additive quantity a by the addition to it of $-m$, and the subtraction from the algebraic sum $a+(-m)$ of $+s$, is equal to the effect produced on a by either subtracting from it m and s in succession, or by subtracting at once the sum $m+s$.

b$''''$. Lastly, let m, the minuend, and s, the subtrahend, be both subtractive quantities: the expression of the remainder $-m-(-s)$ is to be reduced, so as to contain only one symbol of operation between m and s.

Since the subtractive quantity $(-m)$ is to be diminished by the subtractive quantity $(-s)$, it is evident that the subtractive quantity which remains is equal to the excess of m over s, supposing m the greater. This remainder is indicated by the expression $m-s$; therefore $-m-(-s)$ may be written $-(m-s)$.

To determine the effect of the symbol $-$, which precedes this expression, upon the other $-$, which is placed between the quantities within the parenthesis, let the subtractive quantity $-(m-s)$ be connected by means of its sign $-$ with an additive quantity, a.

The result of the algebraic addition or numerical subtraction is obtained by first subtracting m from a; the expression of this remainder is $a-m$; but this quantity is too little by the quantity s, for the subtrahend is not m, but $m-s$; therefore the remainder $a-m$ must be augmented by s. It thus becomes $a-m+s$.

Consequently the expression $-(m-s)$ may be otherwise written, without the parenthesis, $-m+s$ or $s-m$.

If $m>s$, the quantity $-(m-s)$ is subtractive; but if $m<s$, it is additive.

Combining, as before, the symbols $+$ and $-$, it follows from this discussion that,

$$\left.\begin{array}{l} \pm m-(+s)=\pm m-s \\ \mp m-(-s)=\mp m+s \end{array}\right\} \text{or that } \pm m-(\pm s)=\pm m\mp s.$$

In words; that the expression of the difference of two monomials is composed of the minuend and subtrahend connected by a symbol of operation; that when the subtrahend is an additive quantity (or has the sign $+$), the connecting symbol is $-$; and that when the subtrahend is a subtractive quantity (or has the sign $-$), the connecting symbol is $+$.

The sign which, in the expression of the difference, is placed before the subtrahend, is the reverse of that which the subtrahend has before subtraction. The sign of the minuend remains unchanged.

Whence, to subtract one monomial quantity from another, change the sign of the subtrahend, and by means of the changed sign connect the minuend and subtrahend together, as in addition.

c. Let m, m' be two monomial quantities whose product is required:

If m, m' are both additive quantities, the product mm' is an additive quantity. This is the case of arithmetic.

If the multiplicand m is an additive quantity, and the multiplier m' a subtractive quantity, the product $(+m)\times(-m')$ is a subtractive quantity.

For the expression $(+m)\times(-m')$ or $m\times(-m')$ indicates that the multiplicand m is to be subtracted as many times as the multiplier m' contains simple units; or that m' repetitions of the quantity m are to be subtracted. Now m' repetitions of m are expressed by the product $m'm$ or mm' (Part. I. Art. 66), and the condition that this quantity mm' is subtractive is expressed by the symbol $-$ prefixed to the product mm'. Whence $(+m)\times(-m')=-mm'$.

If m is subtractive and m' additive, this case may be reduced to the preceding by changing the order of the factors; or, since $(-m)$ taken once is $-m$; twice, is $-2m$ . . . . it follows that the subtractive quantity $(-m)$ taken m' times is the subtractive quantity mm', that is, $-mm'$.

If m and m' are both subtractive, the expression of the product $(-m)\times(-m')$ indicates that the subtractive quantity $-m$ is to be subtracted m' times. Now $-m$ subtracted once is $+m$; subtracted twice, it is $+2m$ . . . , and subtracted m' times, it is $+mm'$.

Therefore $(-m)\times(-m')=+mm'$.

Whence, in conclusion, $\pm m\times\pm m'=+mm'$,

$\pm m\times\mp m'=-mm'$, that is,

the multiplicand and multiplier being both additive quantities or both subtractive quantities, the product is additive. But if the multiplicand is additive and the multiplier subtractive, or if the multiplicand is subtractive and the multiplier additive, then in either case the product is a subtractive quantity.

d. The quotient in algebraic division is a quantity such that multiplied by the divisor it produces the dividend both in respect of quantity and sign.

Let d be the dividend, d' the divisor, and q the quotient; and first, let d and d' be additive, then q is additive.

For if q were subtractive, the product $+d\times(-q)$ would be subtractive (Part II. Art. 4 c), which is contrary to the hypothesis by which $d=d'\times q$ is additive.

2d. If d is additive and d' subtractive, or,

3d. If d is subtractive and d' additive, in either case q is subtractive. For if q is additive in Case 2, the dividend $=(-d')\times(+q)$ must be subtractive;

And if q is additive in Case 3, the dividend $=(+d')\times(+q)$ must be additive, results which are contrary to the hypotheses made respecting d in these cases.

4th. If d and d' are both subtractive, q is additive; for if q were subtractive the dividend $=(-d')\times(-q)$ must be additive, which is contrary to the hypothesis.

$$\text{Whence } \pm d\div\pm d'=+q$$
$$\pm d\div\mp d'=-q.$$

Expressed in words, the conclusion is this: If the dividend and divisor are both additive or both subtractive quantities the quotient is additive; but if the dividend is additive and the divisor subtractive, or if the dividend is subtractive and the divisor additive, then, in each of these cases, the quotient is subtractive.

5. Every quantity expressed by means of algebraic symbols is termed an algebraic quantity or literal quantity. Thus, $4a$ is the algebraic expression of four times the number a; $3a^2$ is the algebraic expression of three times the square of a; a^3b^2, that of the product of the cube of a by the square of b; $b^2-2bc+c^2$, that of the square of b, diminished by twice the product of b by c, and augmented by the square of c.

The numerical value of an algebraic quantity is the number which is obtained by giving particular values to all the letters of which it is composed, and performing the arithmetical operations which are symbolically indicated by it.

This numerical value depends on the particular values which are given to the letters; and it ought, in general, to change as these values are changed. The numerical value, for example, of $4a^2$ is 36 when $a=3$, and 64 when $a=4$.

The numerical value of a polynomial expression is not changed by a change in the arrangement of its terms, provided its proper sign is preserved to each term. Thus the polynomials $7a^4-4a^3b+2a^2c$, $2a^2c-4a^3b+7a^4$, $2a^2c+7a^4-4a^3b$, have the same numerical value; for, from the nature of arithmetical addition and subtraction, the number of individuals is independent of the order of their arrangement.

Of the terms composing a polynomial, some are preceded by the sign +, others by the sign —. The first are named additive, the second subtractive, terms; the first are also named positive, the second negative, terms. The latter denominations are given from the conventions of Article 3; though improper, their use is sanctioned by custom. If the first term of a polynomial has no sign it is considered additive.

If in the polynomial $7a^4-4a^3b+2a^2c$,
or $7a^4+2a^2c-4a^3b$,
a is made $=4$, $b=3$, $c=2$, the numerical value of the polynomial for these particular values of a, b, c, is given by the expression

$$7\times4^4+2\times4^2\times2-4\times4^3\times3.$$

Now, $7\times4^4=7\times4\times4\times4\times4=$ - - 1792
and $2\times4^2\times2=2\times4\times4\times2=$ - - 64

The sum of the additive terms = - +1856
Again, $-4\times4^3\times3=-4\times4\times4\times4\times3=-768$

Taking the difference, it is found that 1088 is the numerical value of the expression $7a^4-4a^3b+2a^2c$, on the hypothesis of $a=4$, $b=3$, $c=2$.

If $a=2$, $b=5$, and $c=3$, the numerical of $7a^4-4a^3b+2a^2c$ is -24.

Ex. 1st. Required the numerical value of $a^3-3a^2b+3ab^2-b^3$ when $a=9$ and $b=5$? .... *Ans.* 64.

2d. Required the numerical value of $(a+b)$ $(a-b)$, when $a=6$ and $b=2$? .... *Ans.* 32.

3d. Required the numerical value of $(a+b)\times a-b$ when $a=6$ and $b=2$? .... *Ans.* 46.

6. Each of the literal factors of a term is called a dimension of the term, and the number of the factors or dimensions is named the degree of the term.

The numeral coefficient of a term is not considered a dimension. Thus, $4a$ is a term of one dimension, or of the first degree; $3bc$ a term of two dimensions, or of the second degree; $5a^2b^3c$, or $5aabbbc$, a term of six dimensions, or of the sixth degree.

The number of dimensions, or the degree of a term, is obtained by forming the sum of the exponents of the letters which enter into the term. In forming this sum it is necessary to be remembered that a letter which has no exponent is considered to have unity for exponent (Part I. Art. 75). Hence the degree of the term $a^3b^2cd^4$ is $3+2+1+4=10$.

A polynomial expression is homogeneous when all its terms are of the same degree. The expressions $a^2+2ab+b^2$, $a^4+2a^3b-a^2bc+4bcde$ are homogeneous; b^3-bc+d, $ab+5c-3d^3$ are not homogeneous.

Terms composed of the same letters raised to the same powers are called like or similar: $3bc$ and $5bc$, $2a^2bd$ and $7a^2bd$, b^3c^2d and $9b^3c^2d$ are like terms; but $3b^2c$ and $5bc^2$, $2a^3bd$ and $5ab^3d$ are not like terms; for, although composed of the same letters, the same letters are not affected with the same exponents.

A polynomial which contains two or more like terms can be simplified. The polynomial $8ab^2+6a^2c-5ab^2+a^2c-b^2c^2$, for example, which contains the like terms $8ab^2$ and $-5ab^2$, $6a^2c$ and a^2c, may have the arrangement of its terms changed thus (Art. 5):

$$8ab^2-5ab^2+6a^2c+a^2c-b^2c^2.$$

$$\text{Now } 8ab^2-5ab^2=3ab^2 \quad \text{(Part I. Art. 43),}$$

$$\text{and } 6a^2c+a^2c=7a^2c \quad \text{(Part I. Art. 26);}$$

$$\text{therefore } 8ab^2+6a^2c-5ab^2+a^2c-b^2c^2=3ab^2+7a^2c-b^2c^2.$$

In like manner the polynomial $2ab^2c^3-4ab^2c^3-6ab^2c^3+5ab^2c^3-ab^2c^3$ may be arranged thus:

$$2ab^2c^3+5ab^2c^3-4ab^2c^3-6ab^2c^3-ab^2c^3.$$

The sum of the additive terms is $7ab^2c^3$.

The sum of the subtractive terms is $11ab^2c^3$.

The given polynomial is hence reduced to $7ab^2c^3-11ab^2c^3$, or to $-4ab^2c^3$ (Art. 3).

Polynomial quantities containing terms which have several but not all their factors common may be simplified in the same manner.

Taking, for example, the binomial

$$mb^2c^3d+nb^2c^3d,$$

the first term expresses m repetitions of the quantity b^2c^3d, and the second term expresses n repetitions of the same quantity; therefore the two terms express $m+n$ repetitions of b^2c^3d, an expression which may be written

$$(m+n)b^2c^3d.$$

The binomial $a^6b^4c^2-a^3b^4c^2$ being composed of b^4c^2 into a^6, less b^4c^2 into a^3, the two terms express a number of repetitions of the common quantity b^4c^2, which would be obtained by subtracting the numerical value of a^3 from that of a^6. This subtraction is indicated by the algebraic difference a^6-a^3 or $a^3(a^3-1)$ (Part I. Art. 75 & 76). Therefore $a^6b^4c^2-a^3b^4c^2=(a^6-a^3)b^4c^2$, or $(a^3-1)a^3b^4c^2$.

In the case, therefore, of polynomials containing terms which have several but not all their factors common, the factors which are common to two or more terms are held to compose like terms, and their other factors are taken as literal coefficients, the sum or the difference of which is indicated by the appropriate sign.

Whence, to effect the reduction of the like terms of a polynomial,

Rule. If the polynomial contains only like additive or like subtractive terms, form them into a single additive or a single subtractive term by adding together their coefficients and annexing the common literal part to the sum. If it contains like additive and subtractive terms, form a single additive term out of all the like terms which are preceded by the sign +, and a single subtractive term out of all the terms, like the former, which are preceded by the sign —; subtract the smaller sum from the greater (or, when the coefficients are literal, indicate the subtraction), and to the remainder prefix the sign of the greater.

This sort of reduction is peculiar to algebra; it affects the coefficients only of the like terms, and not either the common quantities or their exponents.

Examples of reduction of the like terms of a polynomial:

1st. $4a^2b-3a^2c+9ac^2-2a^2b+7a^2c-6b^3$?......*Ans.* $2a^2b+4a^2c+9ac^2-6b^3$.
2d. $-8b^3+7a^3+3b^3-9a^2b-5b^3+ab^2+4b^3$?... *Ans.* $7a^3-9a^2b+ab^2-6b^3$.
3d. $2a^3bc^2-4a^3bc^2+6a^3bc^2-8a^3bc^2+11a^3bc^2$?....................*Ans.* $7a^3bc^2$.
4th. $5a+3b-4c+a-4b-2c+3e$?........................*Ans.* $6a-b-6c+3e$.
5th. $3a^2-4ab+2a^2-3ab+b^2+2ab-5b^2$?..................*Ans.* $5a^2-5ah-4b^2$.
6th. $7abc^2-abc^2-7abc^2-8abc^2+4abc^2$?......................... *Ans.* $-5abc^2$.

SECTION II.

ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION OF INTEGER EXPRESSIONS.

ADDITION.

7. In the addition of algebraic, as in that of arithmetical, quantities the object is to find a single expression which shall be equal to the sum of several others. This, in the case of numbers, is accomplished by combining the addenda into one number, which contains unity as often as all the addenda taken together.

The addition of general symbols of number or quantity is indicated by connecting the addenda together by means of another symbol which denotes the operation of addition. Thus, the sum of ab and cd is $ab+cd$ (Part I. Art. 25, 26), and the sum of ac and $-bd$ is $ac+(-bd)$, or $ac-bd$ (Part II. Art. 4 a).

But the result cannot be reduced to a single quantity unless particular values are given to all the letters.

The sum of a monomial, d, and a polynomial, $a-b+c$, may be indicated thus:

$$(a-b+c)+(+d).$$

To reduce this expression to the form of a single polynomial: the expression $a-b+c$ indicates that the number of units a is to be diminished by the number of units b, and the result to be augmented by the number c. If particular values of a, b, c are given, these operations can be effected, and the result is a number, n (Art. 3).

If a particular value is also given to the additive quantity d, the sum of the given polynomial and monomial quantities is obtained by adding d units to n units. This sum, which is indicated by the expression $n+d$, is that which is required.

So long as the terms are general expressions this reduction cannot be made. Replacing, therefore, n by its value $a-b+c$, the polynomial $n+d$ becomes $a-b+c+d$, which is the expression of the sum required.

If the quantity d is subtractive ($-d$), the algebraic sum of n and $-d$ is $n-d$ (Art. 4 a).

Therefore $a-b+c+(-d)=a-b+c-d$.

The sum of two polynomial expressions, $a^3-3a^2b+3ab^2-b^3$ and $5a^2b-2ab^2+4b^3$, may be indicated thus:

$$(a^3-3a^2b+3ab^2-b^3)+(5a^2b-2ab^2+4b^3).$$

To reduce this expression to a single polynomial: if the values of a, b are given, each of the polynomials can be reduced to a single term, and the sum is obtained by adding these terms together. Without particular values the process is one of indication.

It is required, then, to indicate the combination, by addition, of the quantity $5a^2b-2ab^2+4b^3$ with $a^3-3a^2b+3ab^2-b^3$. Since the sum of the parts is equal to the whole, if the terms of the polynomial $5a^2b-2ab^2+4b^3$ are, one after another, added to the other polynomial until they are all combined with it, the result must be the sum of the two polynomials.

Taking $5a^2b$, the 1st term of the second polynomial,

$$(a^3-3a^2b+3ab^2-b^3)+(+5a^2b)=a^3-3a^2b+3ab^2-b^3+5a^2b.$$

Adding, next, $-2ab^2$, the 2d term of the second polynomial,

$$(a^3-3a^2b+3ab^2-b^3+5a^2b)+(-2ab^2)=a^3-3a^2b+3ab^2-b^3+5a^2b-2ab^2.$$

Lastly, adding the remaining term $4b^3$,

$$(a^3-3a^2b+3ab^2-b^3+5a^2b-2ab^2)+(+4b^3)=a^3-3a^2b+3ab^2-b^3+5a^2b$$
$$-2ab^2+4b^3.$$

The addition of the 2d polynomial to the 1s has been made, term by term, according to the rule of Article 4 a; and the same process must be followed, however many the terms or the number of the polynomials whose sum is required.

When the addenda have been reduced to a single polynomial the process of simplification explained in Article 6 may be employed.

Changing the arrangement of the terms for the purpose of bringing together those which are like, this polynomial may be written thus:

$$a^3-3a^2b+5a^2b+3ab^2-2ab^2-b^3+4b^3.$$

Now $-3a^2b+5a^2b=+2a^2b$, $3ab^2-2ab^2=ab^2$, $-b^3+4b^3=+3b^3$ (Art. 6).

Whence the algebraic sum of the given polynomials is

$$a^3+2a^2b+ab^2+3b^3.$$

As a second example, let it be required to find the algebraic sum of the polynomial quantities $4a+9b-2c$, $2a-3c+4d$, $7b+c-e$.

The addition of the second polynomial to the first, and of the third to the single polynomial composed of the first and second, may be made as in the preceding example, and the polynomial which is obtained as the indication of the sum may be simplified by the reduction of the like terms.

The sum of the proposed polynomials before simplification is $4a+9b-2c+2a-3c+4d+7b+c-e$.

After simplification it is $6a+20b-4c+4d-e$.

If the polynomials are composed of like terms differently arranged, the order of the terms of the 2d, 3d, . . . polynomials may be changed into the order of the 1st; the polynomials being then placed one under another in horizontal lines, the like terms must fall in vertical columns; their reduction is thus rendered more easy.

Ex. Required the sum of $2a^3+3b^2x-5c^2x^2$, $2c^2x^2+5a^3-7b^2x$, and $b^2x-3c^2x^2+a^3$.

$$\begin{array}{rcl} & & 2a^3+3b^2x-5c^2x^2 \\ 2c^2x^2+5a^2-7b^2x & = & 5a^3-7b^2x+2c^2x^2 \\ b^2x-3c^2x^2+a^3 & = & \underline{a^3+\ b^2x-3c^2x^2} \end{array}$$

The sum of the proposed expressions is $\underline{8a^3-3b^2x-6c^2x^2}$

If some of the polynomials contain terms which are not contained in all the others, the change of arrangement may be troublesome. In this case the polynomial addenda may be written, without change of order, in horizontal lines; the like terms are then selected, by inspection of the whole addenda, and reduced into one term; each, as it is made to enter into the result, being marked with a stroke (thus $\cancel{b}$), to show that it has been taken into account. If each term is marked when it is reduced, any omission of terms is avoided, for those unmarked remain to be added.

Ex. Required the sum of the following polynomials:

$$\begin{array}{l} \cancel{7}a^2b-\cancel{8}abc-\cancel{6}b^2c-\ 9c^3+cd^2 \\ +\cancel{6}abc-\cancel{6}a^2b+\ 3c^3-\cancel{4}b^2c+cd^2 \\ \underline{+\cancel{4}a^2b-\ 8c^3+\cancel{9}b^2c-\ 3d^2} \\ \underline{6a^2b+5abc-3b^2c,\ \&c.} \end{array}$$

The sum in this instance is incomplete, the unmarked terms remaining to be combined with it.

The reasoning employed in these instances is applicable to any case of the addition of polynomial expressions. It supplies the following rule for such addition.

Write the terms of the given polynomials one after another, with their proper signs, and make reduction of like terms when such reduction is possible.

It is required to find the algebraic sums of the following expressions:

1st. $a+b$ and $a-b$? .... *Ans.* $2a$.
2d. $3a-4x$ and $5a-8x$? .... *Ans.* $8a-11x$.
3d. $7a-5c+3b$ and $2a-3c-7b$?............... *Ans.* $9a-8c-4b$.
4th. $a+6x-5y+8$ and $-5a-4x+4y-3$? *Ans.* $-4a+2x-y+5$.

5th. $4a+4b+3c$ and $-4x-4y+3z$?

Ans. $4a+4b+3c-4x-4y+3z$.

6th. $5a+4b-3c-7d+8$ and $3a-12b+7c-10d-4$?

Ans. $8a-8b+4c-17d+4$.

7th. $4a^2c-10bde$, $6a^2c-9bde$, and $11a^2c-3bde$?

Ans. $21a^2c-22bde$.

8th. $3a^2+4bc-e^2+10$, $-5a^2+6bc+2e^2-15$, and $-4a^2-9bc$ $-10e^2+21$? .... *Ans.* $-6a^2+bc-9e^2+16$.

9th. $5a+3b-4c$, $2a-5b+6c+2d$, $a-4b-2c+3e$, and $7a+4b$ $-3c-6e$? .... *Ans.* $15a-2b-3c+2d-3e$.

10th. $16a-5b+10c-9d$, $3a+18b-5c-7d+3e$, $-7a-2b$, $-3d$ $+5e-9h$, and $11a-3b+2c+8d+7h$?

Ans. $23a+8b+7c-11d+8e-2h$.

11th. $8a+b$, $2-b+c$, $-3a+5b+2d$, $3d-6b-3c$, and $7c-2d$ $-5a$? .... *Ans.* $2a-b+5c+3d$.

12th. $3a-7f$, $4f-2a$, $3f-3a$, and $2a$? .... *Ans.* 0.

13th. $7x-6y+5z+3-g$, $-x-3y-8-g$, $7g-x+y-3z-1$, $-2x$ $+3y+3z-1-g$, and $x+8y-5z+9+g$?

Ans. $4x+3y+2+5g$.

14th. $6a-4b+3c$, $3a+6b-4c+4d$, $a+2b+2c-d$, and $5a-b+c$ $+2d+e$? .... *Ans.* $15a+3b+2c+5d+e$.

15th. $6a-4b-3c$, $3a-6b+4c-4d$, $a+2b-2c+d$, $5a-b+c+2d$ $-e$? .... *Ans.* $15a-9b-d-e$.

16th. $8a^2-\frac{3}{2}ab-2b^2-\frac{3}{4}$, $7c^2-a^2+\frac{1}{4}ab+2$, and $ab+2b^2+4c^2+\frac{1}{2}$?

Ans. $7a^2-\frac{1}{4}ab+11c^2+1\frac{3}{4}$.

17th. $4ac-15bd+ex$, $11ac+7b^2-19ex$, and $6bd-41a^2-7de$?

Ans. $15ac-41a^2-9bd+7b^2-18ex-7de$.

18th. px^3-qx^2-rx and ax^3-bx^2-x?

Ans. $(p+a)x^3-(q+b)x^2-(r+1)x$.

SUBTRACTION.

8. The difference of two monomial quantities, additive or subtractive, is indicated in the most general manner by the formula $\pm m-(\pm s)=\pm m\mp s$ (Art. 4 b).

If the minuend is a polynomial, as $a+b-c$, and the subtrahend an additive monomial, s, the difference is indicated by the expression $(a+b-c)-(+s)$; which signifies that the number of units expressed by a having been augmented by the number expressed by b, and this sum diminished by the number expressed by c, this result is further to be diminished by the number of units expressed by s.

If particular values are given to a, b, c, and s, the value of the polynomial $a+b-c=m$ can be found, and then the remainder left by the subtraction of s from this value, or $m-s$. But as the particular values of these quantities are not given, the subtraction can only be indicated. Since $m-s$ is the indication of the difference, and $m=a+b-c$, it follows that the polynomial $a+b-c-s$ expresses the difference required.

From similar reasoning, and from Article 4 b, it likewise follows that $(a+b-c)-(-s)=m-(-s)=m+s$,

or that $(a+b-c)-(-s)=a+b-c+s$.

The difference of two polynomial expressions, $a+b-d$ and $c-f+g$, can be indicated thus,

$$(a+b-d)-(c-f+g).$$

Since the subtrahend is composed of the terms $+c$, $-f$, and $+g$, if these terms are one after another subtracted from the minuend and successive remainders, the last remainder must be that which results from the subtraction of $c-f+g$ from $a+b-d$.

$$\text{Now } (a+b-d)-(+c)=a+b-d-c,$$
$$(a+b-d-c)-(-f)=a+b-d-c+f,$$
$$(a+b-d-c+f)-(+g)=a+b-d-c+f-g.$$
$$\text{Therefore } (a+b-d)-(c-f+g)=a+b-d-c+f-g.$$

In this process by successive subtractions the first partial remainder is obtained by changing the sign of the first term of the subtrahend, and connecting this first term, by means of the changed sign, with the minuend.

The second partial remainder is obtained by changing the sign of the second term of the subtrahend, and connecting it, by means of the changed sign, with the first partial remainder, &c. &c.

Whence, generally, the expression of the difference of two polynomials is obtained in the form of a single polynomial by connecting the terms of the subtrahend with the minuend by means of signs contrary to those which the terms of the subtrahend have before subtraction, in accordance with Article 4 b.

Consequently, to subtract a polynomial,

Rule. Having changed the signs of all the terms of the subtrahend, add it to the minuend.

Reduction of the like terms (as practised in addition) can be made after the signs of the terms of the subtrahend have been changed.

It is required to find the algebraic differences of the following expressions, the first polynomial in each example being considered the minuend, and the second the subtrahend:

1st. $a+b$ and $a-b$?..*Ans.* $2b$.

2d. $3a-2b+6$ and $2a-7b-3$?....................*Ans.* $a+5b+9$.

3d. $6a-12b$ and $-5a-10b$?.......................*Ans.* $11a-2b$.

4th. $13a-2b+9c-3d$ and $8a-6b+9c-10d+12$?
Ans. $5a+4b+7d-12$.

5th. $2a-3x+5y-6$ and $6a+4x+5y+4$?...*Ans.* $-4a-7x-10$.

6th. $5a^2+4ab-6xy$ and $11a^2+6ab-4xy$?
Ans. $-6a^2-2ab-2xy$.

7th. $-7f+3m-8x$ and $-6f-5m-2x+3d+8$?
Ans. $-f+8m-6x-3d-8$.

8th. $3h-2k$ and $9l-7-8k$?.................*Ans.* $3h-9l+6k+7$.

9th. $-a-5b+7c-d$ and $4b-3c+2d+3k$?
Ans. $-a-9b+10c-3d-3k$.

10th. $5b-3a+203c+5$ and $-2b-8a+67c+7$?
Ans. $7b+5a+136c-2$.

11th. $5b-8a-3c$ and $7a-3b-2c$?............*Ans.* $15a+8b-c$.

12th. $3c-2l+5c$ and $8l+7c-4l$?.......................*Ans.* $c-6l$.

13th. $3a-5c+3d$ and $7a-5d+8c-2e$?.*Ans.* $8d+2e-4a-13c$.

14th. $5c+3$ and $2c-9-7c$?..........................*Ans.* $10c+12$.

15th. $3a-17b-10b+13a-2a$ and $6b-8a-b-2a+3d+9a-5h$?................................*Ans.* $15a-32b-3d+5h$.

16th. ax^3-bx^2+x and px^3-qx^2+rx ?
Ans. $(a-p)x^3-(b-q)x^2+(1-r)x$.

MULTIPLICATION.

9. Multiplication of algebraic, like multiplication of arithmetical, quantities is an abridged process for determining the sum of many repetitions of the same quantity.

a. If the addendum or multiplicand, m, is added to itself 2, 3 . . . . . m' times, the sums of the repetitions $m+m$, $m+m+m$, . . . $m+m+m$ . . . . to m' terms are indicated by the combinations of symbols;

$$2\times m \text{ or } 2m,\ 3\times m \text{ or } 3m,\ m'\times m \text{ or } m'\cdot m \text{ or } m'm.$$

The numeral factor or coefficient is always prefixed to the literal factor; but the product of two literal factors m, m', is written, indifferently, $m'm$ or mm'.

In effect it is proved (Part I. Art. 71) that the product of any given factors expresses the sum of a certain number of units, and that this sum is not made greater or less by mere changes of arrangement.

b. Let the literal factors m, m' have numeral coefficients, and let the multiplicand be $6m$ and the multiplier $4m'$; the product is indicated by the expression $6m\times 4m'$,

or by $6\times m\times 4\times m'$,
or by $6\times 4\times m\times m'$ (Part I. Art. 71),
or, since $6\times 4=24$, by $24\,mm'$.

c. If $m=m'$, the product mm is written m^2; and by the same convention the product of 3, 4 . . . . n factors equal to m are written respectively,

$$m^3,\ m^4\ .\ .\ .\ .\ .\ m^n.$$

If m^3 or mmm is multiplied by m^5 or $mmmmm$, the product $m^3\times m^5=mmm\times mmmmm$, or $mmmmmmmm$, or m^8, is obtained by adding together 3 and 5, the exponents of the multiplicand and multiplier, and affecting the root m with an exponent equal to the sum.

And if m^e or mmm . . . . to e repetitions of m is multiplied by $m^{e'}$ or $mmmm$ . . . to e' repetitions of m, the product is, in like manner, indicated by the expression $m^{e+e'}$ (Part I. Art. 75).

Hence the product of the monomial factors $5a^2b^3cd^2$, $7ab^2c^5$ is $35a^3b^5c^6d^2$.

For $5a^2b^3cd^2\times 7ab^2c^5=5\times a^2\times b^3\times c\times d^2\times 7\times a\times b^2\times c^5=5\times 7\times a^2\times a\times b^3\times b^2\times c\times c^5\times d^2$.

Now $5\times 7=35$, $a^2\times a=a^3$, $b^3\times b^2=b^5$, $c\times c^5=c^6$.

Making these substitutions,

$$5a^2b^3cd^2\times 7ab^2c^5=35a^3b^5c^6d^2.$$

d. The quantities whose products have been found are additive. Either factor or both factors may, however, be subtractive. The formulæ of Article 4 c, which are,

$$(\pm m)\times(\pm m')=+mm',$$
$$(\pm m)\times(\mp m')=-mm',$$

serve to determine the sign of the product in every case.

Whence, to multiply a literal quantity by any particular number, write the numeral factor on the left of the literal quantity.

To multiply together any unlike literal factors, write them one after another in any order (the alphabetical order of the letters is generally followed).

To multiply together any like literal factors, write each of the letters found in the factors once only, and affect each letter with an exponent equal to the sum of the exponents with which it is affected in all the factors.

If any of the quantities have numeral coefficients, write the product of the coefficients on the left of the product of the literal factors.

The following is, consequently, the rule for the multiplication of one monomial quantity by another:

1st. Multiply together the two coefficients.

2d. Annex to this product all the letters which are found in both the multiplicand and multiplier, affecting each letter with an exponent equal to the sum of the two exponents with which the letter is affected in the two factors.

3d. If a letter is found in one only of the factors, write it in the product with the exponent with which it is affected in that factor.

4th. If the monomials are both additive or both subtractive, the product is additive; but if one of them is additive and the other subtractive, the product is subtractive.

It is required to find the products of the following monomial quantities;

1st. ab and abc?..*Ans.* a^2b^2c.
2d. $-5abc$ and $7ade$?..............................*Ans.* $-35a^2bcde$.
3d. $5axy^2$ and $-3a^2x^3$? ....*Ans.* $-15a^3x^4y^2$.
4th. $-\frac{1}{2}abd$ and $-\frac{1}{4}cdef$?..............................*Ans.* $\frac{1}{8}abcd^2ef$.
5th. $3a^2b^4c$ and $7a^3cd^2$?..............................*Ans.* $21a^5b^4c^2d^2$.
6th. $-7a^4bd^2$ and $13a^3b^2c$?......................*Ans.* $-91a^7b^3cd^2$.
7th. $-\frac{2}{3}ab^4c$ and $-\frac{5}{7}a^2c$?..............................*Ans.* $\frac{10}{21}a^3b^4c^2$.
8th. $21a^3b^2cd$ and $8abc^3$?.........................*Ans.* $168a^4b^3c^4d$.

10. The product of a polynomial multiplicand, $a-b+c$, by a monomial multiplier, m, is indicated by the expression $(a-b+c)m$.

The product can be obtained by writing the multiplicand in a horizontal line writing it again term under term in a second horizontal line, repeating this process as often as the multiplier m contains unity, and reducing the like terms, thus,

$$\left.\begin{array}{ccccl} a & - & b & + & c =1^{\text{st}} \\ a & - & b & + & c =2^{\text{d}} \\ a & - & b & + & c =3^{\text{d}} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ a & - & b & + & c =m^{\text{th}} \end{array}\right\}\text{repetition of the multiplicand.}$$

$$\underline{ma \quad -mb \quad +mc} = m \text{ repetitions of the multiplicand.}$$

The first column containing m repetitions of the additive quantity a, its sum is ma.

The second containing m repetitions of the subtractive quantity b, its sum is $-mb$.

The third containing m repetitions of the additive quantity c, its sum is mc.

Whence the sum of m repetitions of the polynomial $a-b+c$ is $ma-mb+mc$; in other words, the product of $a-b+c$ by m is $ma-mb+mc$.

It is manifest that by this process each term of the multiplicand is repeated as often as the multiplier contains unity; otherwise, that each term of the multiplicand is multiplied by the multiplier, and that the sum of the partial products is the product required.

Making the multiplier a subtractive quantity $(-m)$.

1st. The sum of m terms each equal to a, that is ma, is to be taken subtractively, a condition expressed by writing the quantity thus, $-ma$.

2d. The sum of m terms each equal to $-b$ (or $-mb$) is to be taken subtractively. Now $-mb$, taken subtractively, becomes $+mb$ (Art. 4 b).

3d. The sum of m terms, each equal to c, taken subtractively is $-mc$;

$$\text{therefore } (a-b+c)\times(-m)=-ma+mb-mc;$$
$$\text{now } (a-b+c)\times(+m)=ma-mb+mc;$$
$$\text{therefore } (a-b+c)\times(\pm m)=\pm ma\mp mb\pm mc.$$

The formation of the product of a polynomial by a monomial is hence reduced to a series of operations with monomial factors, in which the rule for the multiplication of such quantities (Art. 9) is followed.

Hence, to multiply a polynomial by a monomial quantity,

Rule. Form the product of each term of the multiplicand by the multiplier, by the rule for the multiplication of monomial factors, and take the algebraic sum of the partial products for the result required.

Examples of the multiplication of polynomial by monomial quantities:

1st. $(6a+3b-5f)\times 5b$?……………………*Ans.* $30ab+15b^2-25bf$.
2d. $(-2b+3c-d)\times -8a$?………………*Ans.* $16ab-24ac+8ad$.
3d. $(7ad-16bc-16acf)\times 10ab$?
Ans. $70a^2bd-150ab^2c-160a^2bcf$.
4th. $(2a^4-3ba^3-5b^2a^2+b^3a)\times a^3$?
Ans. $2a^7-3a^6b-5a^5b+a^4b^3$.
5th. $(a^3-a^2b+ab^2-b^3)\times ab$?………*Ans.* $a^4b-a^3b^2+a^2b^3-ab^4$.
6th. $(a^m-b^n+c^r)$ by a^n ?………………*Ans.* $a^{m+n}-a^nb^n+a^nc^r$.

11. If the multiplicand is a monomial and the multiplier a polynomial quantity, the order of the factors may be changed, and this case reduced to the preceding.

Otherwise, making m the multiplicand and $a+b-d$ the multiplier, the product $m\times(a+b-d)$ can be obtained by taking a repetitions of m, then b repetitions of m, then $-d$ repetitions of m, and forming the algebraic sum of all the terms, thus,

$$\left.\begin{array}{l} m=1^{\text{st}} \\ m=2^{\text{d}} \\ m=3^{\text{d}} \\ \cdot\ \ \cdot\ \ \cdot \\ m=a^{\text{th}} \\ m=a+1 \\ m=a+2 \\ \cdot\ \ \cdot\ \ \cdot\ \ \cdot \\ m=a+b \\ -m=a+b-1 \\ -m=a+b-2 \\ \cdot\ \ \cdot\ \ \cdot\ \ \cdot\ \ \cdot \\ -m=a+b-d \end{array}\right\}\text{ repetition of multiplicand.}$$

The column composed of m contains,

1st. a repetitions of m, taken additively, or am;
2d. b repetitions of m, taken additively, or bm;
3d. d repetitions of m, taken subtractively, or $-dm$.

Since the column contains m as often as the multiplier contains unity, it is evident that the algebraic sum of the column is equal to the product of the multiplicand and multiplier.

The algebraic sum of the column is $am+bm-dm$;
Therefore $m\times(a+b-d)=am+bm-dm$.

It seems unnecessary to repeat the process for a subtractive multiplier, $-(a+b-d)$, or the rule, which is the same as that deduced from the preceding case.

12. Let both multiplicand and multiplier be polynomial expressions, and let the former be denoted by $a-b+c-d$, and the latter by $m-n+r-s$.

If the product is formed by addition, it is evident that the product of the multiplicand by the multiplier is composed of the product of each term of the multiplicand by the polynomial multiplier, and that the product of a term of the multiplicand by the polynomial multiplier is composed of the product of that term by each term of the polynomial multiplier, as in Article 11.

Hence the product of two polynomials is composed of the product of every term of the multiplicand by every term of the multiplier. The partial products are found by the rule of Article 9.

Otherwise, since $a-b+c-d=a+c-b-d=a+c-(b+d)$,
and $m-n+r-s=m+r-n-s=m+r-(n+s)$;
and since $a+c$, or the sum of the additive terms of the multiplicand, can be represented by a single letter, p, and $b+d$, the sum of the subtractive terms, by another letter, q, a multiplicand which is composed of terms of which

some are additive, others subtractive, may be represented by the expression $p-q$.

Similarly, a multiplier composed of terms of which some are additive and others subtractive may be represented by the expression $p'-q'$.

Now the product of $p-q$ by $p'-q'$ is obtained by forming the sum of $p'-q'$ repetitions of $p-q$, or by forming the sum of p' repetitions of $p-q$, then the sum of q' repetitions of $p-q$, and subtracting the second quantity from the first.

Now p' repetitions of $p-q=p'p-p'q$ (Art. 10),
and q' repetitions of $p-q=pq'-qq'$.

Subtracting the second of these expressions from the first,

$$(p-q)\,p'-(p-q)\,q', \text{ or } (p-q)\,(p'-q'), =p'p-p'q-pq'+qq'.$$

The signs of the partial products obtained in this manner agree with the signs obtained from the rule for the multiplication of monomials having the same or opposite signs (Art. 4 c).

This reasoning is applicable to polynomials composed of any number of terms. Whence, to find the product of two polynomial factors,

Rule. Multiply every term of the multiplicand by every term of the multiplier, and make reduction of the similar terms.

a. In forming the product of two polynomial factors, the partial products and their algebraic sum are independent of any particular arrangement of the terms of the multiplicand and multiplier; but the reduction of the like terms among the partial products is facilitated by a particular arrangement of the terms of the factors whenever the same letter raised to different powers is found in several terms of both the factors. This arrangement consists in placing in both factors the term which contains the highest power of the letter which is common to many terms on the left; the term which contains the next lower power of the same letter, second; the term which contains the next lower power, third, &c.; and lastly, on the right, the term or terms which do not contain this letter. Factors arranged in this manner are said to be arranged according to the descending powers of the same letter.

If this order is reversed, the factors are said to be arranged according to the ascending powers of the same letter.

Both arrangements are equally convenient, and sometimes both are combined in the same expression, as in

$$a^5+a^4x+a^3bx^2-a^2cx^3-ac^2x^4+x^5,$$

which is arranged according to the descending powers of a and the ascending powers of x.

Examples of the multiplication of polynomial quantities :

1st. $(a+b)\times(a+b)$? .... *Ans.* $a^2+2ab+b^2$.
2d. $(a+b)\times(a-b)$? .... *Ans.* a^2-b^2.
3d. $(a-b)\times(a-b)$? .... *Ans.* $a^2-2ab+b^2$.
4th. $(a+b)\times(c+d)$? .... *Ans.* $ac+bc+ad+bd$.
5th. $(a+b-c)\times(d-e)$? .... *Ans.* $ad+bd-cd-ae-be-ce$.
6th. $(7l-2m-9)\times(3l-11m)$?
Ans. $21l^2-lm-27l+22m^2+99m$.
7th. $(1-x+x^2-x^3)\times(1+x)$? .... *Ans.* $1-x^4$.
8th. $(a^2+2ab+b^2)\times(a^2-2ab+b^2)$? .... *Ans.* $a^4-2a^2b^2+b^4$.
9th. $(4a^3-5a^2b-8ab^2+2b^3)\times(2a^2-3ab-4b^2)$?
Ans. $8a^5-22a^4b-17a^3b^2+48a^2b^3+26ab^4-8b^5$.
10th. $(3a^2-5bd+cf)\times(-5a^2+4bd-8cf)$?
Ans. $-15a^4+37a^2bd-29a^2cf-20b^2d^2+44bcdf-8c^2f^2$.
11th. $(3a^3-5a^2b-\frac{1}{2}ab^2)\times(-2a^2-3ab+b^2)$?
Ans. $-6a^5+a^4b+19a^3b^2-\frac{7}{2}a^2b^3-\frac{1}{2}ab^4$.

12th. $(a^3+3a^2b+3ab^2+b^3)\times(a^3-3a^2b+3ab^2-b^3)$?

Ans. $a^6-3a^4b^2+3a^2b^4-b^6$.

13th. $(5a^4-2a^3b+4a^2b^2)\times(a^3-4a^2b+2b^3)$?

Ans. $5a^7-22a^6b+12a^5b^2-6a^4b^3-4a^3b^4+8a^2b^5$.

14th. $(3a+5b-\frac{7}{2}c)\times(a-2b+9c)$?

Ans. $3a^2-ab+\frac{47}{2}ac-10b^2+52bc-\frac{63}{2}c^2$.

15th. $(x^2-px+q)\times(x+a)$?

Ans. $x^3-(p-a)x^2+(q-ap)x+aq$.

16th. $(a^m-a^n+a^2)\times(a^m-a)$?

Ans. $a^{2m}-a^{m+n}+a^{m+2}-a^{m+1}+a^{n+1}-a^3$.

12′. Remarks on the multiplication of polynomial quantities:

1st. If the polynomial factors are composed of homogeneous terms, the product is composed of homogeneous terms; and the degree of each term of the product is equal to the sum of the degrees of any two terms, one taken in the multiplicand, the other in the multiplier; for, by the rule for the multiplication of monomial factors, the sum of the exponents in the product of two factors is equal to the sum of the exponents of the two factors.

This remark may be employed to detect errors of calculation with respect to the exponents.

2d. Since the product of two polynomials is obtained by multiplying every term of the multiplicand by every term of the multiplier, it follows that when no reduction of like terms can be made, the number of terms of the product is equal to the product of the number of terms of the multiplicand by the number of terms of the multiplier.

3d. When some of the partial products are like, the whole number of terms of the reduced product may be less than the product of the number of terms in the multiplicand by the number of terms in the multiplier; but among the partial products there are at least two which cannot be reduced with any other; these are, the term arising from the multiplication of that term of the multiplicand which is affected with the highest exponent of one of the letters by that term of the multiplier which is affected with the highest exponent of the same letter, and the term arising from the multiplication of the two terms affected by the lowest exponents of the same letter; for these partial products must contain this letter, the first with a greater and the second with a less exponent than any of the other partial products. Consequently, these terms cannot be similar to any of the other partial products, and therefore cannot be reduced with them.

DIVISION.

13. In this operation the data are a product and one of its factors, and that which is required is to find the other factor.

Let $56a^6b^4c^3$ and $8a^2b^4$ be a product and one of its factors, and let it be required to find the other factor; or, in other words, let it be required to divide $56a^6b^4c^3$ by $8a^2b^4$.

The division is indicated thus, $\frac{56a^6b^4c^3}{8a^2b^4}$, and the required factor or quotient ought to be a quantity such that its product by $8a^2b^4$ is $56a^6b^4c^3$.

Since the product of a polynomial by a monomial is a polynomial, and that the dividend is a monomial, it is evident that the quotient can contain only one term.

Now it follows, from the rule for the multiplication of monomial factors, that 56, the coefficient of the product or dividend, is the product of the coefficients of the divisor and quotient; dividing, therefore, 56 by 8, the coefficient of the quotient is 7. From the same rule it follows, that the exponent, 6, in the dividend is the sum of the exponents of the letter a in the divisor and quotient; therefore the difference $6-2$, or 4, is the exponent of the letter a in the quotient. Also, that the letters which are not common to the divisor and the quotient must be found in the dividend with unchanged exponents; therefore the letter b, which has the same exponent in the dividend and divisor, ought not to enter into the quotient; and, on the contrary, the letter c, which is not in the divisor, ought to be found in the quotient with the same exponent as in the dividend.

Consequently, $\dfrac{56a^6b^4c^3}{8a^2b^4}=7a^2c^3.$

Whence, to divide one monomial quantity by another,

Rule. Divide the coefficient of the dividend by the coefficient of the divisor; the result is the coefficient of the quotient.

To obtain the literal part of the quotient,

1st. When a letter which is common to the two quantities has a greater exponent in the dividend than in the divisor, write it in the quotient with an exponent equal to the difference of these exponents.

2d. When a letter has the same exponent in the dividend and divisor, omit it from the quotient.

3d. When a letter of the dividend is not contained in the divisor, write it in the quotient without change of exponent.

The sign of the quotient is determined by the formulæ

$$\pm d \div \pm d' = +q\,; \quad \pm d \div \mp d' = -q,$$

for which see Article 4 d.

Examples of the division of monomial algebraic quantities:

1st. $48a^7b^3c^2 \div 6a^4b^3$?.................................. *Ans.* $8a^3c^2$.

2d. $8a^6b^2 \div -4a^3b$?.................................. *Ans.* $-2a^3b$.

3d. $-35a^7c^5x^3 \div 7a^3c^3x^2$?.......................... *Ans.* $-5a^4c^2x$.

4th. $8ab^3x^6 \div -12abx$?.................................. *Ans.* $\frac{2}{3}b^2x^5$.

5th. $48a^3b^3c^2d \div 4a^2bcd$?.............................. *Ans.* $12ab^2c$.

6th. $150a^5b^8cd^3 \div 5a^2b^3cd$?.......................... *Ans.* $30a^3b^5d^2$.

7th. $a^mb^nc^r \div a^nb^nc$?.................................. *Ans.* $a^{m-n}c^{r-1}$.

8th. $a^{3m}b^{n+1}c^{r-2} \div a^mb^nc$?.......................... *Ans.* $a^{2m}bc^{r-3}$.

14. In division of monomials,

1st. If the coefficient of the dividend is not divisible by the coefficient of the divisor, or

2d. If the exponent of any letter of the dividend is less than the exponent of the same letter in the divisor, or

3d. If any letter not contained in the dividend is contained in the divisor, the division cannot be effected.

In these cases the division is indicated by writing the dividend over and the divisor under the same horizontal line, as in arithmetic; and, as in arithmetic, the expression of an unexecuted division is termed a fraction.

If the coefficients and literal quantities of both dividend and divisor are resolved into simple factors, and if all the simple factors, numeral and literal, which are common to both the dividend and the divisor are cancelled, then,

1st. When the divisor is reduced to 1, the remaining simple factors of the dividend express the quotient of this dividend and divisor.

2d. When the divisor is not reduced to 1, the remaining simple factors of the dividend and divisor are the numerator and denominator of the fractional expression of the quotient, reduced to its most simple form.

15. Let it be required to divide $8a^6b^2-6a^2b^2c+4a^3b^3$ by $2a^2b$.

Since the product of a polynomial by a monomial quantity is composed of the product of each term of the polynomial by the monomial (Art. 10), it follows, conversely, that if the product is made dividend, and the monomial factor divisor, the polynomial factor becomes the quotient; and that the terms of the quotient must be obtained by dividing each term of the dividend by the divisor.

$$\text{Therefore } \frac{8a^6b^2-6a^2b^2c+4a^3b^3}{2a^2b}=\frac{8a^6b^2}{2a^2b}+\left(\frac{-6a^2b^2c}{2a^2b}\right)+\frac{4a^3b^3}{2a^2b};$$

$$\text{but } \frac{8a^6b^2}{2a^2b}=4a^4b\,;\ \frac{-6a^2b^2c}{2a^2b}=-3bc;\ \frac{4a^3b^3}{2a^2b}=2ab^2;$$

$$\text{therefore } \frac{8a^6b^2-6a^2b^2c+4a^3b^3}{2a^2b}=4a^4b-3bc+2ab^2.$$

The partial divisions are made, in respect both of the quantities and the signs, by the rule for the division of monomials.

Whence, to divide a polynomial by a monomial,

Rule. Divide each term of the dividend by the divisor, as in the case of monomials, and take the algebraic sum of the partial quotients for the quotient required.

Examples of the division of polynomial by monomial algebraic quantities:

1st. $(a^3x^2-5abx^3+6ax^4)\div ax^2$?................*Ans.* $a^2-5bx+6x^2$.

2d. $(-12a^4b+15a^3b^2+24a^2b^3-6ab^4)\div 3ab$?

Ans. $4a^3-5a^2b-8ab^2+2b^3$.

3d. $(10a^4b^3-4a^3b^4+8a^2b^5)\div 2b^3$?........*Ans.* $5a^4-2a^3b+4a^2b^2$.

4th. $(4a^3b^5x^7+42a^5b^3cm^6-6a^4b^8x^7y^4)\div 2a^2b^3$?

Ans. $2ab^2x^7+21a^3cm^6-6a^2b^5x^7y^4$.

5th. $(6a^3b^2-10a^2f+72a^4bx)\div 2a^2$?......*Ans.* $3ab^2-5f+36a^2bx$.

6th. $(a^{2m}b^{n+2}-a^{mn}b^n)\div a^mb^2$?............*Ans.* $a^mb^n-a^{m(n-1)}b^{n-2}$.

16. A monomial is not divisible by a polynomial; for, supposing the quotient either a monomial or polynomial expression, the product of this quotient by the polynomial divisor must contain at least two terms (Art. 12′).

A polynomial is not divisible by a monomial, unless this divides each term of the polynomial; for the dividend must be equal to the product of every term of the quotient by the divisor.

17. When the dividend and divisor are both polynomials the general problem may be stated thus: a given polynomial, denoted by d, is the product of another given polynomial d', by an unknown polynomial q; it is required to find q.

Since $d=d'\times q$; if q is known, d can be reproduced by multiplying the terms of d' successively by q, and forming the algebraic sum of the partial products.

Of these partial products the like are reduced with each other, but such as are unlike any of the others are found in d (or $d'\times q$) uncombined with other quantities, and capable of resolution into the very terms of d' and q of which they are the product.

Now, the partial product which arises from the multiplication of that term of d' which is affected with the highest exponent of one of the letters by that term of q which is affected with the highest exponent of the same letter cannot be reduced with any of the other partial products (Art. 12′). This partial product must, therefore, be itself a term of d, that affected with the highest power of the same letter.

Whence, if that term of d which is affected with the highest exponent of one of the letters is divided by that term of d' which is affected with the highest exponent of the same letter, the result must be that term of q which is affected with the highest exponent of that letter.

Since the dividend d is obtained by adding together the partial products of all the terms of d' by each term of q, it follows that if d' is multiplied by that term of q which has been found, and the product subtracted from d, the remainder must be equal to the product of d' by the unknown terms of q.

The same reasoning may be used in the case of this remainder and d' as has been employed in the case of d and d', and the same conclusion deduced from it, namely, that if that term of the remainder or new dividend which contains the highest power of a letter is divided by that term of d' which contains the highest power of the same letter, the result is a second term of the quotient.

Forming the product of the divisor by this second term, and subtracting it from the second dividend, a second remainder is obtained, which is to be considered as a third partial dividend. From this is obtained, in the same manner, a third term of the quotient.

Continuing this series of operations, it is evident that all the terms of the quotient ought to be obtained by them; and that, if the dividend is the exact product of the divisor and quotient, a remainder $=0$ must be at length obtained.

It follows from the preceding reasoning, and from Article 12, that if, instead of taking those terms of the dividend and divisor which involve the same letter with the highest exponent, the terms which involve the same letter with the lowest exponent are taken, the terms of the quotient can be in like manner obtained.

Ex. Divide $51a^2b^2+10a^4-48a^3b-15b^4+4ab^3$ by $4ab-5a^2+3b^2$.

Selecting $+10a^4$ and $-5a^2$, the terms which involve the letter a with the highest exponent, it follows, from the preceding discussion, that $10a^4$ is derived, without reduction, from the multiplication of $-5a^2$ by a term of the quotient.

Therefore $+10a^4\div(-5a^2)=-2a^2$ is a term of the quotient.

$$\begin{array}{lcl}
\text{The dividend is} & \cdot & 51a^2b^2+10a^4-48a^3b-15b^4+4ab^3. \\
(4ab-5a^2+3b^2)(-2a^2) & = & \underline{-8a^3b+10a^4-6a^2b^2\qquad\qquad} \\
\text{The first remainder} & = & \underline{57a^2b^2\quad *\quad -40a^3b-15b^4+4ab^3.}
\end{array}$$

Dividing $-40a^3b$, the term of the first remainder or first partial dividend which involves the highest power of a, by $-5a^2$, the result, $(-40a^3b)\div(-5a^2)=8ab$, is a term of the quotient.

$$\begin{array}{lcl}
\text{The first partial dividend is} & & 57a^2b^2-40a^3b-15b^4+4ab^3 \\
(4ab-5a^2+3b^2)8ab & = & \underline{32a^2b^2-40a^3b+24ab^3\qquad} \\
\text{The second remainder} & = & \underline{25a^2b^2\quad *\quad -15b^4-20ab^3.}
\end{array}$$

Dividing $25a^2b^2$ (that term of the 2d remainder or partial dividend which contains the highest power of a) by $-5a^2$, the result, which is $-5b^2$, is a term of the quotient.

$$\begin{array}{lcl}
\text{The second partial dividend is} & & 25a^2b^2-\ 15b^4-20ab^3 \\
(4ab-5a^2+3b^2)\,(-5b^2) & = & \underline{-20ab^3+25a^2b^2-\ 15b^4} \\
\text{The third remainder} & = & \underline{\quad *\qquad *\qquad *\quad}
\end{array}$$

The remainder being zero, $-2a^2+8ab-5b^2$, or $8ab-2a^2-5b^2$, is the quotient required.

The accuracy of this result may be verified by multiplying the divisor by it. The product ought to be equal to the dividend.

The division might have been performed by selecting the terms of the dividend and divisor which contain the highest powers of b, or the terms which involve the lowest powers of a or of b. Thus the terms which involve the highest powers of b are $-15b^4$ and $+3b^2$, the term of the quotient obtained from which is $-5b^2$, a term obtained by the preceding operation.

18. In each of the preceding partial divisions it is necessary to seek that term of the dividend which is affected with the highest exponent of one of the letters, and to divide it by that term of the divisor which is affected with the highest exponent of the same letter.

This trouble would be avoided if care were taken at first to arrange the terms of the dividend and the divisor in such a manner that the exponents of the same letter were highest in the terms on the left, and diminished term by term from left to right.

Arranged in this manner, the first term of the dividend and the first term of the divisor are always the two terms which it is necessary to divide one by the other, to obtain a term of the quotient. The same happens in all the partial divisions, since the partial quotients and the products of the divisor by these quotients are constantly arranged.

The dividend of the example given in this article, arranged according to the descending powers of a, is

$$10a^4-48a^3b+51a^2b^2+4ab^3-15b^4,$$

and the divisor, arranged in the same manner, is,

$$-5a^2+4ab-3b^2.$$

The quantities being arranged thus, the calculation may be made as follows:

$$\begin{array}{lll}
\text{Divisor.} & \text{Dividend.} & \text{Quotient.} \\
-5a^2+4ab+3b^2\,) & 10a^4-48a^3b+51a^2b^2+4ab^3-15b^4 & (\,-2a^2+8ab-5b^2 \\
 & \underline{10a^4-\ 8a^3b-\ 6a^2b^2} & \\
 & \quad -40a^3b+57a^2b^2+\ 4ab^3 & \\
 & \quad \underline{-40a^3b+32a^2b^2+24ab^3} & \\
 & \qquad\qquad 25a^2b^2-20ab^3-15b^4 & \\
 & \qquad\qquad \underline{25a^2b^2-20ab^3-15b^4} &
\end{array}$$

If the dividend and divisor are arranged according to the descending powers of b (or ascending powers of a) the same result is obtained, but in a different order.

$$\begin{array}{lll}
3b^2+4ab-5a^2\,) & -15b^4+\ 4ab^3+51a^2b^2-48a^3b+10a^4 & (\,-5b^2+8ab-2a^2 \\
 & \underline{-15b^4-20ab^3+25a^2b^2} & \\
 & \qquad 24ab^3+26a^2b^2-48a^3b & \\
 & \qquad \underline{24ab^3+32a^2b^2-40a^3b} & \\
 & \qquad\qquad -\ 6a^2b^2-\ 8a^3b+10a^4 & \\
 & \qquad\qquad \underline{-\ 6a^2b^2-\ 8a^3b+10a^4} &
\end{array}$$

After the subtraction of $(3b^2+4ab-5a^2)$ $(-5b^2)$ from the dividend, the remainder or partial dividend is

$$24ab^3+26a^2b^2-48a^2b+10a^4.$$

If the terms of the divisor and this partial dividend are arranged anew, according to the descending powers of a, and the calculation is completed, thus,

$$\begin{array}{lll}
-5a^2+4ab+3b^2\,) & 10a^4-48a^3b+26a^2b^2+24ab^3 & (-2a^2+8ab \\
 & \underline{10a^4-\ 8a^3b-\ 6a^2b^2} & \\
 & \quad -40a^3b+32a^2b^2+24ab^3 & \\
 & \quad \underline{-40a^3b+32a^2b^2+24ab^3} &
\end{array}$$

a partial quotient, $=-2a^2+8ab$, is obtained; therefore the complete quotient is $-5b^2-2a^2+8ab$, or $-2a^2+8ab-5b^2$, as before.

2d. Ex. Divide a^3-b^3 by $a-b$.

$$\begin{array}{lll}
a-b\,) & a^3-\ b^3 & (\,a^2+ab+b^2 \\
 & \underline{a^3-a^2b} & \\
 & \quad a^2b-b^3 & \\
 & \quad \underline{a^2b-ab^2} & \\
 & \qquad\quad ab^2-b^3 & \\
 & \qquad\quad \underline{ab^2-b^3} &
\end{array}
\qquad\qquad
\begin{array}{l}
a^2+ab+b^2 \\
\underline{a\ -\ b} \\
a^3+a^2b+ab^2 \\
\underline{\quad -a^2b-ab^2-b^3} \\
\underline{a^3\quad * \qquad * \ -b^3}
\end{array}$$

In this example the multiplication of the divisor by the different terms of the quotient gives rise to terms which are not contained in the dividend, and which terms it becomes necessary to divide by the first term of the divisor. Forming the product of the divisor and quotient, it is found that these terms are comprehended amongst the partial products arising from the multiplication of each term of the quotient (taken as multiplicand) by each term of the divisor (employed as multiplier), and that their disappearance from the complete product is caused by their being reduced with like quantities which are affected with contrary signs, as in the case of $+a^2b+(-a^2b)$, the algebraic sum of which is zero.

19. From the preceding discussion and examples it follows that if a polynomial dividend and divisor are arranged according to the powers of the same letter, and the first term of the dividend is divided by the first term of the divisor, the result (which is obtained by the rule for the division of monomials) is the first term of the quotient; that if the product of the divisor by this term is subtracted from the dividend, and the first term of the remainder is divided, as before, by the first term of the divisor, the result is the second term of the quotient, &c.

It also follows that the terms of the quotient can be obtained without any particular arrangement of the terms of the dividend and divisor, or by arranging the terms according to the highest or the lowest powers of the same letter; and that if, after arranging the terms, obtaining a term of the quotient, and subtracting the product of the divisor by this term from the dividend, the remainder or partial dividend and divisor are arranged according to the powers of some other letter, and the division is concluded, the result is the same as that which would be obtained without making any change in the arrangement of the divisor and partial dividend.

When the terms of the dividend and divisor have been arranged according to the powers of the same letter, this arrangement is observed throughout the calculation, because there is no reason for changing it, and because the first terms of the dividend and divisor after each operation are those which give the next term of the quotient; whereas, if the division were conducted with respect to another letter, it would be necessary to seek anew the terms affected with the highest or the lowest exponents of that letter.

The arrangement of terms being made according to the powers, either ascending or descending, of the same letter, if the first term of the whole dividend or of any partial dividend is not divisible by the first term of the divisor, the division is impossible, or, in other words, there is no polynomial whose product by the divisor is equal to the dividend. Hence, in any dividend and divisor, selecting the terms which involve the highest powers of any, the same, letter, if the term of the dividend is not exactly divisible by the term of the divisor the division is impossible.

Taking, for example, the polynomials $12a^3-5a^2b+7ab^2-11b^3$, and $4a^2-8ab+3b^2$;

On comparison of the terms in a the division appears possible; but taking $-11b^3$ and $+3b^2$, the terms which contain the highest powers of b, it appears that the division is impossible, for $-11b^3$ is not exactly divisible by $3b^2$.

A polynomial dividend cannot be divided by a divisor, polynomial or monomial, containing a letter which is not also contained in the dividend; for it is impossible that any quantity multiplied by one which depends on a particular letter should give a product which is independent of that letter.

The dividend being written as numerator and the divisor as denominator, a polynomial division which cannot be executed may be represented fractionally. This expression, as in the case of fractions whose terms are monomial, admits of simplification. To make the simplification it is requisite to divide the terms of the fraction by a common measure, or by their greatest common measure.

Rule for the division of polynomials:

Arrange the terms of the dividend and divisor according to the powers of the same letter.

Divide the first term of the dividend by the first term of the divisor; the result is the first term of the quotient.

Multiply the divisor by this term, and subtract the product from the quotient.

Divide the first term of the remainder by the first term of the divisor; the result is the second term of the quotient.

Multiply the divisor by the second term of the quotient, and subtract the product from the first remainder.

Repeat the process of dividing the first terms of the successive remainders by the first term of the divisor, and of subtracting the products of the divisor by the partial quotients from these remainders, until there is obtained a remainder equal to zero, or a remainder which cannot be divided by the divisor.

The algebraic sum of the partial quotients is the quotient required.

Examples of the division of polynomial algebraic quantities:

1st. $(bc^3-c^3x)\div(b-x)$?..................................*Ans.* c^3.

2d. $(a^3+a^2b-ab^2-b^3)\div(a-b)$?..............*Ans.* $a^2+2ab+b^2$.

3d. $(3a^5+16a^4b-33a^3b^2+14a^2b^3)\div(a^2+7ab)$?
Ans. $3a^3-5\,a^2b+2ab^2$.

4th. $(a^7-6a^6b^3+14a^5b^6-12a^4b^9)\div(a^3-2a^2b^3)$?
Ans. $a^4-4a^3b^3+6a^2b^6$.

5th. $(a^8-16z^8)\div(a^2-2z^2)$?.........*Ans.* $a^6+2a^4z^2+4a^2z^4+8z^6$.

6th. $(2a^4-13a^3b+31a^2b^2-38ab^3+24b^4)\div(2a^2-3ab+4b^2$?
Ans. $a^2-5ab+6b^2$.

7th. $(4c^4-9b^2c^2+6b^3c-b^4)\div(2c^2-3bc+b^2)$?
Ans. $2c^2+3bc-b^2$.

8th. $(-1+a^3n^3)\div(-1+an)$?..................*Ans.* $1+an+a^2n^2$.

9th. $(a^6+2a^3z^3+z^6)\div(a^2-az+z^2)$?......*Ans.* $a^4+a^3z+az^3+z^4$.

10th. $(\frac{1}{3}-6z^2+27z^4)\div(\frac{1}{3}+2z+3z^2)$?............*Ans.* $1-6z+9z^2$.

11th. $(a^6-16a^3x^3+64x^6)\div(a^2-4ax+4x^2)$?
Ans. $a^4+4a^3x+12a^2x^2+16ax^3+16x^4$.

12th. $(14a^5-27a^4b+21a^3b^2-3a^2b^3-2ab^4)\div(2a^2-3ab+2b^2)$?
Ans. $7a^3-3a^2b-ab^2$.

13th. $(51a^2b^2+10a^4-48a^3b-15b^4+4ab^3)\div(4ab-5a^2+3b^2)$?
Ans. $-2a^2+8ab-5b^2$.

14th. $(21x^3y^2+25x^2y^3+68xy^4-40y^5-56x^5-18x^4y)\div(5y^2-8x^2-6xy)$?......................*Ans.* $-8y^3+4xy^2-3x^2y+7x^3$.

15th. $(95a-73a^2+56a^4-25-59a^3)\div(-3a^2+5-11a+7a^3)$?
Ans. $8a-5$.

16th. $(a^{m+n}b^n-4a^{m+n-1}b^{2n}-27a^{m+n-2}b^{3n}+42a^{m+n-3}b^{4n})\div(a^n b^n-7a^{n-1}b^{2n})$?............*Ans.* $a^m+3a^{m-1}b^n-6a^{m-2}b^{2n}$.

20. When the terms of the dividend and divisor are arranged according to the powers of the same letter, it may happen that several terms are affected with the same power of that letter. In such a case it becomes necessary to arrange these terms according to the powers of some other letter.

Suppose, for example, that a polynomial contains the terms $abx^2+a^2x^2-b^2x^2$, all of which contain x^2, the coefficients of x^2 may be arranged according to the powers of a, and the terms in x^2 written either thus $(a^2+ab-b^2)\,x^2$,

or thus,
$$\begin{array}{r|l} a^2 & x^2 \\ +\,ab & \\ -\,b^2 & \end{array}$$

Assuming that the dividend and divisor are arranged according to the powers of the same letter, x for example, and that the literal coefficients of the terms in x are arranged according to the powers of some other letter, the division is made by the rule for the division of polynomials; for in each partial division the first term of the dividend ought to be the product of the first term of the divisor by that term of the quotient which is to be found by that partial division. All the terms of the quotient ought thus to be obtained, arranged respectively in the same manner as the dividend and divisor.

To prove this, let a polynomial dividend and divisor be arranged according to the descending powers of x. Let x^4 be the highest power of x in the dividend, and let the terms which contain x^4 be represented by A, those which contain x^3 by B, x^2 by C, x by D, and the terms into which x does not enter by E; that is, let the dividend be denoted by

$$Ax^4+Bx^3+Cx^2+Dx+E.$$

Also, let the highest power of x in the divisor be x^2, and let the divisor be expressed by

$$A'x^2+B'x+C'.$$

Then, since the highest exponent of x in the dividend and divisor, respectively, are 4 and 2, the highest in the quotient must be 2; and the quotient ought to have the form $A''x^2+B''x+C''$.

Again, the product $A'x^2\times A''x^2$ cannot be reduced with the other products of the divisor by the quotient (Art. 12′); it must therefore be equal to Ax^4. Whence, if Ax^4 is divided by $A'x^2$, the result must be the part $A''x^2$ of the quotient; and since $x^4\div x^2=x^2$, the division of Ax^4 by $A'x^2$ is reduced to the division of A by A′.

Whence, if A, A′ are themselves polynomials, composed of one or several letters, their quotient must be found by the rule for the division of polynomials, which requires their arrangement according to the powers of some letter contained in them.

The term $A''x^2$ being obtained, the whole divisor $A'x^2+B'x+C'$ is multiplied by it, and the product subtracted from the dividend $Ax^4+Bx^3+Cx^2+Dx+E$.

The remainder is a second partial dividend, upon which the same process is to be repeated, and so on.

Example. Given the dividend $(12b^2-29bc+15c^2)a^3+(23b^2-31b^2c-9bc^2+15c^3)a^2+(10b^4-6b^2c^2)a$, and the divisor $(3b-5c)a+2b^2$; required the quotient?

First partial division of the polynomial coefficient of the highest power of a in the dividend by the polynomial coefficient of the highest power of a in the divisor:

$$\begin{array}{l}
3b-5c)12b^2-29bc+15c^2(4b-3c\\
\qquad\quad\underline{12b^2-20bc\qquad\qquad}\\
\qquad\qquad\qquad -9bc+15c^2\\
\qquad\qquad\qquad \underline{-9bc+15c^2}
\end{array}$$

Since also $a^3\div a=a^2$, it follows that $(4b-3c)a^2$ is the first part of the quotient;

And since $(3b-5c)a\times(4b-3c)a^2=(12b^2-29bc+15c^2)a^3$, the term of the dividend containing a^3 is destroyed by the product of the first terms of the divisor and quotient.

Next, $2b^2\times(4b-3c)a^2=(8b^3-6b^2c)a^2$.

Subtracting this from the coefficient of a^2 in the dividend, $(15b^3-25b^2c-9bc^2+15c^3)a^2$ is the first term of the second partial dividend.

Second partial division of the polynomial coefficient of a^2 by the coefficient of the highest power of a in the divisor:

$$\begin{array}{l}
3b-5c)15b^3-25b^2c-9bc^2+15c^3(5b^2-3c^2\\
\qquad\quad\underline{15b^3-25b^2c\qquad\qquad\qquad\qquad}\\
\qquad\qquad\qquad\qquad -9bc^2+15c^3\\
\qquad\qquad\qquad\qquad \underline{-9bc^2+15c^3}
\end{array}$$

Therefore, since $a^2\div a=a$, $(5b^2-3c^2)a$ is the second part of the quotient.

Now $(3b-5c)a\times(5b^2-3c^2)a=(15b^2-25b^2c-9bc^2+15c^3)a^2$; therefore the term in a^2 of the second partial dividend is destroyed by the subtraction of the product of the first term of the divisor by the second term of the quotient.

The product of the second term of the divisor by the second term of the quotient, $2b^2\times(5b^2-3c^2)a=(10b^3-6b^2c^2)a$.

This product being equal to the term in a of the dividend, the second remainder is zero, and the exact quotient is $(4b-3c)a^2+(5b^2-3c^2)a$.

Calculation described in the preceding detail:

Divisor. Dividend. Quotient.

$$\begin{array}{r|l} 3b & a+2b^2 \\ -5c & \end{array}\Bigg) \begin{array}{r|l|l|l} 12b^2 & a^3+23b^3 & a^2+10b^4 & a \\ -29bc & -31b^2c & -6b^2c^2 & \\ +15c^2 & -9bc^2 & & \\ & +15c^3 & & \end{array} \Bigg(\begin{array}{r|l|l} 4b & a^2+5b^2 & a \\ -3c & -3c^2 & \end{array}$$

$$\begin{array}{r|l|l|l} 12b^2 & a^3+8b^3 & a^2 & \\ -29bc & -6b^2c & & \\ +15c^2 & & & \\ \hline * & +15b^3 & a^2+10b^4 & a \\ & -25b^2c & -6b^2c^2 & \\ & -9bc^2 & & \\ & +15c^3 & & \\ & 15b^3 & a^2+10b^4 & a \\ & -25b^2c & -6b^2c^2 & \\ & -9bc^2 & & \\ & +15c^3 & & \\ \hline & * & * & \end{array}$$

Example 2.

Divisor. Dividend. Quotient.

$$\begin{array}{r|l} 3b & a+b^2-2b \\ -5 & \end{array}\Bigg) \begin{array}{r|l|l|l|l} 6b & a^4-7b^2 & a^3-3b^3 & a^2+4b^3 & a+b^2-2b \\ -10 & +23b & +22b^2 & -9b^2 & \\ & -2 & -31b & +5b & \\ & & +5 & -5 & \end{array}\Bigg(\begin{array}{r|l|l} 2a^3-3b & a^2+4b & a+1 \\ +4 & -1 & \end{array}$$

Example 3.

$$\begin{array}{r|l} a^2 & x^2+a^3x-a^2b^2 \\ -ab & \end{array}\Bigg) \begin{array}{r|l|l|l|l} a^4 & x^4+a^5 & x^3-a^5b & x^2+a^4b^3 & x-a^2b^6 \\ -a^3b & -a^4b & -2a^4b^2 & +2a^3b^4 & \\ +a^2b^2 & +a^3b^2 & -ab^5 & & \\ -ab^3 & +a^2b^3 & & & \end{array}\Bigg(\begin{array}{r|l|l} a^2 & x^2-a^2b & x+b^4 \\ +b^2 & -ab^2 & \end{array}$$

21. If a polynomial dividend contain different powers of a letter which is not contained in the divisor, the dividend and divisor may be arranged according to the powers of some other letter which is common to both, and the division made by the methods already explained.

Otherwise, suppose that the dividend contains different powers of x, and that this letter does not enter into the divisor.

If the dividend is arranged according to the powers of x (the highest being x^3) it may be expressed by the polynomial Ax^3+Bx^2+Cx+D, in which A, B, C, D, are quantities mónomial or polynomial, which do not involve x. Let the polynomial divisor, which is also independent of x, be represented by P. Then the divisor multiplied by the quotient must reproduce the dividend; and since the divisor does not contain x, the quotient must be a polynomial affected with the same powers of x as are found in the dividend. The quotient is therefore of the form

$$A'x^3+B'x^2+C'x+D.$$

Suppose this quotient determined, and let each term of it be multiplied by the divisor; the products are

$$A'Px^3+B'Px^2+C'Px+D'P.$$

These products, involving different powers of x, cannot be reduced with each other; whence they are equal, respectively, to the terms Ax^3+Bx^2+Cx+D of the dividend;

Therefore $A'P=A$ and $A'=A+P$;
$B'P=B$ and $B'=B+P$;
$C'P=C$ and $C'=C+P$.

Whence, in order that a polynomial arranged with respect to a letter may be exactly divisible by a polynomial independent of that letter, it is necessary that each of the coefficients of the different powers of the first polynomial be exactly divisible by the second polynomial.

The coefficients of the different powers of the letter in the quotient are the successive quotients obtained by dividing the coefficients of the polynomial dividend by the polynomial divisor.

Ex. Divide $3a^2b^3-3abc^3-2b^3c^2+b^5-3a^2bc^2+3ab^3c-a^2c^3+bc^4+a^2b^2c$ by b^2-c^2?

The dividend (arranged according to the powers of a, and the coefficients according to the powers of b) is

$$(3b^3+b^2c-3bc^2-c^3)a^2+(3b^3c-3bc^3)a+b^5-2b^3c^2+bc^4;$$

$$\text{Now } \frac{3b^3+b^2c-3bc^2-c^3}{b^2-c^2}=3b+c;$$

$$\frac{3b^3c-3bc^3}{b^2-c^2}=3bc;$$

$$\frac{b^5-2b^3c+bc^4}{b^2-c^2}=b^3-bc^2;$$

consequently the quotient is $(3b+c)a^2+3bca+b^3-bc^2$.

22. If x^2-a^2 is divided by $x-a$, the quotient is $x+a$.
x^3-a^3 - $x-a$, - x^2+ax+a^2.
x^4-a^4 - $x-a$, - $x^3+ax^2+a^2x+a^3$.
x^5-a^5 - $x-a$, - $x^4+ax^3+a^2x^2+a^3x+a^4$.

In these quotients it is to be observed,

1st. That all the terms are additive.

2d. That the first and last terms are formed by subtracting 1 from the exponents of x and a in the dividend.

3d. That, of the intermediate terms beginning with the second, the exponents of x are successively diminished by 1, and the exponents of a augmented by 1, in such a manner that the sum of the exponents of x and a in each term is constant.

Denoting by m any additive whole number, it may be inferred by analogy that $\dfrac{x^m-a^m}{x-a}=x^{m-1}+x^{m-2}a+x^{m-3}a^2+\ldots\ldots+xa^{m-2}+a^{m-1}$.

In effect, any number of the terms of this quotient can be found by division; but, without resorting to this kind of proof, the accuracy of the result can be verified by multiplication; for

$$(x^{m-1}+x^{m-2}a+x^{m-3}a^2+\ldots\ldots+xa^{m-2}+a^{m-1})\times x=x^m+x^{m-1}a+x^{m-2}a^2+\ldots\ldots$$
$$+x^2a^{m-2}+xa^{m-1}.$$

$$(x^{m-1}+x^{m-2}a+x^{m-3}a^2+\ldots\ldots+xa^{m-2}+a^{m-1})\times(-a)=-x^{m-1}a-x^{m-2}a^2-\ldots\ldots$$
$$-x^2a^{m-2}-xa^{m-1}-a^m.$$

Now, it is evident that the second term of the first partial product, or first line, is equal and of a contrary sign to the first term of the second partial product or second line, and that the sum of these two terms is zero.

In like manner the third term of the upper line is destroyed by the second term of the lower line, the fourth by the third, and so on to $+xa^{m-1}$, the last term of the upper line, which is destroyed by $-xa^{m-1}$, the last term but one of the lower line.

Whence, if the quotient of x^m-a^m is multiplied by the divisor $x-a$, the dividend x^m-a^m is reproduced. Consequently

$$\frac{x^m-a^m}{x-a}=x^{m-1}+ax^{m-2}+a^2x^{m-3}+\ldots.+a^{m-2}x+a^{m-1}.$$

23. Let it be required to divide $x^m+px^{m-1}+qx^{m-2}+\dots+tx+u$ by $x-a$?

Calculation:

$$x-a)\,x^m+px^{m-1}+qx^{m-2}+\dots+tx+u\;\Big(x^{m-1}+\left.\begin{matrix}a\\+p\end{matrix}\right|x^{m-2}+\left.\begin{matrix}a^2\\+ap\\+q\end{matrix}\right|x^{m-3}+\dots+\begin{matrix}a^{m-1}\\+pa^{m-2}\\+qa^{m-3}\\\cdots\\\cdots\\+t\end{matrix}$$

$$\underline{x^m-ax^{m-1}\qquad\qquad}$$

$$*\quad\left.\begin{matrix}a\\+p\end{matrix}\right|x^{m-1}+qx^{m-2}$$

$$\underline{\left.\begin{matrix}a\\+p\end{matrix}\right|x^{m-1}\left.\begin{matrix}-a^2\\-ap\end{matrix}\right|x^{m-2}\qquad\qquad}$$

$$*\qquad\left.\begin{matrix}a^2\\+ap\\+q\end{matrix}\right|x^{m-2}+rx^{m-3}$$

$$\underline{\left.\begin{matrix}a^2\\+ap\\+q\end{matrix}\right|x^{m-2}\left.\begin{matrix}-a^3\\-a^2p\\-aq\end{matrix}\right|x^{m-3}\qquad\qquad}$$

$$*\qquad\left.\begin{matrix}a^3\\+a^2p\\+aq\\+r\end{matrix}\right|x^{m-3}+sx^{m-4}$$

$$\cdot\quad\cdot\;\cdot\;\cdot\;\cdot\;\cdot$$

$$\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot$$

Detail and conclusion:

Dividing x^m by x, the first term of the quotient is x^{m-1}.

The product of the divisor by this quotient is x^m-ax^{m-1}; subtracting this product from the dividend, that term of the remainder which is affected with the highest power of x is $(a+p)x^{m-1}$.

Dividing $(a+p)x^{m-1}$ by x, the quotient is $(a+p)x^{m-2}$; and that term of the remainder which is affected with the highest power of x is $(a^2+ap+q)x^{m-2}$. Dividing this term by x, the third term of the quotient is $(a^2+ap+q)x^{m-3}$.

Proceeding in this manner, and regarding always as a single term all the quantities which involve the same power of x, it becomes evident that each term of the quotient is obtained from the preceding term by multiplying it by a, adding to the product the term of the dividend which involves the same power of x, and then dividing by x.

Hence it follows that the part of the quotient which is independent of x is

$$a^{m-1}+pa^{m-2}+qa^{m-3}+\dots\dots+t.$$

If the divisor is multiplied by this quantity, its product by x will destroy the part in x of the last partial dividend, and the remainder will be

$$a^m+pa^{m-1}+qa^{m-2}+\dots+ta+u.$$

Here the division ends, and the remainder is the original dividend, except that x is replaced by a.

When this remainder is zero the division is exact. Whence, if a is a quantity such that, being substituted for x, it renders the polynomial,

$$x^m+px^{m-1}+qx^{m-2}+\dots.tx+u=0,$$

the polynomial is divisible by $x-a$.

SECTION III.

SIMPLIFICATION OF FRACTIONAL EXPRESSIONS. GREATEST COMMON MEASURE. REDUCTION OF FRACTIONAL EXPRESSIONS TO THE SAME DENOMINATOR. ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION OF FRACTIONAL EXPRESSIONS.

24. When the division of one algebraic quantity by another cannot be executed, it is indicated by making the dividend the numerator and the divisor the denominator of an algebraic fraction.

If particular values, expressed by whole numbers, are given to all the letters composing an algebraic fraction, the numerator and denominator (whether monomial or polynomial) can be each reduced to a single number; then, according as the numerical value of the numerator is or is not a multiple of that of the denominator, the value of the algebraic fraction is expressed by a whole number or an arithmetical fraction.

If the particular values of some of the letters or all the letters which compose an algebraic fraction are expressed by arithmetical fractions, either or both of the terms of the numerical value of that algebraic fraction may be arithmetical fractions. In such cases, since the numerator is a dividend and the denominator a divisor, the reduction of the numerical value of an algebraic fraction to the form of a common arithmetical fraction can be made by the rule for the division of fractions.

Examples:

1st. In the fractional expression $\frac{ab}{cd}$ let $a=6$, $b=4$, $c=2$, $d=3$;

$$\text{then } \frac{ab}{cd}=\frac{6\times 4}{2\times 3}=\frac{24}{6}=4.$$

2d. In the fraction $\frac{m^2np}{rs^2t}$ let $m=3$, $n=2$, $p=5$, $r=7$, $s=4$, $t=6$;

$$\text{then } \frac{m^2np}{rs^2t}=\frac{3^2\times 2\times 5}{7\times 4^2\times 6}=\frac{3\times 5}{7\times 4\times 4}=\frac{15}{112}.$$

3d. In the fraction $\frac{a^3b-a^2c+d}{b^2+b-ac^2}$ let $a=2$, $b=5$, $c=3$, $d=8$;

$$\text{then } \frac{a^3b-a^2c+d}{b^2+b-ac^2}=\frac{2^3\times 5-2^2\times 3+8}{5^2+5-2\times 3^2}=\frac{40-12+8}{25+5-18}=\frac{36}{12}=3.$$

4th. In the fraction $\frac{abd}{cfg}$ let $a=\frac{3}{5}$, $b=\frac{2}{3}$, $d=4$, $c=\frac{1}{6}$, $f=\frac{1}{3}$, $g=2\frac{1}{4}$;

then $\frac{abd}{cfg}=\frac{\frac{3}{5}\times\frac{2}{3}\times 4}{\frac{5}{6}\times\frac{1}{3}\times 2\frac{1}{4}}=\frac{\frac{8}{5}}{\frac{5}{8}}$; but $\frac{8}{5}\div\frac{5}{8}=\frac{8}{5}\times\frac{8}{5}=\frac{64}{25}$, and $\frac{64}{25}=2\frac{14}{25}$ $\therefore \frac{abd}{efg}=2\frac{14}{25}$.

25. The terms of an arithmetical fraction being whole numbers, it is proved (Part I. Art. 149) that the value of a fraction is not changed by the multiplication or the division of its terms by the same number, and from this principle are derived processes for the simplification of fractions, and for their reduction to the same denominator.

This general principle and its consequences may be employed in any calculation made with algebraic fractions whose terms are expressed by whole numbers; but as each of the terms of an algebraic fraction may have for numerical value an arithmetical fraction, it becomes necessary to prove generally, that if the terms of an algebraic fraction are both multiplied or both divided by the same quantity, the value of the fraction remains constant.

Let, therefore, n and d be any quantities, whole or fractional, and q their quotient, so that

$$\frac{n}{d}=q, \text{ or } n=dq,$$

and let m be any quantity whatever; it is to be proved that $\frac{n}{d}=\frac{n\cdot m}{d\cdot m}$, and that $\frac{n\cdot m}{d\cdot m}=\frac{n}{d}$.

$$\text{Since } n=dq,\ nm=dqm \text{ or } q\times dm,$$

$$\therefore \frac{nm}{dm}=\frac{q\cdot dm}{dm}=q;$$

$$\text{but } q=\frac{n}{d} \therefore \frac{n}{d}=\frac{nm}{dm}, \text{ and } \frac{nm}{dm}=\frac{n}{d}.$$

Therefore the proposition is established.

26. An algebraic fraction, considered in the most general manner, is the expression of an unexecuted division. If the numerator is a multiple of the denominator, the quotient is exact; if the numerator is equal to the denominator, the quotient is 1; if the numerator is not a multiple of the denominator, or if it is less than the denominator, the result must retain the fractional form, but it frequently admits of simplification.

If it is required to divide $12a^4b^2cd$ by $8a^2bc^2$, the quotient cannot be expressed by an integer monomial, for 12 is not divisible by 8, and the exponent of c is less in the dividend than in the divisor.

The quotient is therefore indicated by the expression

$$\frac{12a^4b^2cd}{8a^2bc^2}.$$

But this expression may be simplified, or reduced to lower terms; for the factor 4 is common to the coefficients 12 and 8, the factor b is common to b^2 and b, and the factor c to c and c^2.

These common factors being suppressed, the result is

$$\frac{3a^2bd}{2c}.$$

This simplification is effected by suppressing in the numerator and denominator all the factors which are common to them both, or by dividing the terms of the fraction by a divisor composed of the product of all the factors which are common to both terms of the fraction. This divisor, by Article 139, Part I, is the greatest common measure of the terms of the fraction; and, by Article 25, the division of the terms of the fraction by this quantity does not change the value of the fraction.

Since division of the numerator and denominator by their greatest common measure takes away all their common factors, and a fraction whose numerator and denominator contain no common factors is said to be in the lowest terms, it follows that a monomial fraction is reduced to the lowest terms by dividing its terms by their greatest common measure.

27. When the numerator and denominator of an algebraic fraction are polynomials, it is evident that monomial factors which are common to all the terms of the numerator and denominator, and polynomial factors which are common to both the numerator and denominator, may be suppressed; for the suppression of these factors amounts to the division of both the terms of the fraction by the product of the common factors; an operation by which the value of the fraction is not changed (Art. 25).

It is also evident that if all the common factors of two quantities are suppressed, these quantities cannot both be exactly divided by any integer divisor greater than 1.

The product of all the factors, numerical, monomial, and polynomial, which are common to two or more algebraic expressions of the integer form, is termed the greatest common measure of these expressions.

Whence an algebraic fraction is reduced to the lowest terms by dividing its numerator and denominator by their greatest common measure.

28. When the integer algebraic quantities are monomial the greatest common measure of this definition is that of Part I. Articles 138 and 139; it is obtained by forming the product of all the simple factors, numerical and literal, which are common to the quantities.

The greatest common measure of polynomial expressions is not obtained with the same facility; its determination depends on these principles:

First, the greatest divisor common to two integer expressions is not changed by the multiplication or by the division of either of these expressions by any integer quantity which contains no factor common to itself and to the other expression.

For it is evident that thus the prime factors which are common to the two expressions remain unchanged; and it is the product of these common factors which is the greatest common measure of the two expressions.

Second, the greatest common measure of two polynomials is the same as the greatest common measure of the less of the two polynomials and the remainder resulting from the division of the greater by the less.

Let A, B denote the two polynomials, A being the greater; also let A be divided by B, and let the integer quotient be Q, and the remainder R; it is to be proved that D, the greatest common measure of A, B, is also the greatest common measure of B and R.

$$\text{Since } \frac{A}{B}=Q+\frac{R}{B},\ A=BQ+R.$$

$$\text{Let } A\div D=A',\ B\div D=B',\ R\div D=R',$$

$$\text{then } \frac{A}{D}=\frac{B}{D}Q+\frac{R}{D},\ \text{or } A'=B'Q+R'.$$

In the expression $A'=B'Q+R'$, B′ and R′ have no common factor; for if they have, it must measure $B'Q+R'$, and by consequence A′ (Part I. Art. 105); therefore A′, B′ would still contain a common factor, and D, which is the greatest common measure of A, B, would not contain all the common factors of these quantities, which is contrary to the definition of the greatest common measure.

Since B′ and R′, which are the quotients of B and R by D, cannot have a common factor, it follows that the greatest common measure of B and R is equal to D; therefore it is the same as that of the quantities A and B.

29. As an application of these principles, let it be required to find the greatest common measure of the quantities

$$X=48a^2b^3x^6-120a^3b^3x^5+12a^4b^3x^4-12a^6b^3,\ \text{and}$$

$$Y=48a^3bx^3-88a^4bx^6-64a^5bx^5-8a^6bx^4.$$

The factor $12a^2b^3x^2$ is common to all the terms of X, and the factor $8a^3bx^4$ to all the terms of Y.

Now, the greatest common measure of the monomials $12a^2b^3x^2$ and $8a^3bx^4$ is $4a^2bx^2$ (Art. 26), therefore $4a^2bx^2$ is the product of all the monomial factors common to X, Y.

Dividing X by $12a^2b^3x^2$, and denoting the result by A,

$$A=4x^4-12ax^3+a^2x^2-a^4.$$

Also, dividing Y by $8a^3bx^4$, and denoting the result by B,

$$B=6x^3-11ax^2-8a^2x-a^3.$$

The expressions A, B contain only the polynomial factors common to X, Y; consequently, the greatest common measure of A, B being found, it will be sufficient to multiply this by $4a^2bx^2$ to obtain the greatest common measure of X, Y.

The question is now reduced to this: to find the greatest common measure of two polynomials, A, B, which contain no common monomial factors.

By the definition (Art. 27) the greatest common measure is the product of all the factors which are common to A, B.

Therefore, if the quantity B exactly divides A, B is the greatest common measure of the polynomials A, B.

In attempting the division of A by B, it is found that $4x^4$, the first term of A, is not exactly divisible by $6x^3$, the first term of B. To render the division possible A may be multiplied by 6, the coefficient of the first term of B. The greatest common measure of A and B cannot be affected by this multiplication, since 6 contains no factor which is common to the terms of B, this quantity in effect containing no common monomial factor. It is, however, sufficient to multiply A by 3, for thus the first term of $A\times 3$ is $12x^4$, a quantity divisible by $6x^3$.

$A\times 3$ or $12x^4-30ax^3+3a^2x^2-3x^4$ is to be divided by B.

The quotient found by dividing $12x^4$ by $6x^3$ is $2x$.

Multiplying B by $2x$, and subtracting the product from $A\times 3$, the remainder is $-8ax^3+19a^2x^2+2a^3x-3a^4$, and this ought to contain all the factors of B which are found in A.

Multiplying again by 3, to render the partial dividend capable of division by B, the quotient is $-4a$, and the remainder $13a^2x^2-26a^3x-13a^4$, which must still contain all the factors common to A, B.

The exponent of x being less in this remainder than in B, B must be taken for dividend and R for divisor.

Before attempting the division of B by R, it is necessary to suppress the factors which are common to all the terms of R; which, since A, B contain no common monomial factors, cannot affect the greatest common measure of these quantities and R. Suppressing the common factor $13a^2$ in all the terms of the remainder, R becomes $x^2-2ax-a^2$.

The division of B by R, thus reduced, is made, without the necessity of introducing any new factor into the dividend B; and since the remainder is zero the conclusion is, that $x^2-2ax-a^2$ is the greatest common measure of A and B.

Calculation :

$$
\begin{array}{rl}
 & 4x^4-10ax^3+a^2x^2-a^4 \\
 & 3 \\ \cline{2-2}
6x^3-11ax^2-8a^2x-a^3) & 12x^4-30ax^3+3a^2x^2-3a^4(2x \\
 & 12x^4-22ax^3-16a^2x^2-2a^3x \\ \cline{2-2}
 & -8ax^3+19a^2x^2+2a^3x-3a^4 \\
 & 3 \\ \cline{2-2}
6x^3-11ax^2-8a^2x-a^3) & -24ax^3+57a^2x^2+6a^3x-9a^4(-4a \\
 & -24ax^3+44a^2x^2+32a^3x+4a^4 \\ \cline{2-2}
 & 13a^2x^2-26a^3x-13a^4 \\ \cline{2-2}
\multicolumn{2}{r}{13a^2x^2-26a^3x-13a^4=13a^2(x^2-2ax-a^2)=13a^2\times R} \\
x^2-2ax-a^2) & 6x^3-11ax^2-8a^2x-a^3(6x-a \\
 & 6x^3-12ax^2-6a^2x \\ \cline{2-2}
 & ax^2-2a^2x-a^3 \\
 & a^2x-2a^2x-a^3 \\ \cline{2-2}
\end{array}
$$

The preceding detail and example show in what manner the greatest common measure of two polynomials is obtained by successive divisions. Two divisions have sufficed in the given example; but if the second division had been inexact it must have led to another remainder of a degree inferior to the divisor, and there must have been a third division following the second, in the same manner as the second is made to follow from the first. If the third division also is inexact, a fourth division is made to follow the third; if the fourth is inexact, a fifth is made to follow the fourth, &c.

Proceeding in this manner, if a remainder equal to zero is found, the divisor of the last operation is the greatest common measure required; for let $R, R', R'', \ldots R^{N-\prime}, R^N$ be the successive remainders, and let R^N be that remainder or divisor from the division of $R^{N-\prime}$ by which a remainder equal to zero is obtained. Then R^N divides itself and $R^{N-\prime}$; and since R^N cannot be divided by any divisor greater than itself, it is the greatest common measure of R^N and $R^{N-\prime}$.

But by the second principle (Art. 28) the greatest common measure of R^N and $R^{N-\prime}$ is also the greatest common measure of $R^{N-\prime}$ and $R^{N-\prime\prime}$ . . . . of R' and R, of R and B, of B and A.

To find the greatest common measure of two integer polynomials,

Rule. Arrange the terms of both polynomials according to the powers of the same letter:

Suppress in both quantities the monomial factors which are common to all the terms of each polynomial; and if the monomial factor which is common to all the terms of the first polynomial, and that which is common to all the terms of the second, has a greatest common measure, reserve it as a factor of all the terms of the greatest common measure of the two quantities.

Make the polynomial, which contains the highest degree of the letter according to the powers of which the quantities are arranged, dividend, and the other divisor. Multiply, if necessary, the dividend by that number or quantity which renders the first term of the dividend divisible by the first term of the divisor.

Then divide the dividend by the divisor, and continue the division until a remainder $=0$ or of a degree inferior to the divisor is obtained, (multiplying at the same time the partial dividends, if necessary, as for the first division.)

Suppress monomial factors which are common to all the terms of the remainder.

Make the remainder (thus prepared) divisor, the divisor dividend, and proceed as with the first division. Continue in this manner to divide the last divisor by the last remainder, until a remainder equal to zero, or a remainder independent of the principal letter, is obtained. In the first case the last divisor is the greatest common measure required; in the second the quantities have no common measure, unless the monomial factors suppressed before the first division have a common measure. If they have, this, in the first case, is a factor of every term of the greatest common measure; in the second it is itself the greatest common measure of the proposed quantities.

Examples of the reduction of polynomial fractional expressions to the lowest terms by division of the terms by their greatest common measure:

1st. $\frac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$. The greatest common measure of the terms is a^2-x^2, and the fraction in the lowest terms is $\frac{a^2+x^2}{a-x}$.

2d. $\frac{9a^4b-9a^3b^2+3a^2b^3-3ab^4}{8a^2b^2-10ab^3+2b^4}$. The greatest common measure is $ab-b^2$, and the reduced fraction is $\frac{9a^3+3ab^2}{8ab-2b^2}$.

3d. $\frac{6a^3-17a^2b+22ab^2-15b^3}{6a^2-17ab+12ab^2}$. The greatest common measure is $2a-3b$, and the reduced fraction is $\frac{3a^2-4ab+5b^2}{3a-4b}$.

4th. $\frac{4a^2-12ab+9b^2}{4a^2-9b^2}$. The greatest common measure is $2a-3b$, and the reduced fraction is $\frac{2a-3b}{2a+3b}$.

5th. $\frac{12a^3b^2+38a^2b^3+16ab^4-10b^5}{15a^5+10a^4b+4a^3b^2+6a^2b^3-3ab^4}$. The greatest common measure is $3a^2+2ab-b^2$, and the reduced fraction is $\frac{4ab^2+10b^3}{5a^3+3ab^2}$.

REDUCTION OF FRACTIONAL EXPRESSIONS TO THE SAME DENOMINATOR.

30. If the letters which compose the terms of algebraic fractions express whole numbers, the numerator and denominator can be each

reduced to a single whole number. Consequently, if unity is considered to be divided into as many equal parts as the numerical value of the denominator contains simple units, and the numerator to express as many of these parts as its numerical value contains simple units, it is evident that the elementary operations which it may be necessary to perform with literal fractions may be performed by the rules for arithmetical fractions, attention being given to the conventions established in Articles 3 and 4. However, as a letter may express an arithmetical fraction, the numerical values of the terms of an algebraic fraction may, either or both, be arithmetical fractions. Since the rules for performing the elementary operations of arithmetic with fractional expressions are demonstrated for those fractions only which have whole numbers for their terms, it seems necessary to investigate the rules anew, regard being had to this condition.

It has been proved (Art. 25) that if the terms of any fraction are both multiplied or both divided by the same number, the value of the fraction is not changed; and this principle has been applied to the reduction of fractions to the lowest terms (Art. 26). It supplies the means also of reducing fractions which have different denominators to equivalent fractions having a common denominator; for it is evident that the product of all the denominators of any given fractions is a multiple of all the denominators of these fractions;

That this multiple can be divided by each denominator;

That if it is divided by the denominator of the first fraction, and the terms of this fraction are multiplied by the quotient, the result is an equivalent fraction having the product of all the denominators for its denominator;

That if the multiple of all the denominators is divided by the denominator of the second fraction, and the terms of this fraction are multiplied by the quotient, the result is a fraction equivalent to the second fraction, and having the product of all the denominators for its denominator;

And that in this manner any number of fractional expressions may be reduced to the same denominator.

Whence, to reduce fractions which have different denominators to equivalent fractions having the same denominator,

Rule. Multiply the terms of each fraction by the denominators of all the other fractions; the results are equivalent fractions reduced to the same denominator.

Examples of the reduction of fractions having different denominators to equivalent fractions having a common denominator:

1st. $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$?.................................*Ans.* $\frac{adf}{bdf}, \frac{cbf}{bdf}, \frac{ebd}{bdf}$.

2d. $\frac{a-b}{a+b}, \frac{2a-3c}{a-b}$?...*Ans.* $\frac{a^2-2ab+b^2}{a^2-b^2}, \frac{2a^2-3ac+2ab-3bc}{a^2-b^2}$.

3d. $\frac{a}{b-c}, \frac{a-b}{b^2}$?....................*Ans.* $\frac{ab^2}{b^3-b^2c}, \frac{ab-ac-b^2+bc}{b^3-b^2c}$.

4th. $\frac{1+x^2}{1-x^2}, \frac{1-x^2}{1+x^2}$?..................*Ans.* $\frac{1+2x^2+x^4}{1-x^4}, \frac{1-2x^2+x^4}{1-x^4}$.

5th. $\frac{ax+1}{m^2n-2}, \frac{bx^2}{mn}$? ....*Ans.* $\frac{amnx+mn}{m^3n-2mn}, \frac{bm^2nx^2-2bx^2}{m^3n^2-2mn}$.

6th. $\frac{a}{a+b}, \frac{b}{a-b}, \frac{1}{d}$?......*Ans.* $\frac{a^2d-abd}{a^2d-b^2d}, \frac{abd+b^2d}{a^2d-b^2d}, \frac{a^2-b^2}{a^2d-b^2d}$.

a. If the factors (monomial and polynomial) which are common to two or more of the denominators, and all the factors which are found in one only of the denominators, are each taken once only and multiplied together, the product is a multiple of all the denominators, and less (unless all the denominators are prime to each other) than the product of all the denominators. The monomial factors are easily found by inspection of the quantities; the polynomial can be obtained by seeking the greatest common measure of all the denominators, taken two and two. But this is a very laborious process if the number of fractions is considerable.

Examples:

1st. $\frac{3a^4}{40b^2c}, \frac{7b^6}{18a^2c^3}, \frac{11c^5}{45a^2b^2}$?..*Ans.* $\frac{27a^6c^2}{360a^2b^2c^3}, \frac{140b^8}{360a^2b^2c^3}, \frac{88c^8}{360a^2b^2c^3}$.

2d. $\frac{a^2b^2}{a+x}, \frac{a^3bc}{a-x}, \frac{ab^2c^2}{a^2-x^2}$?...*Ans.* $\frac{a^3b^2-a^2b^2x}{x^2-x^2}, \frac{a^4bc+a^3bcx}{a^2-x^2}, \frac{ab^2c^2}{a^2-x^2}$.

3d. $\frac{a^2}{by^2}, \frac{b^2x}{y}, \frac{4c}{b}, \frac{by}{b^2y^3}$?......................*Ans.* $\frac{a^2}{by^2}, \frac{b^3xy}{by^2}, \frac{4cy^2}{by^2}, \frac{1}{by^2}$.

4th. $\frac{a}{a+b}, \frac{b}{a^2-ab+b^2}, \frac{c}{a^3+b^3}$?*Ans.* $\frac{a^3-a^2b+ab^2}{a^3+b^3}, \frac{ab+b^2}{a^3+b^3}, \frac{c}{a^3+b^3}$.

ADDITION OF FRACTIONAL EXPRESSIONS.

31. It is a self-evident principle that if two quantities are equal, and both are operated on in the same manner, the results are equal.

Let, therefore, $a=bv$, $a'=bv'$, (a, b, v, a', b', v', being any quantities, whole or fractional, monomial or polynomial,) and let both members of these equations be divided by b, which has no factor common to it and a or a'.

Then $\frac{a}{b}=v$; $\frac{a'}{b}=v'$. Whence v, v' represent the values of the fractions $\frac{a}{b}$, $\frac{a'}{b}$, or the quotients resulting from the division of a, a' by b.

$$\text{Since } a=bv, \text{ and } a'=bv',$$
$$a+a=b\cdot v+b\cdot v'=b(v+v');$$
$$\therefore \frac{a+a'}{b}=v+v'.$$
$$\text{But } \frac{a}{b}=v;\ \frac{a'}{b}=v';\ \text{whence also } \frac{a}{b}+\frac{a'}{b}=v+v';$$
$$\therefore \frac{a}{b}+\frac{a'}{b}=\frac{a+a'}{b}.$$

Whence, to obtain the sum of two fractions which have the same denominator,

Rule. Form the algebraic sum of the numerators, and divide it by the common denominator.

Examples of the addition of algebraic fractions having the same denominator:

1st. $\frac{ad}{bd}+\frac{bc}{bd}$?..*Ans.* $\frac{ad+bc}{bd}$.

2d. $\frac{a-b}{a^2-b^2}+\frac{a+b}{a^2-b^2}$?....................*Ans.* $\frac{a+b+a-b}{a^2-b^2}=\frac{2a}{a^2-b^2}$.

3d. $\frac{a}{b}+\frac{c}{b}+\frac{d}{b}+\frac{e}{b}$?..................................*Ans.* $\frac{a+c+d+e}{b}$.

4th. $\frac{a^2-ab}{2b^2c}+\frac{3ab}{2b^2c}+\frac{b}{2b^2c}$?..........................*Ans.* $\frac{a^2+2ab+b^2}{2b^2c}$.

Suppose that the fractions have different denominators, and that

$$\frac{a}{b}=v,\ \frac{a'}{b'}=v',$$

or that $a=bv,\ a'=b'v'$.
Since $a=bv,\ ab'=bb'\cdot v$,
and since $a'=b'v',\ a'b=bb'\cdot v'$,

$$\therefore ab'+a'b=bb'\cdot v+bb'\cdot v'=bb'(v+v');$$

$$\therefore \frac{ab'+a'b}{bb'}=v+v'.$$

But $\frac{a}{b}=v$, $\frac{a'}{b'}=v'$; wherefore $\frac{a}{b}+\frac{a'}{b'}=v+v'$, and consequently $\frac{a}{b}+\frac{a'}{b'}=\frac{a'b+a'b}{bb'}$.

Whence, to find the sum of two fractions which have different denominators,

Rule. Reduce the fractions to a common denominator; form the algebraic sum of the numerators, and divide it by the common denominator.

Examples of the addition of algebraic fractions which have different denominators:

1st. $\frac{a}{m}+\frac{b}{n}$?.. *Ans.* $\frac{an+bm}{mn}$.

2d. $\frac{a}{b}+\frac{c}{d}+\frac{e}{f}$?.................................. *Ans.* $\frac{adf+bcf+bde}{bdf}$.

3d. $\frac{3a}{5b}+\frac{c}{4d}+h$?.............................. *Ans.* $\frac{12ad+5bc+20bdh}{20bd}$.

4th. $\frac{a+b}{a-b}+\frac{a-b}{a+b}+2$?.................................. *Ans.* $\frac{4a^2}{a^2-b^2}$.

5th. $\frac{a}{b^2c}+\frac{d}{bf}+\frac{e}{cg}$?........................... *Ans.* $\frac{afg+bcdg+b^2ef}{b^2cfg}$.

6th. $\frac{1}{a}-\frac{1}{b}+\frac{1}{c}$?.................................... *Ans.* $\frac{bc-ac+ab}{abc}$.

7th. $\frac{a}{bx}+\frac{c}{dx^2}+\frac{e}{fx^3}$?.......................... *Ans.* $\frac{adfx^2+bcfx+bde}{bdfx^3}$.

8th. $\frac{1}{1+x}+\frac{1}{1-x}$?.. *Ans.* $\frac{2}{1-x^2}$.

9th. $\frac{2a}{3bc}+\frac{5df}{8b^2c}-\frac{de}{6b^2c^2}$?.................... *Ans.* $\frac{16abc+15cdf-4de}{24b^2c^2}$.

10th. $\frac{a}{b^n}+\frac{c}{b^{n-r}}+\frac{d}{b^{n-2r}}$?.......................... *Ans.* $\frac{a+cb^r+db^{2r}}{b^n}$.

SUBTRACTION OF FRACTIONAL EXPRESSIONS.

32. The difference of the expressions $\frac{a}{b}=v$ and $\frac{a'}{b}=v'$ is indicated by the expression $\frac{a}{b}-\frac{a'}{b}=v-v'$;

and if from $a=bv$
$a'=bv'$ is subtracted,

$$a-a'=bv-bv'=b(v-v');$$

therefore $\frac{a-a'}{b}=v-v'=\frac{a}{b}-\frac{a'}{b}$.

Whence, to obtain the difference of two fractions which have a common denominator,

Rule. Subtract the numerator of the subtrahend from the numerator of the minuend, and divide the remainder by the common denominator.

Examples of the subtraction of algebraic fractions which have the same denominator:

1st. $\frac{a}{b}-\frac{c}{b}$?..*Ans.* $\frac{a-c}{b}$.

2d. $\frac{a^2+ab}{m^2n}-\frac{ab-a^2}{m^2n}$?..*Ans.* $\frac{2a^2}{m^2n}$.

3d. $\frac{ax-x^2}{a-b}-\frac{bx+x^2}{a-b}$?..................................*Ans.* $\frac{ax-bx}{a-b}=x$.

4th. $\frac{m^2n}{m^2-n^2}-\frac{mn^2}{m^2-n^2}$?.......................................*Ans.* $\frac{mn}{m+n}$.

Let the fractions whose difference is required be $\frac{a}{b}=v$; $\frac{a'}{b'}=v'$, which have different denominators.

The expression of the difference of these fractions is $\frac{a}{b}-\frac{a'}{b'}=v-v'$.

Since $\frac{a}{b}=v$, $a=bv$, and $ab'=bb'\cdot v$,

and since $\frac{a'}{b'}=v'$, $a'=b'v'$, and $a'b=bb'\cdot v'$;

therefore $ab'-a'b=bb'v-bb'\cdot v'=bb'(v-v')$,

and $\frac{ab'-a'b}{bb'}=v-v'$.

Whence $\frac{a}{b}-\frac{a'}{b'}=\frac{ab'-a'b}{bb'}$.

Consequently, to find the difference of two fractions which have not the same denominator,

Rule. Having reduced the fractions to the same denominator, subtract the numerator of the subtrahend from the numerator of the minuend, and divide the remainder by the common denominator.

Examples of the subtraction of algebraic fractions which have different denominators:

1st. $\frac{a}{b}-\frac{c}{d}$?..*Ans.* $\frac{ad-bc}{bd}$.

2d. $\frac{a+x}{a}-\frac{a}{a+x}$?...*Ans.* $\frac{2ax+x^2}{a^2+ax}$.

3d. $\frac{a+b}{a-b}+\frac{a-b}{a+b}$?..*Ans.* $\frac{4ab}{a^2-b^2}$.

4th. $\frac{3a+2b}{c}-\frac{5bd-2a-3d}{4cd}$?............*Ans.* $\frac{12ad+3bd+2a+3d}{4cd}$.

5th. $\frac{a+z}{a^2-z^2}-\frac{a-z}{a+z}$?....................................*Ans.* $\frac{3az-a^2-z^2}{a^2-z^2}$.

6th. $\frac{1+x^2}{1-x^2}-\frac{1-x^2}{1+x^2}$?..*Ans.* $\frac{4x^2}{1-x^4}$.

7th. $\frac{1}{a^{m-n}}-\frac{1}{a^m}$?..*Ans.* $\frac{a^n-1}{a^m}$.

8th. $\frac{a^2}{e^{m-r-1}}-\frac{b^3d}{e^{m+1}}$?..............................*Ans.* $\frac{a^2e^{r+2}-b^3d}{e^{m+1}}$.

33. The sum or difference of an integer and a fraction can be obtained by means of the preceding rules.

Example 1st. Required the sum of a^2b and $\frac{ac-bc^2}{ad}$?

$$a^2b=\frac{a^2b}{1}=\frac{a^2b}{1}\times\frac{ad}{ad}=\frac{a^3bd}{ad},$$

$$\therefore a^2b+\frac{ac-bc^2}{ad}=\frac{a^3bd}{ad}+\frac{ac-bc^2}{ad}=\frac{a^3bd+(ac-bc^2)}{ad}=\frac{a^3bd+ac-bc^2}{ad}.$$

Example 2d. From ax^2 subtract $\frac{bx+cd-ab}{ac^3}$?

$$ax^2=\frac{ax^2}{1}=\frac{ax^2}{1}\times\frac{ac^3}{ac^3}=\frac{a^2c^3x^2}{ac^3};$$

$$\therefore ax^2-\frac{bx+cd-ab}{ac^3}=\frac{a^2c^3x^2}{ac^3}-\frac{bx+cd-ab}{ac^3}=\frac{a^2c^3x^2-(bx+cd-ab)}{ac^3}=$$
$$\frac{a^2c^3x^2-bx-cd+ab}{ac^3}.$$

This process is analogous to that in arithmetic (Part I. Art. 159) for the reduction of a mixed number to an improper fraction.

Regarding $a^2b+\frac{ac-bc^2}{ad}$ and $ax^2-\frac{bx+cd-ab}{ac^3}$ as expressions which it is required to reduce to the form of improper fractions, it is evident that the reduction is made by multiplying the integer by the denominator of the fraction; adding the numerator of the fraction to the product, if the integer and fraction are connected by the sign +; subtracting the numerator of the fraction from the product, if the integer and fraction are connected by the sign —; and giving this sum or difference the denominator of the fraction for its denominator.

A fraction expresses a quotient; therefore a fraction which has the negative sign may be considered as the quotient of a dividend and divisor which are of contrary signs. In the processes of reducing to a common denominator, and adding or subtracting, the negative sign is considered to affect the numerator only.

MULTIPLICATION OF FRACTIONAL EXPRESSIONS.

34. Employing still the notation of Article 31, let $\frac{a}{b}=v$, $\frac{a'}{b'}=v'$.

If these expressions are multiplied together, $\frac{a}{b}\times\frac{a'}{b'}=vv'$.

Again, since $\frac{a}{b}=v$, $a=bv$,

and since $\frac{a'}{b'}=v'$, $a'=b'v'$.

Multiplying, $aa'=bv\cdot b'v'=bb'\cdot vv'$;

dividing by bb', $\frac{aa'}{bb'}=vv'$.

Whence $\frac{a}{b}\times\frac{a'}{b'}=\frac{aa'}{bb'}$.

Consequently, the product of two fractions is a fraction whose numerator is the product of the numerators, and its denominator the product of the denominators, of the given fractions.

In the formula $\frac{a}{b}\times\frac{a'}{b'}=\frac{aa'}{bb'}$, let $b'=1$; then $\frac{a}{b}\times a'=\frac{aa'}{b}$.

If aa' and b are both divided by a' the value of the expression remains constant.

$$\text{Now } aa'\div a'=a \text{ and } b\div a'=\frac{b}{a'},\text{ therefore } \frac{a}{b}\times a'=\frac{a}{\frac{b}{a'}}.$$

Whence the product of a fractional by an integer expression is obtained by either multiplying the numerator or dividing the denominator of the fraction by the integer.

Again, in the formula $\frac{a}{b}\times\frac{a'}{b'}=\frac{aa'}{bb'}$, let $b=1$; then $a\times\frac{a'}{b'}=\frac{aa'}{b'}$.

Consequently an integer is multiplied by a fractional expression by multiplying the integer by the numerator of the fraction, and dividing the product by its denominator.

From these formulæ is deduced the following rule for the multiplication of algebraic expressions of the fractional form:

1st. To multiply an integer by a fractional expression: multiply the integer expression by the numerator, and divide the product by the denominator of the fractional expression.

2d. To multiply a fractional by an integer expression: multiply the numerator or divide the denominator of the fractional multiplicand by the integer multiplier.

3d. To multiply one fractional expression by another: multiply together the numerators of the fractional factors for the numerator of the product, and their denominators for its denominator.

Examples of the multiplication of algebraic fractions:

1st. $a\times\frac{b}{c}$?..*Ans.* $\frac{ab}{c}$.

2d. $a^3b^2c\times\frac{ab}{c^2d}$?..*Ans.* $\frac{a^4b^3}{cd}$.

3d. $a^mb^n\times\frac{1}{ab^2c^3}$?..*Ans.* $\frac{a^{m-1}b^{n-2}}{c^3}$.

4th. $\frac{mx}{n^2}\times am$?..*Ans.* $\frac{am^2x}{n^2}$.

5th. $\frac{a^2b^3}{c^2d}\times cd$?..*Ans.* $\frac{a^2b^3}{c}$.

6th. $\frac{a^2c^2}{a^2-c^2}\times a-c$?..*Ans.* $\frac{a^2c^2}{a+c}$.

7th. $\frac{a}{b}\times\frac{c}{d}$?..*Ans.* $\frac{ac}{bd}$.

8th. $\frac{a+b}{c}\times\frac{a-b}{d}$?..*Ans.* $\frac{a^2-b^2}{cd}$.

9th. $\frac{a}{b}\times\frac{c}{d}\times\frac{e}{f}$?..*Ans.* $\frac{ace}{bdf}$.

10th. $\frac{a^2+b^2}{a^2-b^2}\times\frac{a-b}{a+b}$?..*Ans.* $\frac{a^2+b^2}{a^2+2ab+b^2}$.

11th. $\left(b+\frac{bx}{a}\right)\times\frac{a}{x}$?..*Ans.* $\frac{ab+bx}{x}$.

12th. $\frac{18b^3c^2d^6}{7a^5}\times\frac{4a^6}{9b^5c^3d^9}$?..............................*Ans.* $\frac{8a}{7b^2cd^3}$.

13th. $\left(\frac{7l}{h^5}+\frac{2l^3}{h^4}-\frac{3al^2}{h^3}+7\right)\times-\frac{8h^4}{l^5}$?..*Ans.* $\frac{24ah}{l^3}-\frac{56}{hl^4}-\frac{56h^4}{l^5}-\frac{16}{l^2}$.

14th. $\frac{a^2-b^2}{c^2-d^2}\times\frac{c^2-2cd+d^2}{a^2+2ab+b^2}$?................*Ans.* $\frac{ac-bc-ad+bd}{ac+bc+ad+bd}$.

15th. $\frac{x^2-9x+20}{x^2-6x}\times\frac{\frac{1}{2}x-3}{x-5}$?..............................*Ans.* $\frac{x-4}{2x}$.

16th. $\left(a+\frac{b}{c}\right)\times\left(a+\frac{b}{c}\right)$?..........................*Ans.* $a^2+\frac{2ab}{c}+\frac{b^2}{c^2}$.

17th. $\left(a+\frac{b}{c}\right)\times\left(d+\frac{e}{f}\right)$?..................*Ans.* $\frac{acdf+ace+bdf+be}{cf}$.

18th. $\frac{x^2+3x+2}{x^2+2x+1}\times\frac{x^2+5x+4}{x^2+7x+12}$?..........................*Ans.* $\frac{x+2}{x+3}$.

DIVISION OF FRACTIONAL EXPRESSIONS.

35. Again, let $\frac{a}{b}=v$; $\frac{a'}{b'}=v'$;

$$\therefore \frac{a}{b}\div\frac{a'}{b'}=v\div v'=\frac{v}{v'}.$$

Since $a=bv$ and $a'=b'v'$,

$$\frac{a}{a'}=\frac{bv}{b'v'}.$$

If both members of this equation are multiplied by $\frac{b'}{b}$,

$$\frac{a}{a'}\times\frac{b'}{b}=\frac{bv}{b'v'}\times\frac{b'}{b},$$

$$\text{or } \frac{ab'}{a'b}=\frac{v}{v'};$$

$$\therefore \frac{a}{b}\div\frac{a'}{b'}=\frac{ab'}{ba'}=\frac{a}{b}\times\frac{b'}{a'}.$$

The quotient of one fraction by another is, consequently, obtained by multiplying the dividend by the reciprocal of the divisor.

In the formula $\frac{a}{b}\div\frac{a'}{b'}=\frac{a}{b}\times\frac{b'}{a'}$, let $b'=1$;

$$\text{then } \frac{a}{b}\div a'=\frac{a}{ba'}.$$

If a and ba' are both divided by a' the value of the fraction $\frac{a}{ba'}$ is not changed; now $a\div a'=\frac{a}{a'}$, and $ba'\div a'=b$;

$$\text{therefore } \frac{a}{b}\div a'=\frac{\frac{a}{a'}}{b}.$$

Whence, the dividend being a fractional and the divisor an integer expression, the quotient is obtained by either multiplying the denominator or dividing the numerator of the dividend by the divisor.

Lastly, in the formula $\frac{a}{b}\div\frac{a'}{b'}=\frac{ab'}{ba'}$ let $b=1$;

$$\text{then } a\div\frac{a'}{b'}=\frac{ab'}{a'}.$$

Whence, when the dividend is an integer and the divisor a fractional expression, the quotient is obtained by multiplying the dividend by the denominator of the divisor, and dividing the product by its numerator.

From these cases is derived the following rule for division when either the dividend or the divisor, or both the dividend and the divisor, are fractional expressions:

1st. To divide a fractional by an integer expression: multiply the denominator or divide the numerator of the dividend by the divisor.

2d. To divide an integer by a fractional expression: multiply the dividend by the denominator of the divisor, and divide the product by its numerator.

3d. To divide one fractional expression by another: multiply the dividend by the reciprocal of the divisor.

Examples of the division of algebraic fractions:

1st. $\frac{a}{b}\div c$? .... *Ans.* $\frac{a}{bc}$.

2d. $\frac{am}{bn}\div m$? .... *Ans.* $\frac{a}{bn}$.

3d. $d\div\frac{a}{b}$? .... *Ans.* $\frac{bd}{a}$.

4th. $\frac{a}{b}\div\frac{c}{d}$? .... *Ans.* $\frac{ad}{bc}$.

5th. $\frac{ab}{c}\div\frac{1}{c}$? .... *Ans.* ab.

6th. $\frac{a^2+b^2}{c+d}\div\frac{a-b}{c-d}$? .... *Ans.* $\frac{a^2c+b^2c-a^2d-b^2d}{ac+ad-bc-bd}$.

7th. $\left(\frac{a}{b}+\frac{c}{d}\right)\div\left(\frac{e}{f}+\frac{g}{h}\right)$? .... *Ans.* $\frac{adfh+bcfh}{bdeh+bdfg}$.

8th. $\left(a+\frac{b}{c}\right)\div\left(d+\frac{e}{f}\right)$? .... *Ans.* $\frac{acf+bf}{cdf+ce}$.

9th. $\left(\frac{a}{a+b}+\frac{b}{a-b}\right)\div\left(\frac{a}{a-b}-\frac{b}{a+b}\right)$? .... *Ans.* 1.

10th. $\left(1+\frac{n-1}{n+1}\right)\div\left(1-\frac{n-1}{n+1}\right)$? .... *Ans.* n.

11th. $\left(\frac{a+x}{a-x}+\frac{a-x}{a+x}\right)\div\left(\frac{a+x}{a-x}-\frac{a-x}{a+x}\right)$? .... *Ans.* $\frac{a^2+x^2}{2ax}$.

12th. $\left(a-\frac{b^2}{2a}\right)\left(a-\frac{a^2+b^2}{a+b}\right)\div\left(1-\frac{a}{a+b}\right)$?

Ans. $\frac{2a^3-2a^2b-ab^2+b^3}{2a}$.

13th. $\left(\frac{a}{b}+\frac{c}{d}+\frac{e}{f}\right)\div\left(\frac{g}{h}+\frac{i}{k}+\frac{l}{m}\right)$? ..... *Ans.* $\frac{(adf+bcf+bde)hkm}{(gkm+him+hkl)bdf}$.

14th. $\left(\frac{a}{a-b}+\frac{b}{a+b}\right)\div\left(\frac{a}{a-b}-\frac{b}{a+b}\right)$? .... *Ans.* $\frac{a^2-2ab-b^2}{a^2+b^2}$.

15th. $\left(\frac{c^2}{d^2}-\frac{c^3}{a+b}\right)\div\left(\frac{c^2}{a+b}-\frac{c^4}{dh^2}\right)$? .... *Ans.* $\frac{(a+b-cd^2)h^2}{d^2h^2-(a+b)c^2d}$.

16th. $\left(1-\frac{a}{b}+\frac{b}{a}\right)\left(\frac{a+b}{2a}+\frac{a-b}{2b}\right)\div\left(a-2b+\frac{b^2}{a}\right)\left(\frac{a}{a+b}+\frac{b}{a-b}\right)$?

Ans. $\frac{(a^2-ab-b^2)(a+b)}{2ab^2(a-b)}$.

OF THE EXPRESSIONS a^0, a^{-p}, $\frac{A}{0}$, $\frac{0}{0}$.

36. The product of two powers of the same root is obtained by affecting that root with an exponent equal to the sum of the exponents of the multiplicand and multiplier (Part I. Art. 75), and the quotient arising from the division, one by the other, of two powers of the same root, by affecting that root with an exponent equal to the excess of the exponent of the dividend over the exponent of the divisor (Part I. Art. 100).

In applying this rule to multiplication the numerical values of the exponents of the multiplicand and multiplier do not necessarily enter into consideration; but in its application to division the case is otherwise. It is in effect assumed that the dividend (which may be denoted by a^m) is the product of the divisor (a^n) by some other power (as a^p) of the root a, and that m is greater than n. It may, however, happen

1st. That n is equal to m, or

2d. That n is greater than m.

1st. Let $n=m$.

$$\text{Then } \frac{a^m}{a^n}=a^{m-n}=a^{m-m}=a^0.$$

This expression, a^0, is new, and requires explanation.

Since in every division in which the dividend is equal to the divisor the quotient is 1, therefore

$$\frac{a^m}{a^n}=\frac{a^m}{a^m}=1=a^{m-m}=a^0.$$

Whence $a^0=1$. In the same manner it can be proved that the value of any other root affected with the exponent 0 is 1. It is therefore agreed by convention to regard the expression a^0, or any quantity having zero for exponent, as equal to unity.

The expression of a quantity raised to a power equal to zero owes its existence to the rule of exponents, and not to that of division; for it is evident that the result obtained by the rule of division is 1, since the quotient of any quantity divided by itself is 1.

In the division of one monomial by another, and also in the process of simplifying a fraction, any quantity (such as a^m) which is common to the dividend and the divisor disappears from the quotient or the simplified fraction. Sometimes it is important to preserve a quantity from disappearing, and in such a case the expression a^0 is employed.

2d. Let n be greater than m, and equal to $m+p$.

$$\text{Then } \frac{a^m}{a^n}=\frac{a^m}{a^{m+p}}=\frac{a^m}{a^m\times a^p}=\frac{1}{a^p}.$$

But by the rule of exponents

$$\frac{a^m}{a^{m+p}}=a^{m-(m+p)}=a^{-p}.$$

The expressions $\frac{1}{a^p}$ and a^{-p} being obtained, the first by the common process of division, and the second by a rule derived from that process, are regarded as equivalent; and, in general, every quantity affected with a negative (or subtractive) exponent is considered as equivalent to the quotient of 1 divided by that quantity, with the sign of its exponent changed.

Consequently any quantity may be removed from the denominator to the numerator of a fraction by changing the sign of its exponent.

37. It is sometimes convenient to employ quantities which have subtractive exponents; and therefore it becomes necessary to determine in what manner quantities with such exponents are to be combined, in any calculation, with each other, and with quantities whose exponents are additive.

Now, if it is required to multiply a^m by a^{-n}, it follows, from Article 36, that

$$a^m \times a^{-n} = a^m \times \frac{1}{a^n} = \frac{a^m}{a^n} = a^{m-n};$$

And if it is required to multiply a^{-m} by a^{-n}, that

$$a^{-m} \times a^{-n} = \frac{1}{a^m} \times \frac{1}{a^n} = \frac{1}{a^{m+n}} = a^{-m-n}.$$

Also, if it is required to divide a^m by a^{-n},

$$a^m \div a^{-n} = a^m \div \frac{1}{a^n} = a^m \times \frac{a^n}{1} = a^m \times a^n = a^{m+n};$$

And if it is required to divide a^{-m} by a^{-n},

$$a^{-m} \div a^{-n} = \frac{1}{a^m} \div \frac{1}{a^n} = \frac{1}{a^m} \times \frac{a^n}{1} = \frac{a^n}{a^m} = a^{n-m} \text{ or } a^{-(m-n)}.$$

Whence the rules for the multiplication and division of quantities which have exponents of which some are additive and others subtractive, or of which all are subtractive, are comprehended (with the interpolation of the word "algebraic") in the rules for the multiplication and division of quantities with additive exponents; namely, that the exponent of the product of two powers of the same root is the algebraic sum of the exponents of the factors, and the exponent of the quotient of two powers of the same root the algebraic difference between the exponents of the dividend and the divisor.

38. a^m is the expression of the letter a written m times as a multiplier; a^{-m} represents the same letter, a, written m times as a divisor; and a^0 indicates that a is the same number of times a factor of the dividend and of the divisor.

Since zero, multiplied by itself any number of times, gives a product equal to zero, it follows that

$$0^m = 0, \text{ and therefore that } \frac{0^m}{0^m} = \frac{0}{0}.$$

If, in the expression $a^{m-m} = \frac{a^m}{a^m}$, a is made equal to zero, (which is permitted, since a expresses any number,)

$$a^0 = \frac{a^m}{a^m} \text{ becomes}$$

$$0^0 = \frac{0}{0}.$$

Effecting the division of zero by zero, any number whatever may be assumed as quotient; for any number multiplied by zero gives for product zero, which is in this case the dividend.

The expression 0^0 appears, therefore, to involve an indefinite number of numerical values; but the expression $\frac{0}{0}$ has in many cases a fixed and determinate value.

Thus the fraction $\frac{Aa^0}{a^0}$, for the hypothesis $a=0$, becomes $\frac{A \times 0}{0} = \frac{0}{0}$.

Writing the fraction $\frac{Aa^m}{a^n}$ thus, Aa^{m-n}, and making $a=0$, Aa^{m-n} becomes $A \times 0^{m-n}$.

In the case of $m > n$, or $m = n + d$,

$$A \times 0^{m-n} \text{ becomes } A \times 0^d = 0.$$

In the case of $m<n$, or $m=n-d$,

$$A\times 0^{m-n} \text{ becomes } A\times 0^{-d}=\frac{A}{0^d},$$

an expression which is to be interpreted hereafter.

In the case of $m=n$, if both terms of the fraction are divided by a^m, the expression is reduced to A.

RELATIVE VALUES OF $\frac{a}{b}$ AND $\frac{a+m}{b+m}$.

39. Let it, in concluding the consideration of algebraic fractions, be required to find the effect produced on a fraction by augmenting or by diminishing both its terms by the same quantity.

To both terms of the fraction $\frac{a}{b}=v$ let the quantity m be added; the value v of the fraction will vary, becoming either greater or less.

Denoting the variation by δ,

$$\frac{a+m}{b+m}=v+\delta;$$

in which equation δ expresses the difference of the fractions $\frac{a}{b}$ and $\frac{a+m}{b+m}$, a difference unknown both in quantity and in sign, since the difference of the fractions is not known, and it is also unknown whether $\frac{a+m}{b+m}$ is greater or less than $\frac{a}{b}$.

It is to be conceived, therefore, that δ represents the unknown number, and the sign by which it is to be connected with the primitive value, v.

This remark applies to every unknown quantity.

$$\text{Since } \delta+v=\frac{a+m}{b+m},$$

$$\text{and } v=\frac{a}{b},$$

$$\text{by subtraction } \delta=\frac{a+m}{b+m}-\frac{a}{b}.$$

$$\text{Reducing to a common denominator } \delta=\frac{b(a+m)}{b(b+m)}-\frac{a(b+m)}{b(b+m)},$$

$$\text{or } \delta=\frac{ba+bm-ab-am}{b(b+m)},$$

$$\text{or } \delta=\frac{bm-am}{b(b+m)}=\frac{m(b-a)}{b(b+m)},$$

$$\text{or } \delta=\frac{m}{b}\times\frac{b-a}{b+m}.$$

Make, 1st., $a>b$, 2d. $a=b$, 3d. $a<b$. Then

1st. When a is greater than b, the difference $b-a$ is subtractive; dividing this difference by the absolute number, $b+m$, the quotient is subtractive, and multiplying the quotient by $\frac{m}{b}$, the result is still subtractive. Therefore, when a is greater than b, the fraction $\frac{a}{b}$ is greater than $\frac{a+m}{b+m}$; and instead of

$$\frac{a+m}{b+m}=v+\delta, \text{ the proper expression is}$$

$$\frac{a+m}{b+m}=v-\delta.$$

Whence the value of an improper fraction is diminished by adding the same quantity to both its terms.

2d. When $a=b$,

$$\frac{a+m}{b+m} \text{ becomes } \frac{a+m}{a+m};$$

$$\text{and since } \frac{a}{a}=1, \text{ and } \frac{a+m}{a+m}=1, \ \delta=0.$$

3d. When a is less than b;
In this case $b-a$ is additive, and therefore also

$$\frac{m}{b}\times\frac{b-a}{b+m};$$

Whence δ is additive; $\frac{a+m}{b+m}=v+\delta$;

And the conclusion is, that the value of a proper fraction is increased by adding the same quantity to both its terms.

It deserves notice that to the hypotheses

$$a>b, \ a=b, \ a<b,$$

correspond δ subtractive, $\delta=0$, δ additive;

and that, consequently, the difference, δ, changes from subtractive to additive in passing through zero.

Suppose, next, that the quantity m is subtracted from both terms of the fraction $\frac{a}{b}$, and that

$$\frac{a-m}{b-m}=v+\delta,$$

$$\frac{a}{b}=v.$$

Proceeding as with $\frac{a+m}{b+m}=b+\delta$, $\frac{a}{b}=v$, it is found that $\delta=\frac{m}{b}\times\frac{a-b}{b-m}$.

Make, 1st. $b>m$, with $a>b$, $a=b$, $a<b$.
Make, 2d. $b=m$, with $a>b$, $a=b$, $a<b$.
Make, 3d. $b<m$, with $a>b$, $a=b$, $a<b$.

1st. When b is greater than m, the denominator $b-m$ is additive. Therefore δ is additive when a is greater than b, equal to zero when a is equal to b, and subtractive when a is less than b.

Whence, if from both the terms of a fraction a number less than the denominator of that fraction is subtracted, if the fraction is improper, its value is augmented; if the numerator and denominator are equal, the value of the fraction is not changed; and if the fraction is proper, its value is diminished.

2d. When $b=m$, the denominator $b-m=0$.

Therefore when $a>b$, $\delta=\frac{m}{b}\times\frac{a-b}{b-m}=1\times\frac{a-b}{0}$.

For example, if $a=3$ and $b=2$, $\delta=\frac{1}{0}$.

When $a=b$, $\delta=\frac{m}{b}\times\frac{a-b}{b-m}=1\times\frac{0}{0}=\frac{0}{0}$.

When $a<b$, $\delta=\frac{m}{b}\times\frac{a-b}{b-m}=1\times\frac{-(b-a)}{0}$.

If $a=2$ and $b=3$, $\delta=-\frac{1}{0}$, a result which, except in sign, is the same as that found for $b=m$, $a>b$.

3d. When $b<m$, the denominator $b-m$ is subtractive. In this case it is found that

when $a>b$, δ is subtractive,
when $a=b$, $\delta=0$,
when $a<b$, δ is additive.

Therefore, when from both terms of a fraction a number greater than its denominator is subtracted, if the fraction is improper, its value is diminished; if the numerator is equal to the denominator, the value of the fraction is not changed; if the fraction is proper, its value is augmented.

40. For the purpose of giving some interpretation of the expression $\frac{1}{0}$, let it be required to divide 1 by $1-a$.

$$\begin{array}{l}
1-a)1\qquad(1+a+a^2+a^3+a^4+\ \dots\ +a^{n-1}+\frac{a^n}{1-a} \\
\quad\underline{1-a} \\
\qquad\quad a \\
\qquad\quad\underline{a-a^2} \\
\qquad\qquad\quad a^2 \\
\qquad\qquad\quad\underline{a^2-a^3} \\
\qquad\qquad\qquad\quad a^3 \\
\qquad\qquad\qquad\quad\underline{a^3-a^4} \\
\qquad\qquad\qquad\qquad\quad a^4 \\
\qquad\qquad\qquad\qquad\quad\underline{a^4-a^5} \\
\qquad\qquad\qquad\qquad\qquad\quad a^5
\end{array}$$

Effecting the division, the quotient is composed of a series of which the first term is a^0 or 1, the second a, the third a^2, the fourth a^3. . . .; and with whatever term the division is left off, the remainder is a raised to a power one degree higher than that of the last term of the quotient.

Now, if a is made equal to 1,

$$\frac{1}{1-a}=\frac{1}{1-1}=\frac{1}{0};$$

and the second member of the equation,

$$1+a+a^2+a^3+\ \dots\ +a^{n-1}+\frac{a^n}{1-a}, \text{ becomes}$$

$$1+1+1+1\ +\ \dots\ +1\quad+\frac{1}{0}.$$

Whence $\frac{1}{0}=1+1+1+1+\ \dots\ +\frac{1}{0}$.

This result shows that with whatever term the division is terminated, the fraction which must be added to the quotient, in order to complete it, is equal to the expression $\frac{1}{0}$, so that

$$\frac{1}{0}=1+\frac{1}{0}$$

$$\frac{1}{0}=2+\frac{1}{0}$$

$$\frac{1}{0}=3+\frac{1}{0}$$

$$\cdot\quad\cdot$$

Now any finite number is increased by the addition of a whole additive number to it; and the number which is not increased by such addition is considered to be infinite.

Therefore the expression $\frac{1}{0}$ is considered that of a number which is infinitely great. It is generally denoted by the character ∞; thus

$$\frac{1}{0}=\infty.$$

An expression such as $1+a+a^2+a^3+a^4+\ \dots$ is termed an infinite series.

SECTION IV.

OF EQUATIONS OF THE FIRST DEGREE.

41. Algebra is a language composed of symbols, by the use of which the process of reasoning employed to demonstrate the existence of a property belonging to numbers, or to find the solution of a problem, is followed more readily than it can be if the process is conducted through the medium of common language.

To illustrate this observation by an example,

Let it be required to divide the number 100 into three parts, such that the first part shall exceed the second part by 5, and the second part shall exceed the third part by 10.

Since the first part is to exceed the second part by 5, the second part must be equal to the first part less 5; and since the second part is to exceed the third part by 10, the third part must be equal to the second part less 10; therefore the third part must be equal to the first part less 15.

Consequently the sum of the three parts is equal to the first part, augmented by the first part less 5, augmented by the first part less 15; that is, the sum of the three parts is equal to three times the first part less 20, or three times the first part are equal to the sum of the three parts augmented by 20.

But the sum of the three parts is 100; therefore three times the first part are equal to 100 augmented by 20, or 120. Whence the first part must be equal to one third of 120, or 40.

Consequently the second part is equal to 40 less 5, or 35, and the third part is equal to 40 less 15, or 25.

To resolve the same question algebraically,

Let the first part be denoted by x; then, by the question, the second part is $x-5$, and the third part $(x-5)-10$, or $x-15$; therefore the sum of the three parts is $x+x-5+x-15$, or $3x-20$.

Now, by the question, the sum of the three parts is 100; therefore $3x-20=100$.

But if $3x-20=100$, $3x$ alone must be equal to $100+20$; that is, $3x=120$.

$$\therefore x=\frac{1}{3} \text{ of } 120=\frac{120}{3}=40=1\text{st part.}$$

$$\text{Whence } x-5=40-5=35=2\text{d part;}$$

$$x-15=40-15=25=1\text{st part.}$$

42. If a question enunciated in common language is resolved by means of algebra, the solution is composed of two distinct parts.

In the first part of the solution the relations which the question establishes between the given and the unknown quantities are translated from common language into the language of algebra, and formed into expressions which must be considered equal to each other.

In the second part the values of the unknown quantities are deduced from the expressions obtained by the first part of the solution.

When two algebraic expressions contain one or more unknown quantities, whose values must be determined in such a manner that the two expressions shall become equal, they are connected by the symbol $=$, as if they were already equal; and the two expressions thus connected obtain the name Equation.

If x denotes an unknown quantity, $2x+3=x+7$ is an equation. The quantity $2x+3$ being the same as $x+7+x-4$, it is evident that $2x+3$ cannot be made equal to $x+7$ except by giving x the value 4.

When two expressions, whose values can be proved equal, are connected by the symbol $=$, they form an equality. Thus by putting $(a+b)\ (a-b)=a^2-b^2$ an equality is formed, for the product of $a+b$ by $a-b$ is a^2-b^2.

When two quantities connected by the symbol = are equal, and expressed in precisely the same manner, they form an identical equation or identity.

For example, $\left.\begin{array}{r}4+2=4+2,\\ a^2+b=a^2+b,\end{array}\right\}$ are identical equations.

From these definitions it follows, that when the term Equation is employed, it ought to be understood that the expression contains one or more unknown quantities, and that the values which are to be determined for the unknown quantity or quantities ought to be such that, when substituted for the letters which represent them, the equations shall be changed into equalities; that the term Equality is applied to expressions in which the quantities separated by the symbol, =, are such as can be proved equal or have been proved equal to one another; and that the identity is a self-evident equality.

In practice the terms Equation and Equality are often used one for the other. This is caused by the habit of considering the unknown quantity as the representative of the value which is to be found for it, and tacitly substituting the value for the symbol by which it is represented.

43. The sum of the exponents of the unknown quantities, taken in that term in which the sum is greatest, marks what is called the degree of an equation.

Thus, $ax+b=c$ is an equation of the first degree.

$\left.\begin{array}{l}ax^2+bx+c=d,\\ axy+bx+cy+d=e,\end{array}\right\}$ are equations of the second degree.

$\left.\begin{array}{l}ax^3+bx^2+cx+d=e,\\ x^2y+bxy+cx+dy+e=f,\\ xyv+axy+byv+cx+dy+ev+f=g,\end{array}\right\}$ are equations of the third degree.

$x^4+bx^3+cx^2+dx+e=f$, is an equation of the fourth degree.

. . . .

44. Equations are distinguished into numerical and literal equations. The first are such as contain only particular numbers, with the exception of the unknown quantity, which is always represented by a letter. Thus,

$\left.\begin{array}{l}5x-4=3x+6,\\ x^2-2x=8,\end{array}\right\}$ are numerical equations.

In literal equations the data are represented by letters:

$\left.\begin{array}{l}ax+b=c,\\ ax^2+bx+c=d,\end{array}\right\}$ are literal equations.

When the data are represented by letters the first letters of the alphabet are used; and, for the sake of distinction, unknown quantities are always represented by the last letters, x, y, z, t, v, . . .

45. When of the unknown quantities contained in any equations values have been found such that by the substitution of each value for its symbol the members of the equations are made equal, the equations are said to be resolved.

Hence, if in any equations the unknown quantities are replaced by their values, and all the calculations indicated in the two members of each equation are performed, the equations are rendered identical.

Equations reduced thus to identities are said to be verified or satisfied.

46. The nature and conditions of the questions which may be proposed for solution by algebra are so various that it is not possible to reduce under any precise rule the method to be followed in the first part of the algebraic solution of a question; that is, in translating the conditions of a question from common language into the language of algebra.

The following directions, although, from the nature of the subject, very general, may be found useful.

1st. Express the known quantities either by numbers or letters, and the unknown quantities by letters.

2d. Consider the question resolved; and by means of these expressions and the appropriate algebraic symbols indicate the same reasonings and operations as it would be necessary to employ in order to verify the value of the unknown quantity if that value were determined.

Two different algebraic expressions having the same assumed value, and involving the symbol of the unknown quantity, are formed by these means; and the equation of the question is obtained by putting these expressions equal to each other.

If the question involves more unknown quantities than one, and these are not given in terms of each other, for each unknown quantity the algebraic translation of the question must contain (as will be hereafter shown, Art. 50,) two different expressions of the same assumed value.

47. The second part of the solution of a question, in which the values of the unknown quantities are deduced from the equations obtained by the first part of the solution, is made by fixed and invariable rules.

To resolve an equation it is necessary to disengage the unknown quantity from the known quantities with which it is combined, for the purpose of obtaining an equation of which the unknown quantity alone shall form one member (generally the first) and given quantities the other member.

The unknown quantity may be combined with the known quantities by

1st. Addition, as $x+a=b$,
2d. Subtraction, as $x-c=d$,
3d. Multiplication, as $mx=n$,
4th. Division, as $\frac{x}{d}=q$,

or by several of these forms of combination, or by all of them.

a. It has been already stated as a general principle that if two quantities are equal, and the same arithmetical operation is performed upon them both, the results are equal.

Therefore, if from both members of the first equation a is subtracted, the remainders are equal; that is,

$$x+a-a=b-a,$$
or
$$x=b-a.$$

b. And in like manner, if to both members of the second equation c is added, the sums are equal; or,

$$x-c+c=d+c,$$
or
$$x=d+c.$$

Therefore, to free the unknown quantity from known quantities connected with it by the sign of addition, subtract these quantities from both members of the equation;

And to free the unknown quantity from known quantities connected with it by the sign of subtraction, add these quantities to both members of the equation.

Hence, a term which is connected additively or subtractively with one member of an equation may be cancelled in that member, provided it is introduced into the other member with a contrary sign.

By this means the known quantities which are connected with the unknown quantity by the signs + or — are transferred to the other side of the equation.

An unknown quantity found in the second member is transferred to the first by the same process, which is called Transposition of terms.

A term found in both members with the same sign disappears by the effect of transposition.

Examples:

1st. Given $x-12=8$?......*Ans.* $x=20$.
2d. - $x+a=bc$?......*Ans.* $x=bc-a$.
3d. - $7x-30=6x-10$?......*Ans.* $x=20$.
4th. - $x+a^2-ab=a^2-bc$?......*Ans.* $x=ab-bc$.

c. If both members of the third equation are divided by m, the quotients are equal; or

$$\frac{mx}{m}=\frac{n}{m},$$
$$\text{or } x=\frac{n}{m}.$$

d. And if both members of the fourth equation are multiplied by d, the products are equal; or

$$\frac{x}{d}\times d=qd,$$
$$\text{or } x=qd.$$

Whence, to free the unknown quantity from known quantities connected with it by the sign of multiplication, divide both members of the equation by the multiplier of the unknown quantity;

And to free the unknown quantity from known quantities connected with it by the sign of division, multiply both members of the equation by the divisor of the unknown quantity.

Consequently, the known quantities which are connected with the unknown quantity by multiplication or division are disengaged by dividing or multiplying all the terms of the equation by these quantities.

By these principles a given quantity which is a factor of all the terms of an equation, or of both members of an equation, may be cancelled.

The signs of all the terms of an equation may be changed by multiplying every term by -1.

The denominators (being given quantities) of any terms, whether in x or not, may be taken away.

Examples:

1st. Given $5x=20$?..*Ans.* $x=4$.

2d. - - $ax=b-c$?..................................*Ans.* $x=\frac{b-c}{a}$.

3d. - - $\frac{x}{4}=6$?..*Ans.* $x=24$.

4th. - - $\frac{x}{a}=b^2+c$?................................*Ans.* $x=ab^2+ac$.

5th. - - $4x=8a-64$?..........................*Ans.* $x=2a-16$.

6th. - - $-x=b^2-5bc$?.........................*Ans.* $x=5bc-b^2$.

7th. - - $x=\frac{b}{m}-\frac{d}{n}$?..............................*Ans.* $x=\frac{nb-md}{mn}$.

8th. - - $\frac{x}{a}=\frac{b}{a^2}-\frac{c}{a^3}$?..............................*Ans.* $x=\frac{ab-c}{a^2}$.

48. Equations in which the unknown quantity is connected with the given quantities by several or all of these forms of combination are resolved by the successive application of the appropriate rules.

Thus in the equation $3x+5=20-2x$, the terms 5 and $2x$ are transposed by Articles 47 a, b, and the coefficient of x is made to disappear by Article 47. c. The operation is represented thus,

$$3x+5=20-2x$$
$$\text{by transposition - } 3x+2x=20-5,$$
$$\text{or - } 5x=15$$
$$\text{and by division - } x=\frac{15}{5}=3.$$

Similarly, in the equation $ax+b-cx=d-ax+g$, the unknown quantity ax is transferred to the first member of the equation, and the given quantity b to the second, by the rules of Article 47. After these transpositions the equation is,

$$ax-cx+ax=g-b.$$

To disengage the known quantities a and $-c$ from x, it is to be observed that $ax-cx+ax=2ax-cx=(2a-c)x$. Now, if both members of the equation are divided by $2a-c$, the coefficient of the unknown quantity, x alone is left in the first member, and the second becomes $\frac{g-b}{2a-c}$;

$$\text{whence } x=\frac{g-b}{2a-c};$$

Consequently, when the terms in x have been transferred to one side of the sign $=$, and the given quantities connected with x by the sign $+$ or $-$ to the other side, the given quantities connected with x by multiplication may be formed into a coefficient of x. Dividing both members of the equation by this coefficient, an equation is obtained of which the first member is x, and the second is composed of known quantities.

To obtain the value of x in the equation

$$\frac{x}{3}+\frac{9}{2}=\frac{21}{4}-\frac{x}{6},$$

it is necessary, by Rule 47, to multiply the terms of the equation by 3, to multiply the terms of the resulting equation by 2, &c., in the manner following,

$$\frac{x}{3}+\frac{9}{2}=\frac{21}{4}-\frac{x}{6}.$$

Multiplying by 3, - $x+\frac{27}{2}=\frac{63}{4}-\frac{3x}{6}$.

- - 2, - $2x+27=\frac{126}{4}-\frac{6x}{6}$.

- - 4, - $8x+108=126-\frac{24x}{6}$.

- - 6, - $48x+648=756-24x$.

Transposing, - - $48x+24x=756-648$,

or $72x=108$.

$$\therefore x=\frac{108}{72}=\frac{3}{2}=1\tfrac{1}{2}.$$

By dividing the denominators, when possible, instead of multiplying the numerators, the result is simplified. Thus the equations successively obtained are,

From multiplication of 1st equation by 3, $x+\frac{27}{2}=\frac{63}{4}-\frac{x}{2}$.

- - 2d equation by 2, $2x+27=\frac{63}{2}-x$.

- - 3d equation by 4, $4x+54=63-2x$.

Transposing - - - $4x+2x=63-54$.

or $6x=9$

$$\therefore x=\frac{9}{6}=\frac{3}{2}=1\tfrac{1}{2}.$$

The most simple method, however, of removing the denominators, when several terms of an equation have a fractional form, is to reduce all the terms to the same denominator (which does not disturb the equation), and to cancel the denominators, by Article 47 d.

In reducing the terms to the same denominator, the abridged processes of Article 30 a. may be employed.

Resuming the last example; the least common multiple of the denominators is 12; reducing all the terms to the common denominator 12,

$$\frac{4x}{12}+\frac{54}{12}=\frac{63}{12}-\frac{2x}{12}.$$

$$\therefore 4x+54=63-2x.$$

$$6x=9 \text{ and } x=\tfrac{9}{6}=1\tfrac{1}{2}.$$

49. These principles (Arts. 47 and 48) apply to equations of all degrees, and are themselves sufficient for the resolution of equations of the first degree with one unknown quantity.

General rule for the resolution of equations of the first degree with one unknown quantity.

1st. Remove the denominators of all the terms which have a fractional form, and execute in both members of the equation all the algebraic operations which are indicated. Each member of the equation is thus rendered an integer polynomial.

2d. Transpose to one member all the terms which contain the unknown quantity, and to the other member the known terms.

3d. Reduce to a single term all the terms in x, if the equation is numerical; and if literal, reduce all these terms to a single product composed of two factors, of which one is x, and the other the algebraic sum of all the quantities by which x is multiplied.

4th. Divide both terms of the equation by the numerical or literal coefficient of x, and perform the division of the second term, if it is possible.

Find the value of x in each of the following equations:

1st. $18x-61=13x-31$?............ *Ans.* $x=6$.

2d. $3x+\frac{5x}{4}=34$?............ *Ans.* $x=8$.

3d. $\frac{x}{5}+\frac{x}{3}=x-7$?............ *Ans.* $x=15$.

4th. $3x-5=23-x$?............ *Ans.* $x=7$.

5th. $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=13$?............ *Ans.* $x=12$.

6th. $\frac{x}{4}+\frac{x}{5}-\frac{x}{6}=3$?............ *Ans.* $x=10\frac{10}{17}$.

7th. $x+\frac{x}{2}-\frac{x}{3}=4x-17$?............ *Ans.* $x=6$.

8th. $5-\frac{x+4}{11}=x-3$?............ *Ans.* $x=7$.

9th. $x+\frac{3x-5}{2}=12-\frac{2x-4}{3}$?............ *Ans.* $x=5$.

10th. $\frac{x+1}{3}+\frac{x+3}{4}=\frac{x+4}{5}+16$?............ *Ans.* $x=41$.

11th. $5x-\frac{5x}{2}+12=\frac{4x}{3}+26$?............ *Ans.* $x=12$.

12th. $13\frac{3}{4}-\frac{x}{2}=2x-8\frac{3}{4}$?............ *Ans.* $x=9$.

13th. $2x+7+\frac{3x}{2}=6x-23$?............ *Ans.* $x=12$.

14th. $\frac{5x}{12}-\frac{4x}{3}-13=\frac{7}{8}-\frac{13x}{6}$?........................*Ans.* $x=11\frac{1}{10}$.

15th. $ax+c=bx+d$?..................................*Ans.* $x=\frac{d-c}{a-b}$.

16th. $2ax-bx+2ab=4a^2-ab-3ax$?........*Ans.* $x=\frac{4a^2-3ab}{5a-b}$.

17th. $(3a-x)(a-b)+2ax=4b(a+x)$? ....*Ans.* $x=\frac{7ab-3a^2}{a-3b}$.

18th. $\frac{a+3x}{4a}-\frac{7a-5x}{6b}+3-\frac{9x}{4}=\frac{x}{ab}+\frac{5x}{6b}$? *Ans.* $x=\frac{39ab-14a^2}{27ab-9b+12}$.

19th. $\frac{x}{a}-1-\frac{dx}{c}+3ab=0$? ....*Ans.* $x=\frac{ac(1-3ab)}{c-ad}$.

20th. $\frac{a^2x}{b-c}+dc=bx-ac$?...........*Ans.* $x=\frac{abc-ac^2+bcd-c^2d}{b^2-bc-a^2}$.

21st. $\frac{ax}{b}-c=\frac{mx}{n}+d$?..........................*Ans.* $x=\frac{bcn+bdn}{an-bm}$.

22d. $\frac{ax}{a-b}+4b=\frac{cx}{3a+b}$?..............*Ans.* $x=\frac{8ab^2+4b^3-12a^2b}{3a^2+ab-ac+bc}$.

23d. $\frac{3bx}{2a^2}-\frac{x-b}{a+b}=\frac{bx-a^2}{a^2-b^2}-\frac{x}{4a}$?...*Ans.* $x=\frac{4a^2(a^2+ab-b^2)}{3a^3-6a^2b+ab^2+6b^3}$.

24th. $\frac{(a+b)(x-b)}{a-b}-3a=\frac{4ab-b^2}{a+b}-2x+\frac{a^2-bx}{b}$?

Ans. $x=\frac{a^4+3a^3b+4a^2b^2-6ab^3+2b^4}{b(4a^2+2ab-2b^2)}$.

OF EQUATIONS OF THE FIRST DEGREE INVOLVING TWO UNKNOWN QUANTITIES.

50. When two equations of the first degree involving two unknown quantities are given, the method employed to determine the values of these quantities is to deduce from the two proposed equations a new equation containing only one unknown quantity, and from which the value of this unknown quantity may be obtained.

The process by which one of the unknown quantities is banished is named Elimination.

The elimination of one of the unknown quantities from two equations which contain two unknown quantities can be effected in different ways.

To explain these, let

$$ax+by=c \qquad - \quad - \quad - \quad - \quad - \quad - \quad 1$$

$$a'x+b'y=c' \qquad - \quad - \quad - \quad - \quad - \quad 2$$

be two equations which contain the unknown quantities, x, y, and the known quantities, a, a', b, b', c, c', which may be either additive or subtractive.

Assuming that there are two numbers which, being substituted for x, y, satisfy equations 1, 2, and considering x, y as the representatives of these numbers, the equations 1, 2 may be regarded as equalities.

Now these equations give $x=\frac{c-by}{a}$ - - - - 3

$$x=\frac{c'-b'y}{a'} \qquad - \quad - \quad - \quad - \quad 4$$

The two values of x must be equal; whence

$$\frac{c-by}{a}=\frac{c'-b'y}{a'} \qquad - \quad - \quad - \quad - \quad 5$$

Thus an equation of the first degree, which ought to be satisfied by the quantity y, is obtained. The value of y can consequently be deduced from equation 5, by Article 49.

For, removing the denominators $a'c-a'by=ac'-ab'y$,

$$\text{and transposing } ab'y-a'by=ac'-a'c,$$
$$\text{or } (ab'-a'b)y=ac'-a'c,$$
$$\therefore y=\frac{ac'-a'c}{ab'-a'b}.$$

Substituting this value of y in either of the equations 3, 4,

$$x=\frac{c-by}{a}=\frac{c}{a}-\frac{b}{a}y \text{ becomes}$$
$$x=\frac{c}{a}-\frac{b}{a}\times\frac{ac'-a'c}{ab'-a'b},$$
$$\text{or } x=\frac{ab'c-a'bc-abc'+a'bc}{aab'-aa'b},$$
$$\text{or } x=\frac{a(b'c-bc')}{a(ab'-a'b)}=\frac{b'c-bc'}{ab'-a'b}.$$

It is indifferent which of the equations 3, 4 is employed to determine the value of x; for the value of y, which is deduced from equation 5, ought to render the two members of equation 5 identical; and the two expressions of the value of x from equations 3 and 4 form the members of equation 5.

Consequently the values of the unknown quantities x, y are

$$x=\frac{b'c-bc'}{ab'-a'b},$$
$$y=\frac{ac'-a'c}{ab'-a'b}.$$

From the manner in which these values have been found, it is certain that they must satisfy the two proposed equations, for they satisfy equations 3 and 4; and as equations 3 and 4, by the removal of their denominators and the transposition of the terms in y, are made to reproduce the proposed equations 1, 2, it follows that these values of x, y must also satisfy equations 1, 2.

Hence the assumption made in commencing the investigation of the values of x and y, namely, that there are values of x, y which satisfy the two equations, is true.

The same conclusion is obtained by replacing x, y by their values in either of the equations 1, 2.

$$\text{For since } ax+by=c,$$
$$\text{by substitution } a.\frac{b'c-bc'}{ab'-a'b}+b.\frac{ac'-a'c}{ab'-a'b}=c,$$
$$\text{therefore } ab'c-abc'+abc'-a'bc=ab'c-a'bc,$$
$$\text{or } ab'c-a'bc=ab'c-a'bc.$$

The result, which is an identical equation, shows that these values of x, y satisfy the equation $ax+by=c$.

An equation involving two unknown quantities, x, y, can always be reduced to the form $ax+by=c$, in which a expresses the algebraic sum of the known quantities by which x is multiplied, b that of the known quantities by which y is multiplied, and c that of the known quantities which are connected with x, y by the signs + and −.

If it is, besides, made a condition that a, b, c, a', b', c', may express subtractive as well as additive quantities, the equations $ax+by=c$, $a'x+b'y=c'$ are rendered perfectly general.

This method of elimination, which is named the method by Comparison, has the disadvantage of rendering both the members of equation 5 fractional, unless a, a' are equal to 1 or to each other.

If $a=a'$, equation 5 becomes $c-by=c'-b'y$.

$$\text{Whence } y=\frac{c'-c}{b'-b},$$
$$x=\frac{b'c-bc'}{ab'-ab}.$$

If $a=a'=1$, the values of x, y are,

$$x=\frac{b'c-bc'}{b'-b}\,;\ y=\frac{c'-c}{b'-b}.$$

If $b=b'$ or $b=b'=1$, it will be more convenient to eliminate y than x from the proposed equations.

51. Resuming the general equations,

$$ax+by=c \quad - \quad - \quad - \quad - \quad - \quad 1$$
$$a'x+b'y=c' \quad - \quad - \quad - \quad - \quad - \quad 2$$

and considering x, y, as before, to represent two numbers which satisfy these equations, if the value of x,

$$x=\frac{c-by}{a} \quad - \quad - \quad - \quad - \quad - \quad 3$$

which is obtained from the first equation, is substituted for x in the second equation, the result is,

$$a'\times\frac{c-by}{a}+b'y=c' \quad - \quad - \quad - \quad 4$$

This equation contains only the unknown quantity y, consequently the value of y can be determined from it.

$$\text{For by Article 49, } a'c-a'by+ab'y=ac',$$
$$ab'y-a'by=ac'-a'c,$$
$$\text{or } (ab'-a'b)y=ac'-a'c,$$
$$\therefore y=\frac{ac'-a'c}{ab'-a'b} \quad - \quad 5$$

Substituting this value instead of y in equation 3, and reducing

$$x=\frac{b'c-bc'}{ab'-a'b} \quad - \quad - \quad - \quad - \quad - \quad - \quad 6$$

These results correspond with the values of x, y, obtained by the method of comparison; but, notwithstanding this coincidence, it is necessary to prove that this method must give values which satisfy the equations proposed.

Now the value $a=\frac{b'c-bc'}{ab'-a'b}$ being obtained by making $y=\frac{ac'-a'c}{ab'-a'b}$ in equation 3, it follows that these values of x, y must satisfy equation 3.

Multiplying by a, and transposing by to the first member, equation 3 becomes equation 1; therefore the values which have been determined for x, y must satisfy equation 1.

Again, the value $y=\frac{ac'-a'c}{ab'-a'b}$ must satisfy equation 4, from which it is deduced.

But in equation 4 the fraction which is multiplied by a (viz. $\frac{c-by}{a}$) is the second member of equation 3. This fraction therefore gives the value of x when $y=\frac{ac'-a'c}{ab'-a'b}$.

Whence it amounts to the same thing to make

$$y=\frac{ac'-a'c}{ab'-a'b} \text{ in equation 4,}$$

$$\text{or to make } \left.\begin{array}{l} x=\dfrac{b'c-bc'}{ab'-a'b} \\ y=\dfrac{ac'-a'c}{ab'-a'b} \end{array}\right\}\text{in equation 2.}$$

Consequently equation 2 is also satisfied by these values.

This method of elimination is termed the method by Substitution. Like the method by comparison it has the disadvantage of introducting fractional expressions into the first equation, which contains only one unknown quantity, unless one of the coefficients a, a', b, $b'=1$. When this happens the method is very convenient.

Suppose that in equation 1, $a=1$,

$$\text{then } x=c-by \quad - \quad - \quad - \quad - \quad 3$$

$$a'(c-by)+b'y=c', \quad - \quad - \quad - \quad 4$$

$$\text{or} \quad a'c-a'by+b'y=c';$$

$$\therefore (b'-a'b)y=c'-a'c,$$

$$y=\frac{c'-a'c}{b'-a'b},$$

$$x=\frac{b'c=bc'}{b'-a'b}.$$

52. If in the general equations

$$ax+by=c \quad - \quad - \quad - \quad 1$$

$$a'x+b'y=c' \quad - \quad - \quad - \quad 2$$

a is equal to a', the unknown quantity x can be eliminated by subtracting the members of one equation from the other, or adding the members of one equation to the other, according as the terms in x are affected with the same or contrary signs.

Under this hypothesis the result obtained is

$$by \mp b'y=c\mp c',$$

$$\text{or } (b\mp b')y=c\mp c',$$

$$\text{and } \therefore y=\frac{c\mp c'}{b\mp b'}.$$

If a is not equal to a', it is evident that the coefficients of x in equations 1, 2 can be rendered equal by multiplying the terms of equation 1 by a', the coefficient of x in equation 2, and the terms of equation 2 by a, the coefficient of x in equation 1.

When the coefficients of the terms in x have been thus rendered equal, the elimination of x is effected in the same manner as in the particular case of $a=a'$.

$$\text{If from } (ax+by=c)\times a', \text{ or } aa'x+a'by=a'c,$$

$$\text{is taken } (a'x+b'y=c')\times a, \text{ or } aa'x+ab'y=ac',$$

$$\text{the remainder is } (a'b-ab')y=a'c-ac';$$

$$\therefore y=\frac{a'c-ac'}{a'b-ab'}.$$

The value of x may be found by substituting this value of y in equation 1 or equation 2, or by eliminating y in the same manner as x has been eliminated; thus,

$$\text{From equation 1 multiplied by } b', \text{ or } ab'x+bb'y=b'c,$$

$$\text{take equation 2 multiplied by } b, \text{ or } a'bx+bb'y=bc';$$

$$\text{therefore } (ab'-a'b)x=b'c-bc',$$

$$\text{and } x=\frac{b'c-bc'}{ab'-a'b}.$$

This method of elimination is sometimes called the method by Reduction, because the coefficients of the same unknown quantity in both equations are reduced to equality; and sometimes the method by Addition and Subtraction, because, when the coefficients of one unknown quantity have been rendered equal in both equations, that unknown quantity is eliminated by taking the sum or the difference of the two equations according as the terms rendered equal are affected with the same or with contrary signs.

As in the preceding methods, it is necessary to prove that the results obtained by the method of reduction satisfy the proposed equations in x, y.

To render the proof general, let equation 1 be multiplied by m, and equation 2 by n:

$$(ax+by=c)\times m=max+mby=mc \tag{3}$$

$$(a'x+b'y=c')\times n=na'x+nb'y=nc' \tag{4}$$

Subtracting $$(ma-na')x+(mb-nb')y=mc-nc' \tag{5}$$

It is to be shown that equation 5, and either of the proposed equations 1, 2, have the same solutions as the equations 1, 2.

For this purpose let the terms of equation 1 be multiplied by m, and let the product be subtracted from equation 5:

$$(ma-na')x+(mb-nb')y=mc-nc'$$

$$max+mby=mc$$

$$\therefore\ -na'x-nb'y=-nc$$

and dividing all the terms by $-n$,

$$a'x+b'y=c',$$

which is equation 2 reproduced.

Therefore, in the same manner as the system of equations 1 and 5 is a consequence of the system 1 and 2, so, reciprocally, the system 1, 2 is a consequence of the system 1, 5; consequently values of x, y which satisfy equations 1, 5, must also satisfy equations 1, 2.

Now, if the multipliers m, n are chosen in such a manner that $ma-na'=0$, and consequently that equation 5 may not contain x; if from this equation (5) the value of y is deduced, and substituted in equation 1, for the purpose of deducing the value of x, it is evident that values of x, y, which satisfy equations 1, 5, must be obtained; therefore these values must satisfy equations 1, 2 also.

When the value of x is determined in the same manner as that of y; to prove that both values satisfy equations 1, 2, suppose that the terms of equation 1 are multiplied by m', the terms of equation 2 by n', and that the second product is subtracted from the first, the result is,

$$(m'a-n'a')x+(m'b-n'b')y=m'c-n'c' \tag{6}$$

Multiplying now the terms of equation 5 by m', the terms of equation 6 by m, and subtracting the second product from the first,

$$(m'ma-m'na')x+(m'mb-m'nb')y=m'mc-m'nc' \tag{7}$$

$$(mm'a-mn'a')x+(mm'b-mn'b')y=mm'c-mn'c' \tag{8}$$

$$(mn'a'-m'na')x+(mn'b'-m'nb')y=mn'c'-m'nc'$$

or, $$(mn'-m'n)a'x+(mn'-m'n)b'y=(mn'-m'n)c' \tag{9}$$

$$\text{or } a'x+b'y=c'.$$

In like manner if equation 5 is multiplied by n', equation 6 by n, and the second result is subtracted from the first,

$$(n'm-nm')ax+(n'm-nm')by=(n'm-nm')c \tag{10}$$

$$\text{or } ax+by=c.$$

Wherefore the pairs of equations 1, 2, and 5, 6, are mutually consequences of each other.

Now the multipliers m, n, m', n', may be so chosen that equation 5 shall not contain x, and equation 6 shall not contain y, it is therefore evident that the values of x, y, obtained from equations 5, 6, must satisfy equations 1, 2.

It has been assumed that the terms in x have the same signs in both of the equations 1, 2, and also the terms in y, and that the elimination must in both instances be made by subtraction. If the signs are different they can be made the same by affecting the factors m, n with contrary signs.

It has also been assumed that $mn'-m'n$ is not equal to zero.

If $mn'-m'n=0$, equations 9, 10 are reduced to $0=0$.

Whence, in replacing equations 1, 2 by equations 5, 6, it is necessary to choose m, n, m', n', in such a manner that $mn'-m'n$ may not be equal to zero, or, which is the same thing, that mn' may not be equal to $m'n$ or $m:m'::n:n'$.

The operation by which the coefficients of the unknown quantity to be eliminated are made equal is analogous to that for the reduction of fractions to the same denominator, and it admits of the same simplifications.

If, for example, the equations are

$$21y+20x=165$$
$$77y-30x=295,$$

the coefficients of y can be decomposed into the products 7×3 and 7×11. The coefficients of y in both equations are consequently rendered equal by multiplying the terms of the first equation by 11, and the terms of the second equation by 3.

In like manner the coefficients of x are rendered equal by multiplying the terms of the first equation by 3, and the terms of the second by 2.

In eliminating by the method of reduction the equation in x alone, or in y alone, is not incumbered with fractional expressions, as it commonly is when the method of comparison or of substitution is employed.

Whichever method is followed, after the elimination of one unknown quantity, the resulting equation evidently ought always to give the same value of the remaining unknown quantity. Hence, if by means of a process of elimination one unknown quantity is got rid of, and the resulting equation is brought to the form $ax=b$ (a and b being known quantities), it is certain that every other process must give either the equation $ax=b$, or an equation differing from it by only a factor common to both its members; otherwise the values of x would not be the same.

53. It is shown in Article 55 that the preceding methods of elimination can be applied to the determination of the values of three unknown quantities from three independent equations of the first degree, four unknown quantities from four equations, &c. The principle of the methods is the same, namely, that each leads to one equation involving one unknown quantity, and, by consequence, admitting of only one value. By the substitution of this value in the equation which involves the unknown symbol whose value has been determined and another unknown quantity, this equation is converted into an equation with one unknown quantity, which also can have only one value. By the substitution of these values in the equation which involves the two quantities whose values have been determined and a third unknown quantity, the single value of this is obtained, and so on. Consequently it follows that there exists only a single system of values which can satisfy two or more independent equations of the first degree involving the same number of unknown quantities, and that this system of values can be determined by any of the three preceding methods.

54. To find the values of two unknown quantities from two independent simple equations involving these quantities,

Rule. Determine, from each of the proposed equations, the value of the same unknown quantity in terms of the data and the other unknown quantity, and put the one value equal to the other;

Or, find, from one of the proposed equations, the value of one unknown quantity in terms of the data and the other unknown quantity, and substitute this value for its symbol in the other equation;

Or, multiply the terms of the first of the proposed equations by the coefficient of either of the unknown quantities in the second equation, the terms of the second equation by the coefficient of the same unknown quantity in the first equation, and take the difference or the sum of the results according as the coefficients of the unknown quantity which is to be eliminated have the same or different signs.

An equation of the first degree involving only one unknown quantity is obtained by each of the methods.

From this equation determine the value of the remaining unknown quantity, and substitute it in either of the proposed equations; the two results are the values of the two unknown quantities in these equations.

Find the values of x and y in each of the following examples:

1st. $$\left.\begin{array}{r} x+y=15 \\ x-y=7 \end{array}\right\} \quad x=11,\ y=4.$$

2d. $$\left.\begin{array}{r} x+y=10 \\ 2x-3y=5 \end{array}\right\} \quad x=7,\ y=3.$$

3d. $$\left.\begin{array}{r} 2x+3y=13 \\ 5x+4y=22 \end{array}\right\} \quad x=2,\ y=3.$$

4th. $$\left.\begin{array}{r} x=4y \\ 2x+3y=44 \end{array}\right\} \quad x=16,\ y=4.$$

5th. $$\left.\begin{array}{r} 2x+3y=70 \\ 4x+5y=130 \end{array}\right\} \quad x=20,\ y=10.$$

6th. $$\left.\begin{array}{r} 3x-5y=13 \\ 2x+7y=81 \end{array}\right\} \quad x=16,\ y=7.$$

7th. $$\left.\begin{array}{r} 12x+32y=340 \\ 8x+24y=254 \end{array}\right\} \quad x=26,\ y=10\tfrac{1}{4}.$$

8th. $$\left.\begin{array}{l} \frac{2x}{3}+\frac{4y}{5}=64 \\ \frac{5x}{6}+\frac{9y}{10}=77 \end{array}\right\} \quad x=60,\ y=30.$$

9th. $$\left.\begin{array}{l} \frac{3x-5y}{2}+3=\frac{2x+y}{5} \\ 8-\frac{x-2y}{4}=\frac{x}{2}+\frac{y}{3} \end{array}\right\} \quad x=12,\ y=6.$$

10th. $$\left.\begin{array}{l} x-\frac{y-2}{7}=5 \\ 4y-\frac{x+10}{3}=3 \end{array}\right\} \quad x=5,\ y=2.$$

11th. $$\left.\begin{array}{l} 2x-\frac{y+3}{4}=7+\frac{3y-2x}{5} \\ 4y-\frac{8-x}{3}=24\tfrac{1}{2}-\frac{2y+1}{2} \end{array}\right\} \quad x=5,\ y=5.$$

12th. $$\left.\begin{array}{l} 5+\frac{y-8}{5}=\frac{3x+4y+3}{10}-\frac{2x+7-y}{15} \\ \frac{7x+6}{11}=\frac{9y+5x-8}{12}-\frac{x+y}{4} \end{array}\right\} \quad x=7,\ y=9.$$

13th. $$\left.\begin{array}{l} (x+5)\ (y+7)=(x+1)\ (y-9)+112 \\ 2x+10=3y+1 \end{array}\right\} \quad x=3,\ y=5.$$

14th. $$\left.\begin{array}{l} \frac{2x}{3}-4+\frac{y}{2}+x=8-\frac{3y}{4}+\frac{1}{12} \\ \frac{y}{6}-\frac{x}{2}+2=\frac{1}{6}-2x+6 \end{array}\right\} \quad x=2,\ y=7.$$

15th. $$\left.\begin{array}{l} x-\frac{3x+5y}{17}+17=5y+\frac{4x+7}{3} \\ \frac{22-6y}{3}-\frac{5x-7}{11}=\frac{x+1}{6}-\frac{8y+5}{18} \end{array}\right\} \quad x=8,\ y=2.$$

16th. $$\left.\begin{array}{l}x+y=s\\x-y=d\end{array}\right\}\quad x=\frac{s+d}{2},\ y=\frac{s-d}{2}.$$

17th. $$\left.\begin{array}{l}x+y=s\\bx=ay\end{array}\right\}\quad x=\frac{as}{a+b},\ y=\frac{bs}{a+b}.$$

18th. $$\left.\begin{array}{l}x+y=a\\bx+cy=de\end{array}\right\}\quad x=\frac{ca-de}{c-b},\ y=\frac{de-ba}{c-b}.$$

19th. $$\left.\begin{array}{l}ax+by=c\\fx+gy=h\end{array}\right\}\quad x=\frac{cg-bh}{ag-bf},\ y=\frac{ah-cf}{ag-bf}.$$

20th. $$\left.\begin{array}{l}ax+by=c\\mx-ny=d\end{array}\right\}\quad x=\frac{nc+bd}{na+mb},\ y=\frac{mc-ad}{na+mb}.$$

21st. $$\left.\begin{array}{l}7ax=4b\\2cx+3dy=4e\end{array}\right\}\quad x=\frac{4b}{7a},\ y=\frac{28ae-8bc}{21ad}.$$

22d. $$\left.\begin{array}{l}bcx=cy-2b\\b^2y+\dfrac{a(c^3-b^3)}{bc}=\dfrac{2b^3}{c}+c^3x\end{array}\right\}\quad x=\frac{a}{bc},\ y=\frac{a+2b}{c}.$$

23d. $$\left.\begin{array}{l}\dfrac{a}{b+y}=\dfrac{b}{3a+x}\\ax+2by=d\end{array}\right\}\quad x=\frac{2b^2-6a^2+d}{3a},\ y=\frac{3a^2-b^2+d}{3b}.$$

24th. $$\left.\begin{array}{l}x-\dfrac{a+y}{b}=c\\y-\dfrac{a-x}{b}=d\end{array}\right\}\quad x=\frac{a-ab+b^2c+bd}{b^2+1},\ y=\frac{a+ab-bc+b^2d}{b^2+1}.$$

OF EQUATIONS OF THE FIRST DEGREE INVOLVING THREE OR MORE UNKNOWN QUANTITIES.

55. Let there be three unknown quantities, x, y, z, and three equations,

$$ax+by+cz=d \qquad - \quad - \quad 1$$
$$a'x+b'y+c'z=d' \qquad - \quad - \quad 2$$
$$a''x+b''y+c''z=d'' \qquad - \quad - \quad 3$$

And let it be required to assign the values of x, y, z, in terms of the given quantities a, a', a''; b, b' b''; c, c', c''; d, d', d''.

z may be eliminated between equations 1, 2 and equations 1, 3 by one of the methods (Art. 54). Adopting the method by reduction, the equation in x, y, obtained from equations 1, 2, is,

$$(ac'-ca')x+(bc'-cb')y=dc'-cd' \qquad - \quad - \quad 4$$

and from equations 1, 3,

$$(ac''-ca'')x+(bc''-cb'')y=dc''-cd'' \qquad - \quad - \quad 5$$

The values of x, y ought to satisfy equations 4 and 5.

To eliminate y from equations 4, 5, it is necessary to multiply the terms of equation 4 by $bc''-cb''$, and the terms of equation 5 by $bc'-cb'$, and to subtract the second product from the first.

The expression of the remainder (the multiplication being merely indicated) is,

$$\Big\{(ac'-ca')(bc''-cb'')-(ac''-ca'')(bc'-cb')\Big\}x=(dc'-cd')(bc''-cb'')$$
$$-(dc''-cd'')(bc'-cb').$$

Performing the multiplications, reducing the like terms, and dividing every term by the common factor c,

$$(ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a'')x=db'c''-dc'b''+cd'b''-bd'c''$$
$$+bc'd''-cb'd''.$$

$$\therefore\ x=\frac{db'c''-dc'b''+cd'b''-bd'c''+bc'd''-cb'd''}{ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a''} \qquad 6$$

Eliminating x, z; x, y, in the same manner,

$$y=\frac{ad'c''-ac'd''+ca'd''-da'c''+dc'a''-cd'a''}{ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a''} \quad - \quad 7$$

$$z=\frac{ab'd''-ad'b''+da'b''-ba'd''+bd'a''-db'a''}{ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a''} \quad - \quad 8$$

From the investigation of Article 52 it follows that these values of x, y, z, must satisfy equations 1, 4, 5.

Now equation 2 is a consequence of equations 1, 4, and equation 3 is a consequence of equations 1, 5.

Therefore the values of x, y, z, which satisfy equations 1, 4, 5, must also satisfy equations 1, 2, 3.

By generalising the process which has been employed in the case of three equations, a rule for the resolution of a number of equations m, involving m unknown quantities, may be found.

The rule is: Eliminate one unknown quantity between the first and each of the remaining $m-1$ equations; $m-1$ equations of the first degree, involving $m-1$ unknown quantities, are thus obtained.

Proceed, with the $m-1$ equations, to eliminate one unknown quantity between the first of the $m-1$ and each of the remaining $m-2$ equations; $m-2$ equations of the first degree, involving $m-2$ unknown quantities, are thus obtained.

Continue this series of operations with the $m-2$, the $m-3$ . . . . ., the $m-(m-1)$ equations; an equation of the first degree involving only one unknown quantity is the last result.

From the last equation deduce the value of the unknown quantity which it contains.

Substitute this value in the last equation but one; the value of a second unknown quantity is determined.

Substitute these values in the last equation but two; the value of a third unknown quantity is obtained; and by returning thus through the equations upwards to the $(m-1)$th, and lastly to the mth, the values of all the unknown quantities are determined.

Example 1. Given $10x-20y+30z=60$,

$$8x+12y-16z=80,$$

$$27x-18y+45z=234,$$

to find the values of x, y, z.

In the first of these equations the factor 10 is common to all the terms; also, in the second the factor 4, and in the third the factor 9, is common to all the terms.

Suppressing the common factors, the proposed equations are reduced to

$$x-2y+3z=6 \quad - \quad - \quad - \quad - \quad 1$$

$$2x+3y-4z=20 \quad - \quad - \quad - \quad - \quad 2$$

$$3x-2y+5z=26 \quad - \quad - \quad - \quad - \quad 3$$

Replacing, in the general formula,

a, b, c, d, by 1,−2, 3, 6,
a', b', c', d', by 2, 3,−4, 20,
a'', b'', c'', d'', by 3,−2, 5, 26,

and performing the operations indicated,

$$x=\frac{90-48+(-120)-(-200)+208-234}{15-8+(-12)-(-20)+24-27}=\frac{96}{12}=8$$

$$y=\frac{100-(-104)+156-60+(-72)-180}{15-8+(-12)-(-20)+24-27}=\frac{48}{12}=4$$

$$z=\frac{78-(-40)+(-24)-(-104)+(-120)-54}{15-8+(-12)-(-20)+24-27}=\frac{24}{12}=2$$

Or the values of the unknown quantities may be found by the general rule, as follows:

$$\begin{array}{lllr}
\text{Multiply 1st equation by 4, and} & 4x-8y+12z=24 & & \\
\quad\text{- 2d - by 3, -} & \underline{6x+9y-12z=60} & & \\
\text{Therefore, by addition - -} & 10x+y \quad =84 & \text{-} & 4 \\
\text{Multiply 1st equation by 5, and} & 5x-10y+15z=30 & & \\
\quad\text{- 3d - by 3, -} & \underline{9x-6y+15z=78} & & \\
\text{Therefore, by subtraction - -} & 4x+4y \quad =48 & & \\
\qquad\qquad\text{or - -} & x+y \quad =12 & \text{-} & 5
\end{array}$$

$$\begin{array}{ll}
\text{From equation 4 -} & 10x+y=84 \\
\text{subtract equation 5 -} & \underline{x+y=12} \\
\therefore & 9x \quad =72, \text{ and } x=\tfrac{72}{9}=8.
\end{array}$$

Substituting this value of x in equation 5,

$$8+y=12, \text{ and therefore } y=12-8=4.$$

Substituting these values of x, y in the 1st equation,

$$8-8+3z=6\text{; therefore } 3z=6, \text{ and } z=\tfrac{6}{3}=2.$$

Consequently $x=8$, $y=4$, $z=2$ are the values of x, y, z in the proposed equations.

56. If all the unknown quantities do not enter into each equation, the process of elimination, though conducted by the same principles, can be abridged.

$$\begin{array}{llr}
\text{Example. Let} & 7u-13z=87 \quad \text{- - -} & 1 \\
 & 3u+14x=57 \quad \text{- - -} & 2 \\
 & 10y-3x=11 \quad \text{- - -} & 3 \\
 & 2x-11z=50 \quad \text{- - -} & 4
\end{array}$$

If u is eliminated between the first and second equations, the resulting equation will contain only x, z and known quantities; consequently from this and the fourth equation, which also contains only x, z and known quantities, the values of x, z can be determined.

$$\begin{array}{lllr}
\text{Since } 7u-13z=87 & \text{- -} & 21u-39z=\ 261 & \\
\text{and since } 3u+14x=57 & \text{- -} & \underline{21u+98x=\ 399} & \\
\text{Therefore, by subtraction} & \text{- -} & 98x+39z=\ 138 & \text{-}\ 5 \\
\left.\begin{array}{l}\text{Multiply equation 4 by 49, and}\\ \text{subtract the result from equation 5}\end{array}\right\} & & \underline{98x-539z=2450} & \\
 & & \underline{578z=-2312} &
\end{array}$$

$$\therefore z=-\frac{2312}{578}=-4$$

Substituting this value of z in equation 4,

$$2x-11z=50 \text{ becomes } 2x-11\times(-4)=50,$$
$$\therefore 2x+44=50,\ 2x=50-44=6, \text{ and } x=\tfrac{6}{2}=3.$$

Substituting 3 for x in equation 2,

$$3u+14x=57 \text{ becomes } 3u+14\times3=57, \text{ or } 3u+42=57,$$
$$\therefore 3u=57-42=15, \text{ and } u=\tfrac{15}{3}=5.$$

Lastly, substituting 3 for x in equation 3,

$$10y-3x=11 \text{ becomes } 10y-3\times3=11, \text{ or } 10y-9=11$$
$$\therefore 10y=11+9=20, \text{ and } y=\tfrac{20}{10}=2.$$

Whence $u=5$, $y=2$, $x=3$, $z=-4$, are the values of u, y, x, z, which satisfy the proposed equations.

57. In resolving equations of the first degree, if the number of equations is equal to the number of unknown quantities, it follows, from Article 53, that there is one system of values of the unknown quantities, and only one, which satisfies the proposed equations.

This general conclusion is to be taken with some exceptions, which are noticed in Article 80.

If the number of equations is greater than the number of unknown quantities, (if, for example, there are five equations between three unknown quantities, x, y, z,) taking three of these equations, a single system of values can be determined for x, y, z. These values being found, it will be requisite to try whether they verify the two remaining equations also. Now, unless these equations have been chosen in a particular manner, the verification cannot be made, and consequently there do not exist values of the unknown quantities which can at once verify all the five equations.

If, on the other hand, there are more unknown quantities than equations, (for example, five unknown quantities, u, v, x, y, z, and three equations,) arbitrary values may be given to two of the unknown quantities, u, v; then, by means of the three equations, the corresponding values of x, y, z may be determined.

Changing the values of u, v, another system of values may be found for x, y, z; and by thus changing the arbitrary values of u, v, an indefinite number of values may be obtained for x, y, z;

Or, without assigning particular values to u, v, the three equations may be resolved as if u, v were known quantities, and x, y, z unknown.

In this manner values of x, y, z, involving the quantities u, v, are obtained; and from these can be deduced those values of x, y, z which correspond to particular values of u, v.

Find the values of the unknown quantities in each of the following examples:

1st. $$\left.\begin{array}{r}2x+3y+4z=16\\3x+2y-5z=8\\5x-6y+3z=6\end{array}\right\}\quad x=3,\ y=2,\ z=1.$$

2d. $$\left.\begin{array}{r}7x+5y+2z=79\\8x+7y+9z=122\\x+4y+5z=55\end{array}\right\}\quad x=4,\ y=9,\ z=3.$$

3d. $$\left.\begin{array}{r}2x+3y+4z=160\\5x-6y+7z=100\\8x-9y+10z=160\end{array}\right\}\quad x=30,\ y=20,\ z=10.$$

4th. $$\left.\begin{array}{r}x-2y+3z=3\\2x-3y-9z=1\\3x-5y-4z=6\end{array}\right\}\quad x=20,\ y=10,\ z=1.$$

5th. $$\left.\begin{array}{l}2x-\frac{3y}{4}=93-\frac{1}{2}x-\frac{1}{4}y\\7x-5z=y+x-86\\\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=58\end{array}\right\}\quad x=48,\ y=54,\ z=64.$$

6th. $$\left.\begin{array}{r}6x-4y+5z=2\frac{11}{12}\\4x+3y-7z=1\frac{1}{4}\\12x-6y-3z=3\frac{1}{4}\end{array}\right\}\quad x=\tfrac{1}{2},\ y=\tfrac{1}{3},\ z=\tfrac{1}{4}.$$

7th. $$\left.\begin{array}{r}18x-7y-5z=11\\4\frac{2}{5}y-\frac{2}{3}x+z=108\\3\frac{1}{2}z+2y+\frac{3}{4}x=80\end{array}\right\}\quad x=12,\ y=25,\ z=6.$$

8th. $$\left.\begin{array}{l}y+\frac{z}{3}=\frac{x}{5}+5\\\frac{x-1}{4}-\frac{y-2}{5}=\frac{z+3}{10}\\x-\frac{2y-5}{3}=1\frac{3}{4}-\frac{z}{12}\end{array}\right\}\quad x=5,\ y=7,\ z=-3.$$

9th. $$\left.\begin{array}{r}10x-9y=550\\8x-7z=625\\6y-5z=175\end{array}\right\}\quad x=100,\ y=50,\ z=25.$$

10th. $$\left.\begin{array}{l}x+y=10\\x+z=19\\y+z=23\end{array}\right\}\quad x=3,\ y=7,\ z=16.$$

11th. $$\left.\begin{array}{r}3x+5y=161\\7x+2z=209\\2y+z=89\end{array}\right\}\quad x=17,\ y=22,\ z=45.$$

12th. $$\left.\begin{array}{r}53-\frac{1}{2}x-\frac{1}{2}z=y-109\\\frac{1}{4}x+\frac{1}{8}y=26\\5y=4z\end{array}\right\}\quad x=64,\ y=80,\ z=100.$$

13th. $$\left.\begin{array}{r}x+y+z=a\\my=nx\\pz=qx\end{array}\right\}x=\frac{amp}{mp+np+mq},\ y=\frac{anp}{mp+np+mq},\ z=\frac{amq}{mp+np+mq}.$$

14th. $$\left.\begin{array}{r}ax+by=c\\dn+ey=f\\gy+hz=l\end{array}\right\}x=\frac{ce-bf}{ae-bd},\ y=\frac{af-cd}{ae-bd},\ z=\frac{a(el-fg)-d(bl-cg)}{h(ae-bd)}.$$

15th. $$\left.\begin{array}{r}x+2y+3z+4u=20\\5x-6y+7z-8u=8\\9x-10y-11z+12u=-4\\13x+14y-15z-16u=48\end{array}\right\}\quad u=1,\ x=4,\ y=3,\ z=2.$$

16th. $$\left.\begin{array}{r}x+y+z+u=1\\16x+8y+4z+2u=9\\81x+27y+9z+3u=36\\256x+64y+16z+4u=100\end{array}\right\}\quad x=\frac{1}{4},\ y=\frac{1}{2},\ z=\frac{1}{4},\ u=0.$$

17th. $$\left.\begin{array}{r}\frac{x}{3}+\frac{y}{5}+\frac{2z}{7}=58\\\frac{5x}{4}+\frac{y}{6}+\frac{z}{3}=76\\\frac{x}{2}+\frac{3z}{8}+\frac{u}{5}=79\\y+z+u=248\end{array}\right\}\quad u=50,\ x=12,\ y=30,\ z=168.$$

18th. $$\left.\begin{array}{r}ax+by=l\\cx+du=m\\ex+fz=n\\gy+hz=p\end{array}\right\}\quad u=\frac{bh(em-cn)+gf(am-cl)+bcfp}{d(beh+afg)},$$

$$x=\frac{bhn+f(gl-bp)}{beh+afg},\ y=\frac{afp+h(el-an)}{beh+afg},\ z=\frac{bep+g(an-el)}{beh+afg}.$$

19th. $$\left.\begin{array}{r}x+u=a\\x+y=b\\x+z=c\\y+z=d\end{array}\right\}\quad u=\frac{2a-b-c+d}{2},\ x=\frac{b+c-d}{2},\ y=\frac{b-c+d}{2},\ z=\frac{c+d-b}{2}.$$

20th. $$\left.\begin{array}{r}u+x+y+z=s\\au=bx\\cy=dx\\ez=fx\end{array}\right\}\quad x=\frac{s\times ace}{ace+bce+ade+acf},\ u=\frac{s\times bce}{ace+bce+ade+acf},$$

$$y=\frac{s\times bde}{ace+bce+ade+acf},\ z=\frac{s\times bdf}{ace+bce+ade+acf}.$$

58. The enunciation of a problem ought to state the conditions which the unknown quantities are required to satisfy in such a manner that, if particular values are given to the unknown quantities, it can be readily determined whether these particular values verify the conditions.

In most of the questions whose solutions require the employment of the algebraic calculus the verification consists in this, that after the execution of certain operations on the values of the unknown quantities and on the given quantities the results ought to be certain equalities.

Now if the unknown quantities are denoted by letters, algebraic expressions can be formed, with which, by means of symbols, can be indicated all the calculations which it is necessary to make both with the unknown and with the given quantities for the purpose of finding the expressions which ought to be equal.

Consequently, if these expressions are connected by the sign $=$, there are obtained one or more equations which ought to be satisfied when the true values of the unknown quantities are introduced into them instead of the letters.

Reciprocally, when all the conditions of a problem are expressed in the equations, the values of the unknown quantities which satisfy these equations ought to satisfy also the enunciation of the problem.

Hence the conventions of algebra form a language into which it is possible to translate the enunciation of a problem, and to represent with precision all the relations of magnitude which the given and the unknown quantities have to each other. To establish these relations by means of equations is what is meant by the expressions, to put a problem into equation, or to translate it into the language of algebra.

A rule as precise as the subject seems to admit, for putting problems into equation, has been already given, Art. 46.

It frequently happens that a question which at first appears to involve many unknown quantities can be resolved by means of a smaller number, and sometimes by means of only one unknown quantity.

The last case occurs when it appears that if one of the unknown quantities is found, the others can be obtained from it by simple operations, as by an addition, a subtraction, &c.

PROBLEMS WHICH PRODUCE EQUATIONS OF THE FIRST DEGREE.

59. 1st. The sum of two numbers is 50, and the difference 10; required the numbers?

Let x denote the greater number.

Then, since the difference of the numbers is 10, $x-10$ represents the less.

Therefore $x+x-10$, or $2x-10$, is the sum of the numbers.

But by the question this sum is 50;

$$\therefore 2x-10=50.$$

$$\text{Transposing, } 2x=50+10=60;$$

$$\therefore x=\frac{60}{2}=30 \text{ the greater number.}$$

$$\text{Consequently, } x-10=30-10=20=\text{the less.}$$

$$\text{Verification. } 30+20=50;\ 30-20=10.$$

Otherwise, by employing two unknown quantities.

Let x denote the greater number, y the less.

$$\text{Therefore, by the question} \quad x+y=50$$
$$x-y=10.$$

$$\text{Whence, by addition,} \quad 2x=60 \text{ and } x=30;$$
$$\text{by subtraction, } 2y=40 \text{ and } y=20.$$

This problem may be enunciated more generally as follows:

The sum of two numbers is s, and the difference d; required the numbers?

Making, as before, $x=$the greater number,

$x-d=$the less,

$x+x-d$, or $2x-d=$the sum of the numbers;

$\therefore$ by the question, $2x-d=s$;

$$\therefore 2x=s+d,$$

$$\text{and } x=\frac{s+d}{2}=\frac{s}{2}+\frac{d}{2}=\text{the greater number;}$$

$$\text{also } x-d=\frac{s+d}{2}-d=\frac{s-d}{2}=\frac{s}{2}-\frac{d}{2}=\text{the less.}$$

Hence this theorem; half the sum of two unequal numbers augmented by half the difference of the numbers is equal to the greater number; and half the sum, diminished by half the difference, is equal to the less number.

Substituting 50 and 10, the numbers of the particular case of this problem, for s and d in the formula,

$$x=\frac{s}{2}+\frac{d}{2},\ x-d=\frac{s}{2}-\frac{d}{2};$$

$$\frac{50}{2}+\frac{10}{2}=25+5=30=\text{the greater number};$$

$$\frac{50}{2}+\frac{10}{2}=25-5=20=\text{the less number}.$$

60. 1st. A person has a certain number of fourpenny pieces and twice as many shillings; the whole value being 35 £., what is the number of each sort of coin?

Let $x=$ the number of fourpenny pieces, and therefore $2x$ the number of shillings.

The value in shillings of x fourpenny pieces and $2x$ shillings is $\frac{x}{3}+2x$, or $\frac{7x}{3}$ sh.; also 35 £.=700 sh.

Therefore, by the question,

$$\frac{7x}{3}=700,$$

and $x=300=$number of fourpenny pieces;

$\therefore 2x=600=$number of shillings.

Verification. 300 fourpenny pieces=100 sh.= 5 £.

600 sh.=30 £.

Whence the value of both is - 35 £.

2d. A sum of 1200 £. is to be so divided between A and B that A's share shall be to B's as 2 to 5; what sum ought each to receive?

Let $x=$ the share of A, and $y=$ the share of B.

Then, by the question, $x+y=1200$.........(1),

and $x:y::2:5$. Whence (Part I. Art. 332), $2y=5x$ and $y=\frac{5x}{2}$.

Substituting $\frac{5x}{2}$ for y in eq. 1, $x+\frac{5x}{2}=1200$;

$\therefore 2x+5x=2400$ or $7x=2400$;

and $x=\frac{2400}{7}=342\frac{6}{7}$ £.=342 £. 17 sh. $1\frac{5}{7}$ d.=A's share.

$y=\frac{5x}{2}=342\frac{6}{7}$ £.$\times\frac{5}{2}=857\frac{1}{7}$ £.=857 £. 2 sh. $10\frac{2}{7}$ d.=B's share.

Verification. $342\frac{6}{7}+857\frac{1}{7}=1200$ £.;

also, $342\frac{6}{7}:857\frac{1}{7}::2:5$;

for $342\frac{6}{7}\times5=1714\frac{2}{7}$,

and $857\frac{1}{7}\times2=1714\frac{2}{7}$.

3d. Divide the given number, a, into two parts, which shall be to each other in the proportion of the given numbers m and n.

Let x denote the first part and y the second.

Then, by the conditions of the question, $x+y=a$....(1),

and $x:y::m:n$. Whence, $y=\frac{nx}{m}$.

Substituting $\frac{nx}{m}$ for y in eq. 1, $x+\frac{nx}{m}=a$.

$\therefore mx+nx=ma$,

and $x=\frac{ma}{m+n}=$the first part.

$$y=\frac{n}{m}x=\frac{n}{m}\times\frac{ma}{m+n}=\frac{na}{m+n}=\text{the second part.}$$

Verification. $\frac{ma}{m+n}+\frac{na}{m+n}=\frac{(m+n)a}{m+n}=a.$

$$\frac{ma}{m+n}:\frac{nn}{m+n}::m:n.$$

If 700 is substituted in the general formula for a, $\frac{1}{3}$ for m, and 2 for n,

$$x=\frac{\frac{1}{3}\times 700}{\frac{1}{3}+2}=\frac{\frac{1}{3}\times 700}{\frac{7}{3}}=100\text{ sh.}=300\text{ fourpences};$$

$$y=\frac{2\times 700}{\frac{1}{3}\times 2}=\frac{1400}{\frac{7}{3}}=600\text{ sh.};$$

the answer to the first problem of this Article.

And if 1200 is substituted for a, 2 for m, and 5 for n,

$$x=\frac{2\times 1200}{2+5}=\frac{2400}{7}=342\tfrac{6}{7}\text{ £.};$$

$$y=\frac{5\times 1200}{2+5}=\frac{6000}{7}=857\tfrac{1}{7}\text{ £.};$$

the answer to the second problem.

61. 1st. Two partners bought a horse, part of the price of which they paid in ready money, the one contributing the fifth and the other the seventh part of the whole price. The sum contributed by both being 48 £., what was the price of the horse?

Let x £.=the price of the horse.

Then $\frac{x}{5}$=the part contributed by one partner;

and $\frac{x}{7}$=the part contributed by the other;

$\therefore \frac{x}{5}+\frac{x}{7}$ or $\frac{12x}{35}$=the part of the price contributed by both;

and by the question, $\frac{12x}{35}=48$;

$$\therefore 12x=48\times 35=1680;$$

and $x=140$ £.=price of the horse.

Verification. $\frac{140}{5}+\frac{140}{7}=28+20=48$ £.

2d. It is required to find a number which, being divided by m and by n, the sum of the quotients shall be equal to a given number, a.

Let x represent the number.

Then $\frac{x}{m}$ and $\frac{x}{n}$ are the quotients obtained by dividing x by m and n;

and by the question, $\frac{x}{m}+\frac{x}{n}=a$;

$$\therefore nx+mx=mna,\text{ or }(m+n)x=mna;$$

and $x=\frac{mna}{m+n}$.

Verification. $\frac{mna}{m+n}\div m=\frac{na}{m+n}$=quotient by m.

$\frac{mna}{m+a}\div n=\frac{ma}{m+n}$=quotient by n.

$$\frac{na}{m+n}+\frac{ma}{m+n}=\frac{ma+na}{m+n}=\frac{(m+n)a}{m+n}=a.$$

If in the formula of problem 2, 48 is substituted for a, and 5 and 7 for m and n respectively,

$x=\frac{5\times 7\times 48}{5+7}=\frac{1680}{12}=140$=the price of the horse in the first problem of this Article.

62. 1st. Divide the number 92 into two parts, such that if the one part is divided by 14 and the other part by 6, the sum of the quotients may be 10.

Let x=one of the parts, and $\frac{x}{14}$ its quotient by 14.

Then $92-x$=the other part, and $\frac{92-x}{6}$ its quotient by 6;

$$\text{also, } \frac{x}{14}+\frac{92-x}{6}=\frac{322-2x}{21}=\text{the sum of the quotients.}$$

$$\text{Therefore, by the question, } \frac{322-2x}{21}=10;$$

$$322-2x=210;$$

$$\text{and } x=\frac{112}{2}=56=\text{one of the parts};$$

$$92-x=92-56=36=\text{the other part.}$$

Verification. $56\div14=4$; $36\div6=6$; and $4+6=10$.

2d. Divide the given number, a, into two parts, such that if the first part is divided by m, and the second part by n, the sum of the quotients shall be equal to a given number, b.

Let x denote the first part, and $\frac{x}{m}$ the quotient of the first part by m.

Then $a-x$ is the second part, and $\frac{a-x}{n}$ its quotient by n;

$$\text{also } \frac{x}{m}+\frac{a-x}{n}=\frac{(n-m)x+ma}{mn}=\text{the sum of the quotients.}$$

$$\text{Whence, by the question, } \frac{(n-m)x+ma}{mn}=b;$$

$$\therefore\ (n-m)x=mnb-ma;$$

$$\text{and } x=\frac{m(nb-a)}{n-m}=\text{ the first part};$$

$$a-x=a-\frac{m(nb-a)}{n-m}=\frac{n(a-mb)}{n-m}=\text{the second part.}$$

$$\text{Verification. } \frac{m(nb-a)}{n-m}\div m=\frac{nb-a}{n-m};\ \frac{n(a-mb)}{n-m}\div n=\frac{a-mb}{n-m};$$

$$\text{and } \frac{nb-a}{n-m}+\frac{a-mb}{n-m}=\frac{nb-mb}{n-m}=b.$$

Replacing a by 92, m by 14, n by 6, and b by 10,

$$\left.\begin{aligned} x&=\frac{14(6\times10-92)}{6-14}=\frac{-32\times14}{-8}=56.\\ a-x&=\frac{6(92-14\times10)}{6-14}=\frac{-48\times6}{-8}=36.\end{aligned}\right\}\text{The answers to the first problem of this Article.}$$

63. 1st. In a garrison of 5200 men, consisting of infantry, artillery, and cavalry, the number of infantry is nine times and that of artillery three times greater than the number of cavalry. How many of each corps are there?

Let x=the number of cavalry,
and therefore $3x$=the number of artillery,
$9x$=the number of infantry.
Whence $x+3x+9x=13x$=the whole force.
Therefore, by the question, $13x=5200$.
$\therefore\ x=400$=number of cavalry.
$3x=1200$=number of artillery.
$9x=3600$=number of infantry.
Verification. $400+1200+3600=5200$.

2d. Divide the given number, a, into three parts, such that the second part shall be equal to m times and the third to n times the first part.

Let the first part $=x$; and $\therefore$ $mx=$the second; $nx=$the third;

$\therefore$ $x+mx+nx$, or $(m+n+1)x=$the sum of the parts.

Whence, by the question, $(m+n+1)x=a$.

$$\therefore \quad x=\frac{a}{m+n+1}=\text{the first part};$$

$$mx=\frac{ma}{m+n+1}=\text{the second part};$$

$$nx=\frac{na}{m+n+1}=\text{the third part}.$$

Verification. The sum of the three parts is a.

Let $a=5200$; $m=3$; $n=9$. Then,

$$x=\frac{5200}{3+9+1}=\frac{5200}{13}=400=\text{number of cavalry in problem 1 of this Article};$$

$$mx=\frac{5200\times 3}{3+9+1}=400\times 3=1200=\text{number of artillery};$$

$$nx=\frac{5200\times 9}{3+9+1}=400\times 9=3600=\text{number of infantry}.$$

64. 1st. Three regiments, A, B, C, are required to furnish 594 men for a particular service, the number to be furnished by each being in proportion to its strength. Now the number of men in A is to that in B as 3 to 5, and the number of men in B is to the number in C as 8 to 7. How many men must each regiment supply?

Let a represent the number of men furnished by A.

$$\text{Then } 3:5::x:\frac{5x}{3}=\text{number of men furnished by B};$$

$$\text{and } 8:7::\frac{5x}{3}:\frac{35x}{24}=\text{number of men furnished by C}.$$

$$\text{The sum of these numbers is } x+\frac{5x}{3}+\frac{35x}{24}, \text{ or } \frac{33x}{8}.$$

$$\text{Therefore, by the question, } \frac{33x}{8}=594.$$

Whence $33x=4752$.

And $x=144=$number of men furnished by A;

$\frac{5x}{3}=240=$number of men furnished by B;

$\frac{35x}{24}=210=$number of men furnished by C.

Verification. $144+240+210=594$.

2d. It is required to divide a given number, a, into three parts, such that the first shall be to the second as m to n, and the second to the third as p to q.

Let $x=$the first part.

$$\text{Then } m:n::x:\frac{nx}{m}=\text{the second part};$$

$$\text{and } p:q::\frac{nx}{m}:\frac{nqx}{mp}=\text{the third part}.$$

$\therefore x+\frac{nx}{m}+\frac{nqx}{mp}$ or $\frac{mpx+npx+nqx}{mp}$ or $\frac{(mp+np+nq)x}{mp}=$ the sum of the parts.

$$\text{Whence, by the question, } \frac{(mp+np+nq)\,x}{mp}=a.$$

$$\text{And } x=\frac{mpa}{mp+np+nq}=\text{ the first part;}$$

$$\therefore \frac{nx}{m}=\frac{npa}{mp+np+nq}=\text{ the second part;}$$

$$\frac{nqx}{mp}=\frac{nqa}{mp+np+nq}=\text{ the third part.}$$

Verification. The sum of three parts is $\frac{(mp+np+nq)\,a}{mp+np+nq}$ or a.

Making $a=594$; $m=3$; $n=5$; $p=8$; $q=7$;

$$x=\frac{3\times8\times594}{3\times8+5\times8+5\times7}=\frac{14256}{99}=144;$$

$$\frac{nx}{m}=\frac{5\times8\times594}{3\times8+5\times8+5\times7}=\frac{23760}{99}=240;$$

$$\frac{nqx}{mp}=\frac{5\times7\times594}{3\times8+5\times8+5\times7}=\frac{20790}{99}=210.$$

65. 1st. A wine merchant has two kinds of wine, the one worth 36 d. the other 20 d. per quart. What quantities of these wines must he take to compose a mixture of 50 quarts, which may be sold, without gain or loss, at 30 d. per quart?

Let x qts.=quantity at 36 d. per qt.; and therefore $36\,x$ d. its price.
Then $50-x$=quantity at 20 d. per qt. and $20\,(50-x)$ d. its price;
also $50\times30=1500$ d.=price of 50 qts. of the mixture at 30 d. per qt.;

$$\therefore \text{ by the question } 36\,x+1000-20\,x=1500,$$
$$\text{or } 16\,x=500,$$
$$\text{and } x=31\tfrac{1}{4}=\text{number of quarts at 36 d.;}$$
$$\therefore 50-x=50-31\tfrac{1}{4}=18\tfrac{3}{4}=\text{number of qts. at 20 d.}$$

Verification. $31\frac{1}{4}\times36+18\frac{3}{4}\times20=1125+375=1500$ d.

2d. Let the price of the better sort of wine be a pence per quart, that of the inferior b pence per quart, the quantity of the mixture n quarts, and the price of one quart of the mixture c pence. How much must be taken of each sort?

Let x=quantity, and ax=price of the better sort of wine, then $n-x$= the quantity and $b\,(n-x)$=the price of the inferior sort.
Also n qts $\times\,c$ d.=cn d.=price of the n qts. of mixture.

$$\text{Whence, by the question, } ax+bn-bx=cn;$$

$$\text{and } x=\frac{n\,(c-b)}{a-b}=\text{quantity of the better sort;}$$

$$n-x=n-\frac{n(c-b)}{a-b}=\frac{n(a-c)}{a-b}=\text{quantity of the inferior sort.}$$

$$\text{Verification. } \frac{n(c-b)}{a-b}+\frac{n(a-c)}{a-b}=\frac{n(a-b)}{a-b}=n.$$

Both the particular and the general problem of this Article may be resolved by means of two unknown quantities.

If x is the quantity of the better sort of wine and y the quantity of the inferior,

then, in the first prob. $x+y=50$, and in the second $x+y=n$.
$36\,x+20\,y=1500$ - - $ax+by=cn$.

From these equations the values of x and y can, in both examples, be determined by Article 54.

The solution of the particular can also be obtained from that of the general problem as in the Articles 59—64.

66. 1st. A goldsmith has two sorts of silver; the one sort contains $10\frac{1}{2}$ oz. of pure silver and $1\frac{1}{2}$ oz. of alloy in the lb., the other contains equal quantities of pure silver and alloy, or 6 oz. of each, in the lb. How many pounds

of each sort must he take to make a vase which shall weigh 20 lbs., and contain 9 oz. of pure silver in the pound?

$$10\tfrac{1}{2}\text{ oz.}=\tfrac{21}{24}\text{ lb.}=\tfrac{7}{8}\text{ lb. } 6\text{ oz.}=\tfrac{6}{12}\text{ lb.}=\tfrac{1}{2}\text{ lb.}$$

$20\times 9=180$ oz. $=15$ lb. $=$ weight of pure silver in the vase.

Let $x=$ weight of silver of $10\frac{1}{2}$ oz. fine.

$\therefore\ \frac{7x}{8}$ lb. $=$ weight of pure silver in the x pounds.

Then $20-x=$ weight of silver of 6 oz. fine,

and $\frac{1}{2}(20-x)=$ weight of pure silver in the $(20-x)$ lbs.

Whence, by the question, $\frac{7x}{8}+\frac{20-x}{2}=15$;

$$\therefore 7x+80-4x=120,$$

$$3x=40,$$

and $x=13\frac{1}{3}$ lb. $=$ weight of silver of $10\frac{1}{2}$ oz. fine.

$\therefore 20-x=20-13\frac{1}{3}=6\frac{2}{3}$ lb. $=$ weight of silver of 6 oz. fine.

2d. It is required to find two numbers, such that their sum is a, and that if the one number is multiplied by m and the other by n the sum of the products is b.

Let $x=$ the first number; mx its product by m.

Then $a-x=$ the second; and $n(a-x)$ its product by n.

Whence, by the question, $mx+na-nx=b$;

$$\therefore (m-n)x=b-na.$$

$$x=\frac{b-na}{m-n}=\text{the first number,}$$

$$\text{and } a-x=a-\frac{b-na}{m-n}=\frac{ma-b}{m-n}=\text{the second number.}$$

If $a=20$ lb., $b=15$ lbs., $m=\frac{7}{8}$ lb., $n=\frac{1}{2}$ lb.,

$$x=\frac{15-\frac{1}{2}\times 20}{\frac{7}{8}-\frac{1}{2}}=\frac{5}{\frac{3}{8}}=\frac{40}{3}=13\tfrac{1}{3},$$

$$a-x=\frac{20\times\frac{7}{8}-15}{\frac{7}{8}-\frac{1}{2}}=2\tfrac{1}{2}+\tfrac{3}{8}=6\tfrac{2}{3}.$$

The problems of this Article, like those of the last, may be resolved by means of two unknown quantities.

67. 1st. Saltpetre and sulphur are mixed together in a mass of 80 lbs., and in such proportion that for every 7 parts of saltpetre there are 3 parts of sulphur. How much saltpetre must be added to the mass, and sulphur taken from it, in order that, the whole weight being constant, the proportion of saltpetre to sulphur may be that of 11 parts to 4?

$$7+3:7::80:\frac{80\times 7}{10}=\frac{560}{10}=56=\text{weight of saltpetre.}$$

$$7+3:3::80:\frac{80\times 3}{10}=\frac{240}{10}=24=\text{weight of sulphur.}$$

Let x lbs. $=$ weight added to the saltpetre and taken from the sulphur.

Then $56+x$ lbs. $=$ weight of saltpetre in the new mixture;

and $24-x$ lbs. $=$ weight of sulphur in it.

Whence, by question, $56+x:24-x::11:4$;

$$\therefore 4(56+x)=11(24-x);$$

$$\text{or } 4x+224=264-11x.$$

Transposing, $15x=40$;

$\therefore x=2\frac{2}{3}$ lbs. $=$ weight required;

Whence, $58\frac{2}{3}$ lbs. $=$ wt. of saltpetre, and $21\frac{1}{3}$ lbs. $=$ wt. of sulphur.

2d. Two numbers, a, b, being given; what number must be added to a and subtracted from b that the sum may be to the difference as m to n?

Let x denote the number.

Then, by the question, $a+x:b-x::m:n$.

$$\therefore na+nx=mb-mx;$$

$$\text{and } x=\frac{mb-na}{m+n}.$$

Let $a=56$, $b=24$, $m:n::11:4$.

$$\text{Then } x=\frac{11\times24-4\times56}{11+4}=\frac{40}{15}=2\tfrac{2}{3}.$$

Verifications. 1st. $58\tfrac{2}{3}+21\tfrac{1}{3}=80$,
and $58\tfrac{2}{3}:21\tfrac{1}{3}::11:4$;
for $58\tfrac{2}{3}\times4=234\tfrac{2}{3}$,
and $21\tfrac{1}{3}\times11=234\tfrac{2}{3}$.

$$\text{2d. } a+\frac{mb-na}{m+n}+b-\frac{mb-na}{m+n}=a+b,$$

$$\text{and } a+\frac{mb-na}{m+n}:b-\frac{mb-na}{m+n}::m:n;$$

$$\text{for } \left(a+\frac{mb-na}{m+n}\right)\times n=\frac{mna+mnb}{m+n}.$$

$$\left(b-\frac{mb-na}{m+n}\right)\times m=\frac{mna+mnb}{m+n}.$$

68. 1st. Three masons are employed to build a parapet; the first builds 8 cubic feet in 5 days, the second 9 cubic feet in 4 days, and the third 10 cubic feet in 6 days; in how many days can the three build 756 cubic feet of the parapet?

Let x=the number of days.

Then the first builds $\frac{8}{5}$ feet in 1 day and $\frac{8x}{5}$ feet in x days;

the second builds $\frac{9}{4}$ feet in 1 day and $\frac{9x}{4}$ feet in x days;

the third builds $\frac{5}{3}$ feet in 1 day and $\frac{5x}{3}$ feet in x days.

Therefore, in x days the three build $\frac{8x}{5}+\frac{9x}{4}+\frac{5x}{3}=\frac{331x}{60}$ ft.

$$\text{Whence, by the question, } \frac{331x}{60}=756,$$

$$\text{or } 331\,x=45360;$$

$$\therefore x=137\frac{13}{331} \text{ days, the time required.}$$

2d. Three causes, A, A′, A″, respectively produce, in the times t, t', t'', the effects e, e', e''. In what time will the three acting together produce the effect E?

Let x=the time required by A, A′, A″, to produce the effect E.

Then $\frac{e}{t}$ is the effect produced by A in the unit of time; and

$\frac{ex}{t}$ the effect produced by A in the time x;

Similarly $\frac{e'x}{t'}$ is the effect produced by A′ in the time x.

and $\frac{e''x}{t''}$ is the effect produced by A″ in the time x.

$$\text{Therefore } \frac{ex}{t}+\frac{e'x}{t'}+\frac{e''x}{t''}=E,$$

$$\text{and } x=\frac{Ett't''}{et't''+e'tt''+e''tt'}.$$

If the numbers of the last question are substituted in this formula the result $137\frac{13}{331}$ is obtained from it.

69. 1st. The sum of two numbers is 13, and the difference of their squares 39; required the numbers?

Let x=the greater number, and x^2 its square;
y=the less number, and y^2 its square.

Then, by the question, $x+y=13$;
and $x^2-y^2=39$;

$$\therefore \frac{x^2-y^2}{x+y}=\frac{39}{13}, \text{ or } x-y=3.$$

But $x+y=13$;
therefore $2x=16$ and $x=8$=the greater number,
$2y=10$ and $y=5$=the less.

Verification. $8+5=13$; $8^2-5^2=64-25=39$.

2d. The sum of two numbers is a, and the difference of their squares b, required the numbers?

Let x represent the greater number, and y the less;
therefore, by the question, $x^2-y^2=b$
$x+y=a$;
whence $x-y=\frac{b}{a}$;

$$\therefore 2x=a+\frac{b}{a}=\frac{a^2+b}{a}; \quad x=\frac{a^2+b}{2a}=\text{greater number};$$

$$2y=a-\frac{b}{a}=\frac{a^2-b}{a}; \quad y=\frac{a^2-b}{2a}=\text{less number}.$$

Verification. $\frac{a^2+b}{2a}+\frac{a^2-b}{2a}=\frac{2a^2}{2a}=a.$

$$\left(\frac{a^2+b}{2a}\right)^2-\left(\frac{a^2-b}{2a}\right)^2=\frac{a^4+2a^2b+b^2}{4a^2}-\frac{a^4-2a^2b+b^2}{4a^2}=\frac{4a^2b}{4a^2}=b.$$

70. 1st. The crown of Hiero, king of Syracuse, weighed 20 lbs., and lost 1·25 lb. in water. Let it be assumed that it consisted of gold and silver only, and that 19·64 lbs. of gold lose 1 lb. in water, and 10·5 lbs. of silver in like manner lose 1 lb.; how much gold and how much silver did this crown contain?

$20-1{\cdot}25=18{\cdot}75$ lbs.$=18\frac{3}{4}$ lbs.=weight of the crown in water.

Let x=the quantity of gold contained in the crown;
y=the quantity of silver.

Then $\frac{1864x}{1964}=\frac{466x}{491}$=weight of x lbs. of gold in water;

and $\frac{95y}{105}=\frac{19y}{21}$=weight of y lbs. of silver in water;

Whence, by the question, $\frac{466x}{491}+\frac{19y}{21}=18\frac{3}{4}=\frac{75}{4}$,
$x+y=20$.

From 1st equation, $39144x+37316y=773325$.
From 2d equation, $37316x+37316y=746320$.
$\therefore$ by substraction, $1828x=27005$.

$$\therefore x=\frac{27005}{1828}=14{\cdot}77+\text{lbs.}=\text{weight of gold};$$

$y=20-14{\cdot}77+=5{\cdot}22+$lbs.=weight of silver.

2d. An alliage of two metals, A, B, which weighs p pounds, loses a pounds in water. Now p lbs. of A lose b lbs. in water; and p lbs. of B lose c lbs.; how many of the p lbs. of alliage are of the metal A, and how many of the metal B?

Let x=the number of pounds of the metal A;
y= - - - - B;
$\therefore x+y=p$ - - - - 1

Now $\frac{p-b}{p}x$=the weight in water of p lbs. of the metal A;

and $\frac{p-c}{p}x$=the weight in water of p lbs. of the metal B;

Whence, by the question, $\frac{p-b}{p}x+\frac{p-c}{p}y=p-a$ - - 2

From equation 2, $(p-b)x+(p-c)y=p(p-a)$;
and from equation 1, $(p-c)x+(p-c)y=p(p-c)$;

$\therefore$ by subtraction, $(p-b)x-(p-c)x=p(p-a)-p(p-c)$;
or $(c-b)x=p(c-a)$;

$$\therefore x=\frac{(c-a)p}{c-b}=\text{weight of metal A.}$$

Similarly, from equations 2 and 1,

$$(p-c)y-(p-b)y=p(p-a)-p(p-b),$$

$$\text{and } y=\frac{(b-a)p}{b-c}=\text{weight of metal B.}$$

To resolve the first problem of this Article by these formulæ,
For the alliage, $p=20$; $a=1\frac{1}{4}$;
- - gold, $p=20$; $b=20-\frac{466}{491}$ of $20=\frac{25}{491}$ of $20=\frac{500}{491}$;
- - silver, $p=20$; $c=20-\frac{19}{21}$ of $20=\frac{2}{21}$ of $20=\frac{40}{21}$;

$$\therefore x=\frac{(\frac{40}{21}-\frac{5}{4})\times 20}{\frac{40}{21}-\frac{500}{491}}=\frac{1100}{84}+\frac{9140}{10311}=\frac{1100}{84}\times\frac{10311}{9140}=\frac{567105}{38388},$$
$$=14{\cdot}77+=\text{the weight of gold.}$$

The weight of silver is found in like manner from the formula

$$y=\frac{(\frac{500}{491}-\frac{5}{4})\times 20}{\frac{500}{491}-\frac{40}{21}}.$$

71. 1st. Two numbers, composed of tens and units, are expressed by the same figures; the sum of the two figures is a, and the difference of the numbers b; what are the numbers?

Let x denote the figure which expresses the tens of the first number;
y - - - - - - - units of the first number.

Therefore y denotes the figure which expresses the tens, and x the figure which expresses the units, of the second number.

Also $10x+y$ is the expression of the first number, and $10y+x$ the expression of the second.

Now, by the question, $x+y=a$ - - -
and $10x+y-(10y+x)=b$;
or $9x-9y=b$.

But, from equation 1, $9x+9y=9a$;

$\therefore 18x=9a+b$ and $x=\frac{9a+b}{18}$=the figure which expresses the tens;

$18y=9a-b$ and $y=\frac{9a-b}{18}$=the figure which expresses the units.

As a particular case of this problem, let $a=8$, and $b=54$;

Then $x=\frac{9\times 8+54}{18}=\frac{126}{18}=7$=the figure which expresses the tens;

and $y=\frac{9\times 8-54}{18}=\frac{18}{18}=1$=the figure which expresses the units.

Whence $\left.\begin{array}{l}10x+y=10\times 7+1=71\\10y+x=10\times 1+7=17\end{array}\right\}$ the numbers required.

Verification. $7+1=8=a$;
$71-17=54=b$.

2d. It is required to find a fraction such that if 3 is subtracted from its numerator and denominator, the value is $\frac{1}{4}$; and if 5 is added to its numerator and denominator, the value is $\frac{1}{2}$?

Let x denote the numerator of the fraction required;
y the denominator;

$$\therefore \frac{x}{y} \text{ is the expression of the fraction.}$$

Whence, by question, $\frac{x-3}{y-3}=\frac{1}{4}$, or $4x-12=y-3$ - - - 1

$\frac{x+5}{y+5}=\frac{1}{2}$, or $2x+10=y+5$ - - 2

Multiplying equation 2 by 2, $4x+20=2y+10$;
- - - - - 1 by 1, $4x-12=y-3$;
$\therefore$ by subtraction, $32=y+13$.

Transposing, $y=32-13=19=$denominator of fraction,

and, from equation 2, $x=\frac{y-5}{2}=\frac{19-5}{2}=7=$numerator;

$$\text{therefore } \frac{x}{y}=\frac{7}{19}=\text{fraction required.}$$

Verification. $\frac{7-3}{19-3}=\frac{4}{16}=\frac{1}{4}$; and $\frac{7+5}{19+5}=\frac{12}{24}=\frac{1}{2}$.

3d. Find two numbers, such that if the first is increased by a, and the second by b, the product of the two sums shall exceed the product of the two numbers themselves by c; and that if the first is increased by a', and the second by b', the product of these sums shall exceed the product of the numbers themselves by c'.

Let x denote the first number, and y the second.

Then $x+a$, $x+a'$, $y+b$, $y+b'$, are the sums described in the question; and, by the question, $(x+a)(y+b)$ or $xy+bx+ay+ab=xy+c$;
$(x+a')(y+b')$ or $xy+b'x+a'y+a'b'=xy+c'$;
$\therefore bx+ay+ab=c$ - - - - 1
$b'x+a'y+a'b'=c'$ - - - - 2

Multiplying equation 1 by a', $a'bx+aa'y+aa'b=a'c$
- - - 2 by a, $ab'x+aa'y+aa'b'=ac'$
$\therefore$ by subtraction $(a'b-ab')x+aa'(b-b')=a'c-ac'$,

and by transposition and division $x=\frac{a'c-ac'+aa'(b'-b)}{a'b-ab'}=$1st number.

Similarly, - - - $y=\frac{bc'-b'c+bb'(a-a')}{a'b-ab'}=$2d number.

4th. Three parties, A, B, C, are employed to excavate a ditch; A and B together are able to excavate the whole in 12 days, B and C together in 20 days, A and C together in 15 days; in what time can each party execute the whole work, and in what time can the three parties finish it, working all together?

Representing the whole work by 1, and assuming that the party A can execute the whole in x days, the party B in y days, and the party C in z days; then, in 1 day A executes $\frac{1}{x}$, in 12 days $\frac{12}{x}$, and in 15 days $\frac{15}{x}$ of the whole work; in like manner B in 12 days executes $\frac{12}{y}$, and in 20 days $\frac{20}{y}$ of the whole work; and C in 20 days executes $\frac{20}{z}$, and in 15 days $\frac{15}{z}$ of the whole work;

Whence, by the question, $\frac{12}{x}+\frac{12}{y}=1$, or $12x+12y=xy$ - - 1

$$\frac{20}{y}+\frac{20}{z}=1, \text{ or } 20y+20z=yz \quad - \quad - \quad 2$$

$$\frac{15}{x}+\frac{15}{z}=1, \text{ or } 15x+15z=xz \quad - \quad - \quad 3$$

$$\left.\begin{array}{l} \text{From eq. 1, } - \quad x=\dfrac{12y}{y-12} \\ \text{and from eq. 3, } x=\dfrac{15z}{z-15} \end{array}\right\} \therefore \frac{15z}{z-15}=\frac{12y}{y-12}, \text{ or } 15yz-180z=12yz-180y.$$

$$\therefore\ 3yz+180y=180z,$$

$$\text{or}\quad yz+\ 60y=\ 60z.$$

$$\therefore\ y=\frac{60z}{z+60} \quad - \quad - \quad - \quad 4$$

$$\text{But from equation 2, } y=\frac{20z}{z-20} \quad - \quad - \quad - \quad 5$$

$$\therefore \text{ from equations 4, 5, } \frac{60z}{z+60}=\frac{20z}{z-20}$$

$$\text{or } 60z^2-1200z=20z^2+1200z$$

$$\text{or } 60z-1200=20z+1200$$

$$\text{or } 40z=2400, \text{ and } \therefore\ z=60 \text{ days.}$$

$$\text{But } y=\frac{20z}{z-20} \quad \therefore\ y=\frac{20\times60}{60-20}=\frac{20\times60}{40}=30 \text{ days,}$$

$$\text{and } x=\frac{15z}{z-15} \quad \therefore\ x=\frac{15\times60}{60-15}=\frac{15\times60}{45}=20 \text{ days.}$$

In the formula of problem 2, Article 68, making $E=e=e'=e''=1$, $t=20$, $t'=30$, $t''=60$, the time taken by the three parties to finish the work, when all work together, is given by the expression

$$\frac{20\times30\times60}{30\times60+20\times60+20\times30}=\frac{36000}{1800+1200+600}=\frac{36000}{3600}=10 \text{ days.}$$

$$\left.\begin{array}{lr} \text{Whence the party A can execute the work in} & 20 \\ \qquad\qquad\qquad\quad \text{B} \quad - \quad - \quad - & 30 \\ \qquad\qquad\qquad\quad \text{C} \quad - \quad - \quad - & 60 \\ \text{and A, B, C, together} \quad - \quad - \quad - & 10 \end{array}\right\} \text{days.}$$

72. Additional examples:

1st. What number is that of which the half, the fifth, and the seventh parts are together equal to 59? ……………… *Ans.* 70.

2d. The greater of two numbers is equal to four times the less, and the excess of the greater over the less is 24; required the two numbers?

Ans. 32 and 8.

3d. The sum of 237 £. is to be divided between two persons in such a manner that the one shall receive 1 £. 5 sh. as often as the other receives 1 £.; what is the share of each?

Ans. The 1st. 131 £. 13 sh. 4 d.; the 2d. 105 £. 6 sh. 8 d.

4th. If A and B have between them 1200 £., A and C 1400 £., B and C 1500 £.; how much has each? ………. *Ans.* A has 550 £., B 650 £., C 850 £.

5th. A common of 864 acres is to be divided among three land-owners, A, B, C, so that A's share shall be to B's as 5 to 11, and that C shall receive as much as A and B together. Required the share of each?

Ans. A's share is 135 acres; B's, 297 acres; C's, 432 acres.

6th. Two purses together contain 300 sovereigns, and if 30 sovereigns are taken out of the first purse and put into the second there is the same number in each; how many sovereigns are contained in each purse?

Ans. 180 in the first; 120 in the second.

7th. The rent of an estate is greater by 8 per cent. this year than it was last year. This year's rent is 1890 £.; what was last year's? … *Ans.* 1750 £.

8th. A gamester lost, first, the 6th part, and, second, the 10th part of a certain sum of money; he then gained the third part of the same sum. Supposing that his gain exceeded his loss by 3 £., what was the sum?

Ans. 45 £.

9th. A company of 90 persons consists of men, women, and children; the men are 4 in number more than the women, and the children exceed the number of men and women by 10. How many men, women, and children are there in the company? .... *Ans.* 22 men, 18 women, and 50 children.

10th. A traveller with 42 £. in his pocket found that, after paying out of this sum the expenses of his journey, he had three times as much money left as he had spent; what sum had he spent? .... *Ans.* 10 £. 10 sh.

11th. A labourer is engaged for 48 days on the following conditions: for every day on which he works he is to receive 2 sh. and his board, but for every day on which he does not work he is to pay 1 sh. for his board. If at the close of his engagement he receives 2 £. 2 sh., on how many days must he have worked, and on how many has he been idle?

Ans. He has worked 30 days, and has been idle 18 days.

12th. In the last question, let n = the number of days of the engagement, a the allowance for each day's work, b the charge for board on idle days, c the sum received at the close of the engagement; what are the expressions for the number of working and of idle days?

Ans. The working days $= \frac{c+bn}{a+b}$; the idle days $= \frac{an-c}{a+b}$.

13th. The sum of three numbers is 70; and if the second is divided by the first the quotient is 2 and the remainder 1; but if the third is divided by the second the quotient is 3 and the remainder 3. What are the numbers?

Ans. 7, 15, and 48.

14th. A banker receives yearly 2940 £. for money lent at interest; four fifths of the money is lent at 4 per cent. per annum, and one fifth at 5 per cent. How much money has he lent at interest? .... *Ans.* 70000 £.

15th. A person lends 5500 £. at 4 per cent. per annum simple interest, and $4\frac{1}{2}$ years afterwards 8000 £. at 5 per cent. per annum simple interest. If he leave these two sums constantly at interest, in how many years (reckoning from the time of lending the 5500 £.) will he have drawn the same interest from each? .... *Ans.* 10 years.

16th. Two canal boats are despatched from the same place, the first at 6 o'clock in the morning, the other at 4 in the afternoon; the first goes 4 miles an hour, the second 9. How many hours will the second boat take to overtake the first? .... *Ans.* 8 hours.

17th. In question 16, if n hours = the interval between the starting of the boats, a miles the rate of the first boat per hour, b miles the rate of the second; what is the expression for the time taken by the second boat to overtake the first? .... *Ans.* $\frac{na}{b-a}$ hours.

18th. The garrison of a town threatened with siege abandoned the town, and retreated at the rate of 27 miles per day. Two days afterwards a corps was sent in pursuit, with orders to overtake the fugitives in 6 days. How many miles per day must the second party march to accomplish their orders?

Ans. 36 miles.

19th. From the first of two mortars in a battery 36 shells are thrown before the second is ready for firing. Shells are then thrown from both in the proportion of 8 from the first to 7 from the second; the second mortar requiring as much powder for 3 charges as the first does for 4, it is required to determine after how many discharges of the second mortar the quantity of powder consumed by it is equal to the quantity consumed by the first?

Ans. 189 discharges of the second mortar.

20th. The fore wheels of a carriage are $5\frac{1}{4}$ feet, and the hind wheels $7\frac{1}{4}$ feet in circumference; the fore consequently turn round once for every $5\frac{1}{4}$ feet and the hind wheels once for every $7\frac{1}{8}$ feet of the road passed over. Suppose that the difference of the number of revolutions of the wheels is determined by a mechanical contrivance, and that, during a certain journey, this difference is 2000; what is the length of the journey?

Ans. 39900 feet, or $7\frac{49}{88}$ miles.

21st. If the circumference of the fore wheels is a feet, that of the hind wheels b feet, and the excess of the number of revolutions of the fore over the hind wheels n; what is the expression of the distance travelled?

Ans. $\frac{abn}{b-a}$ feet.

22d. A wine merchant has a pipe of wine (126 gals.) which he cannot sell for less than 15 sh. per gallon; his customers do not wish to pay more than 12 sh.; how much water, then, must he add to the pipe so as to be enabled to sell the mixed wine, without gain or loss, at 12 sh. a gallon?

Ans. $31\frac{1}{2}$ gals.

23d. A draper has a quantity of broad cloth, by selling which at 30 sh. per yard he gains 5 £., but if he sells it at 22 sh. per yard he loses 15 £. on the whole quantity; what are the number of yards and the prime cost per yard?....*Ans*. The number of yards is 50, and the prime cost per yard 26 sh.

24th. An architect engages a certain number of masons; he finds that by giving each m sh. a day, he expends a sh. per day less than the sum allowed for this purpose by the estimate; but that if he gives each n sh. per day he exceeds the allowance by b sh. daily; what are the number of masons engaged, and the fixed daily wages of each?

Ans. The number is $\frac{a+b}{n-m}$, and the daily wages of each $\frac{an+bm}{n-m}$ sh.

25th. A person has 500 coins, consisting of sovereigns and shillings; the value of the 500 is 334 £. 14 sh.; what number has he of each coin?

Ans. 326 sovs. and 174 sh.

26th. A and B have between them 570 £.; if A's money were 3 times and B's 5 times as great as each really is, they would have together 2350 £. How much has each?..................................*Ans*. A has 250 £., B 320 £.

27th. A and B jointly have a fortune of 9800 £.; A invests the sixth part of his property in the funds, and B the fifth part of his, and still each has the same sum remaining. How much has each?

Ans. A has 4800 £., B 5000 £.

28th. There is a fraction such that if 1 is added to the numerator its value is $\frac{1}{3}$, and if 1 is added to the denominator its value is $\frac{1}{4}$. What fraction is it?

Ans. $\frac{4}{15}$.

29th. A wine merchant has two sorts of wine; if he mix 3 gals. of the best and 5 gals. of the worst, he can sell the mixture at 17 sh. 1 d. per gal.; but if he mix $3\frac{3}{4}$ gals. of the best with $7\frac{1}{2}$ gals. of the worst, he can sell the mixture at as low a price as 16 sh. 8 d. per gal. What is the price per gallon of each sort of wine?

Ans. The best 1 £. 3 sh. 4 d. per gal., the worst 13 sh. 4 d. per gal.

30th. 37 lbs. of tin lose 5 lbs. in water, and 23 lbs. of lead lose 2 lbs. in water; a composition of tin and lead weighing 120 lbs. loses 14 lbs. in water. How many lbs. does this composition contain of each metal?

Ans. 74 lbs. of tin, and 46 lbs of lead.

31st. Find two numbers whose difference is to their sum as 2 to 3, and whose sum is to their product as 3 to 5?..........................*Ans*. 2 and 10.

32d. Find three numbers such that the sum of the first and second shall be 56, the sum of the second and third 100, and the sum of the first and third 80?...*Ans*. 18, 38, and 62.

33d. The sums of three numbers, taken two and two, are a, b, c. Required the expressions of the numbers?............*Ans*. $\frac{a+b-c}{2}$, $\frac{a+c-b}{2}$, $\frac{b+c-a}{2}$.

34th. Given the value of 24 qrs. of wheat, 9 qrs. of rye, and 15 qrs. of barley=36 £. 14 sh.; the value of 9 qrs. of wheat, 30 qrs. of rye, and 21 qrs. of barley=40 £. 12 sh.; and the value of 18 qrs. of wheat, 27 qrs. of rye, and 39 qrs. of barley=56 £. 10 sh.; to find the price of a quarter of each kind of grain?......*Ans.* Wheat 18 sh. 8 d., barley 14 sh., and rye 10 sh. 8 d. per qr.

35th. A cistern is filled by 3 pipes, A, B, C; the pipes A, B together fill the cistern in 70 minutes; A, C, together, in 84 minutes; B, C, together, in 140 minutes. In what time will each pipe fill the cistern, and in what time will it be filled if all three pipes are open?

Ans. By A in 105 min., by B in 210 min., by C in 420 min., and by A, B, C together in 60 min.

36th. Supposing that 32 lbs. of sea water contain 1 lb. of salt, how much fresh water must be mixed with these 32 lbs. in order that 32 lbs. of the mixture may contain only 2 oz. of salt, or $\frac{1}{8}$ of the former quantity?.....*Ans.* 224 lbs.

37th. A number is expressed by 3 figures; the sum of the figures is 13; the figure which expresses the simple units of the number is triple the figure which expresses the hundreds; and if 396 is added to the number the sum is the number reversed. Required the number?......................*Ans.* 256.

38th. Three brothers, A, B, C, buy a house for 2000 £.; C can pay the whole price if B give him half of his money; B can pay the whole price if A give him one third of his money; A can pay the whole price if C give him one fourth of his money. How much has each?

Ans. A 1680 £., B 1440 £., C 1280 £.

39th. A silversmith has three bars composed of silver, copper, and tin mixed in different proportions. The pound (avoirdupois) of the first bar contains 7 oz. of silver, 3 oz. of copper, and 6 oz. of tin; the pound of the second contains 12 oz. of silver, 3 oz. of copper, and 1 oz. of tin; and the pound of the third contains 4 oz. of silver, 7 oz. of copper, and 5 oz. of tin. How much of each of these 3 bars must be taken to form a fourth, the pound weight of which shall contain 8 oz. of silver, $3\frac{3}{4}$ oz. of copper, and $4\frac{1}{4}$ oz. of tin?

Ans. 8 oz. of the first bar, 5 oz. of the second, and 3 oz. of the third.

40th. A father leaves his property to be divided among his three children in the following manner: the first is to receive the sum a, plus the nth part of the remainder; the second is to receive the sum $2a$, plus the nth part of what remains after the subtraction of the first part augmented by $2a$; the third is to receive the sum $3a$, plus the nth part of what remains after the subtraction from the property of the first part plus the second part plus $3a$; the property is thus entirely divided. What is its value?

Ans. $\frac{a(6n^2-4n+1)}{n^2-2n+1}$.

OF THE INTERPRETATION OF NEGATIVE RESULTS IN EQUATIONS AND PROBLEMS OF THE FIRST DEGREE.

73. The employment of algebraic symbols in the resolution of problems leads, in some instances, to remarkable results.

These are most intelligibly illustrated and explained through the medium of examples.

Let it be supposed, therefore, that a workman was employed 13 days in summer at certain wages, and 17 days in winter at 2 sh. less per day than in summer; that in summer he forfeited 22 sh. for waste of materials, and in winter received 28 sh. for extra work, and that he each time received the same sum; and let it be required from these conditions to find the workman's day's wages in summer?

To resolve the question,

Let x sh.=the day's wages in summer;

$\therefore x-2$ sh.=the day's wages in winter:

Also $13x$ sh.=wages of 13 days at x sh. per day;

$13x-22$ sh.=sum received in summer;

$17(x-2)=17x-34$ sh.=wages of 17 days at $x-2$ sh. per day,

and $17x-34+28$ or $17x-6$=sum received in winter.

But, by the question, these sums are equal:

$$\text{Therefore } \left\{\begin{matrix} 17(x-2)+28=13x-22, \text{ or} \\ 17x-6=13x-22 \end{matrix}\right\} \quad - \quad 1$$

$$\text{Transposing, } 17x-13x=6-22,$$

$$\text{or } 4x=-16,$$

$$\therefore x=-\frac{16}{4}=-4.$$

The result which is obtained, namely, that the workman's day's wages in summer is -4 sh., has no intelligible meaning.

It is to be inferred from this result, which is termed a Negative solution, that the enunciation of the problem contains impossible conditions.

It is evident, from Article 47, that the changes which are made to obtain the equation $x=-4$ from the equation

$$17(x-2)+28=13x-22,$$

cannot alter the value of the unknown quantity. Now the equation $x=-4$ is verified by substituting -4 for x (since $-4=-4$), and cannot be verified by any other number; therefore the equation $17(x-2)+28=13x-22$ can also be verified by this value alone, of x.

Now, if there exists a value of x which answers the question, this value ought of necessity to verify equation 1; and since the verification cannot be made but by a negative value, it is to be concluded that the question is impossible.

But since the negative value $x=-4$ verifies equation 1, which is the immediate expression of the conditions of the problem; that is,

$$\text{since } 17(x-2)+28=13(x)-22 \quad - \quad 1$$

$$\text{or } 17(-4-2)+28=13(-4)-22 \quad - \quad 2$$

if x is replaced in equation 1 by $-x$, it becomes

$$17(-x-2)+28=13(-x)-22 \quad - \quad 3$$

and since $x=-4$, and consequently $-x=4$, it is evident that the substitution of -4 for x in equation 1, and the substitution of 4 for $-x$ in equation 3, must give precisely the same result.

Therefore, since equation 1 ought to be verified by the value $x=-4$, equation 3 must be verified by the value $x=4$. Hence every question whose enunciation is expressed by equation 3 will have for solution $x=4$.

But as it is absurd to suppose a question which leads to an equation containing only negative quantities in its second member, it becomes necessary to change all the signs of equation 3, and to write it thus,

$$17(x+2)-28=13x+22 \quad - \quad 4$$

In order, therefore, that the question proposed may have the solution $x=4$, it is sufficient to modify its enunciation in such a manner that the workman's day's wages in winter may be 2 sh. greater than in summer; that the sum gained in the 13 days of summer may be augmented by 22 sh., instead of being diminished by that sum; and that the sum gained in the 17 days of winter may be diminished by 28 sh., instead of being augmented by that sum.

With these changes, the question is

A workman was employed 13 days in summer at certain daily wages, and 17 days in winter at 2 sh. a day more than in summer. In summer he received a gratuity of 22 sh. for extra work, and in winter forfeited 28 sh. for waste of materials. Each time he received the same sum. What were his daily wages in summer?

It is evident that the question, thus changed, leads to equation 4, which admits the same value, $x=4$, as equation 3; consequently, this value resolves the question in the precise sense of the new enunciation.

It is to be especially noticed that the known quantities of the first enunciation have remained without alteration of absolute value in the new; only some of them which at first were additive have been rendered subtractive, and conversely.

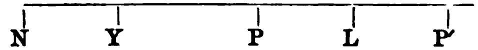
In general, if the enunciation of a problem requires that the unknown quantity x shall have a positive value, and the equation gives a negative value, $x=-a$, it is to be concluded that the problem is impossible.

To rectify the enunciation, x is replaced in the equation by $-x$ (by which means the signs of certain terms are changed), and, without altering the data, the manner of regarding them is modified in such a manner that the changes of signs of the equation correspond exactly to this modification.

Then it is certain, without resolution of the new equation, that the positive value $x=a$, corresponds to the new enunciation, unless it involves some conditions which are not expressible by an equation; as, for example, that the unknown quantity shall be a whole number.

In conclusion, this rectification may be made in many different ways; but it is always necessary to model the new as exactly as possible after the primitive enunciation.

74. Two couriers, A, B, who travel uniformly, A at the rate of 10 miles, and B at the rate of 8 miles an hour, start at the same time for London from Newcastle and York; the distance of Newcastle from London being 270 miles, and of York from London 190 miles, it is required to find at what point in the road A overtakes B?



Suppose that A overtakes B at P, and make $PL=x$,

Then, at P, A has travelled $270-x$ miles,

and B - $190-x$ miles.

Since A travels 10 miles an hour, he takes $\frac{270-x}{10}$ hours to travel $270-x$ miles;

And, in like manner, B takes $\frac{190-x}{8}$ hours to travel $190-x$ miles at the rate of 8 miles an hour.

As A and B start at the same time, the number of hours denoted by $\frac{270-x}{10}$ must be equal to the number denoted by $\frac{190-x}{8}$.

$$\text{Therefore } \frac{270-x}{10}=\frac{190-x}{8} \qquad - \quad - \quad 1$$

$$\text{or } 1080-4x=950-5x;$$

$$\text{Transposing,} \quad 5x-4x=950-1080,$$

$$\text{or } x=-70.$$

The answer is negative; and yet, since A, who is behind B at starting, travels two miles an hour faster than B, A must overtake B, if both proceed far enough.

In forming the equation, it is assumed that A overtakes B between York and London; and in this assumption exists the absurdity which is pointed out by the negative result.

For the purpose of avoiding this absurdity it is necessary to indicate as the place of meeting some point, P', beyond L; then making $LP'=x$, the equation to be resolved is

$$\frac{270+x}{10}=\frac{190+x}{8} \qquad - \quad - \quad 2$$

$$\text{or } \frac{270-(-x)}{10}=\frac{190-(-x)}{8}.$$

Equation 2 is equation 1, with $-x$ substituted for x; therefore, since equation 1 is satisfied by the value $x=-70$, equation 2 must be satisfied by the value $x=70$ (Art. 73). In this manner it is found that the place where A overtakes B is 70 miles beyond London.

Hence it follows that equation 1 gives the solution of the problem, provided the negative value of x is carried along the production of the line NYL; that is, on the side of L opposite to that on which it was first assumed to be situated.

The distance YP may be taken as x; then to obtain PL it becomes necessary to subtract YP from YL. If the remainder is negative, this arises from the circumstance that the place in which A overtakes B is a point, P′, situated beyond L, at a distance, LP′, equal to the remainder taken additively.

75. Two couriers start at the same time from two towns situated on the same road along which they travel. The distance between the two towns and the number of miles travelled in the hour by each courier are known, and hence it is required to find the place of their coming together?

Expressed thus, the question presents two distinct cases:

1st. That in which the couriers go in the same direction.

2d. That in which they go in opposite directions.

1st. Let one courier start from A, the other from B; let both proceed in the same direction from A to C, and suppose that P is the place where the courier from A overtakes the courier from B.

Also, let d denote the given distance AB;
m the number of miles travelled per hour by the courier from A,
m' - - - - - - - B,
x the unknown distance, AP, gone by the courier from A before he overtakes the courier from B. Then the distance travelled by the courier from B is AP—AB, or $x-d$.

Dividing x by m, $\frac{x}{m}$ expresses the time taken by the courier from A to reach P;

And dividing $x-d$ by m', $\frac{x-d}{m'}$ expresses the time taken by the courier from B to reach P.

The times being the same,

$$\frac{x}{m}=\frac{x-d}{m'} \quad - \quad - \quad - \qquad 1$$

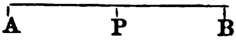
$$\therefore m'x=mx-md.$$

$$\text{Transposing,} \quad mx-m'x=md,$$

$$\text{and } x=\frac{md}{m-m'}.$$

2d. Let the couriers, starting at the same time, one from A, the other from B, advance to meet each other; and let the place of meeting be P.

Adopting the notation of the first case,



and observing that if AP$=x$, BP$=d-x$, the two equivalent expressions of time are

$$\frac{x}{m}, \ \frac{d-x}{m'}. \quad \text{Therefore } \frac{x}{m}=\frac{d-x}{m'} \quad - \quad - \quad 2$$

Equation 2 differs from equation 1 in respect only that m' is replaced by $-m'$; for in $\frac{x-d}{m'}$ replace m' by $-m'$ then,

$$\frac{x-d}{-m'}=\frac{d-x}{m'}.$$

Whence equation 1 is sufficient for the solution of both cases of the problem, provided it is agreed to regard the rate of travelling per hour of the courier from B as changing its sign when the direction in which the courier travels is changed.

From the remarks on case 2, it appears that with the help of negative quantities may be expressed by a single equation, and consequently in one formula, many cases of the same question, each of which would at first appear to require a distinct solution.

For this reason negative quantities are constantly employed in algebraic investigations.

76. It is not possible to include under any general description the class of questions in which the solution of one case may, by a change of signs, be made to include other cases; but all questions in which it is necessary to consider two or more cases which differ from each other, in respect that quantities which are additive for one or more cases are subtractive for another or others, are comprehended in it.

Such are questions in which the same letter is to be regarded as representing a quantity positive or negative, according as it expresses a gain or a loss, a motion in one direction or in a contrary direction, distance on the right of a fixed point or on the left, &c. &c.

The conclusions to be deduced from Articles 73, 74, and 75 are,

1st. The negative value of an unknown quantity may indicate an impossible condition in the enunciation of a question. If so, it is sufficient in this enunciation to give contrary meanings to certain of the quantities, to remove the impossibility; and the negative value before obtained, being taken positively, will give the solution of the amended question.

2d. The negative value of an unknown quantity may also indicate an erroneous supposition; and in this case it gives the solution of the question, provided the quantity which it represents is taken with a meaning contrary to that which had been at first given to it.

3d. If certain quantities are considered positive or negative, according as they are taken to have one meaning or another meaning directly contrary, the different cases of the same question may be combined in a single equation or in one formula.

Example. A father is 45 years of age, his son is 15; in how many years will the age of the son be one fourth of the age of the father? The answer -5 shows that five years ago the age of the son was one fourth of the age of the father.

$$\text{In effect } 15-5=10;\ 45-5=40.$$
$$10=\tfrac{1}{4} \text{ of } 40.$$

OF IMPOSSIBILITY AND INDETERMINATION IN EQUATIONS AND PROBLEMS OF THE FIRST DEGREE.

77. Problem 1. It is required to find a number such that $\frac{1}{3}$ of it augmented by 75, plus $\frac{5}{12}$ of it diminished by 35, may be equal to $\frac{3}{4}$ of it augmented by 49?

Let x denote the number; then, by the question,

$$\frac{x}{3}+75+\frac{5x}{12}-35=\frac{3x}{4}+49 \quad - \quad - \quad 1$$

$$\text{Transposing, } \frac{x}{3}+\frac{5x}{12}-\frac{3x}{4}=49-75+35=9.$$

$$\text{Clearing fractions, } 4x+5x-9x=108,$$
$$\text{or } 0=108.$$

This result, $0=108$, being manifestly absurd, it follows that no value of x can verify equation 1, and, by consequence, that none can satisfy the question.

For this reason the denomination of absurd or impossible is applied to the equation and the question.

The impossibility may be rendered apparent in equation 1 by reducing the like terms of its first member;

$$\text{For } \frac{x}{3}+\frac{5x}{12}=\frac{4x}{12}+\frac{5x}{12}=\frac{9x}{12}=\frac{3x}{4}, \text{ and } 75-35=40,$$

$$\therefore \frac{x}{3}+75+\frac{5x}{12}-35=\frac{3x}{4}+40.$$

$$\text{Whence } \frac{3x}{4}+40=\frac{3x}{4}+49.$$

Consequently the two members must always differ by 9, whatever may be the value assigned to x.

This impossibility is very different from that of Article 73.

The question of that Article was impossible, not because the value of x failed to verify the equation, but because it was negative, and the question excluded a negative value; but in the present question the impossibility is absolute, for no quantity of any kind can give the equality which is established by its enunciation.

78. Problem 2. It is required to find a number, x, such that $\frac{1}{2}(x+10)+\frac{2(x+20)}{3}+\frac{5(x-34)}{6}$ may be equal to $2(x-5)$.

The equation of the problem is,

$$\frac{x+10}{2}+\frac{2(x+20)}{3}+\frac{5(x-34)}{6}=2(x-5) \quad - \quad - \quad 1$$

$$\therefore 3x+30+4x+80+5x-170=12x-60.$$

$$\text{Transposing, } 3x+4x+5x-12x=170-30-80-60, \text{ or } 0=0.$$

To explain this result, let the first member of equation 1 be reduced to the most simple form; it is

$$\frac{3x+30+4x+80+5x-170}{6}, \text{ or } \frac{12x-60}{6}, \text{ or } 2x-10, \text{ or } 2(x-5).$$

Equation 1 is thus reduced to $2(x-5)=2(x-5)$, an expression in which if x is replaced by any number whatever the equation is verified.

Whence equation 1 is an equality which subsists, whatever be the value of x.

In all such cases the equation and the question from which it is deduced are called Indeterminate.

79. Problem 3. It is required to find two numbers such that one half of the first may be equal to one third of the second plus 5, and the second equal to three times one half of the first less 5?

Let x represent the first number, y the second, then $\frac{x}{2}=\frac{y}{3}+5$ (1) and $y=\frac{3x}{2}-5$ (2) are the equations of the problem.

Substituting in the 1st equation the value of y, which is given by the 2d,

$$\frac{x}{2}=\frac{1}{3}\left(\frac{3x}{2}-5\right)+5,$$

$$\text{or } \frac{x}{2}=\frac{x}{2}-\frac{5}{3}+5;$$

$$\therefore 3x=3x-10+30,$$

$$3x-3x=30-10,$$

$$\text{and } \therefore 0=20.$$

From this absurd result, $0=20$, it is evident that no values of x, y can satisfy the two equations; consequently the problem must involve impossible conditions.

Considering equations 1 and 2 individually;

The first, by clearing fractions and transposing terms, becomes

$$3x-2y=30,$$

and the second $3x-2y=10$.

The question therefore requires that the same quantity, $3x-2y$, shall be equal, first, to 30, and second, to 10, which is impossible.

Each of the equations, $3x-2y=30$, $3x-2y=10$, taken singly, is possible, and susceptible of an indefinite number of solutions; but the two conditions of the problem cannot coexist.

Conditions such as these, and also the equations by which they are expressed, are said to be Contradictory or Incompatible.

Let the problem under consideration be changed to this: find two numbers such that one half of the first shall be equal to one third of the second plus 5, and the second equal to three times the half of the first less 15.

The equations of this problem are,

$$\frac{x}{2}=\frac{y}{3}+5 \quad - \; - \; - \quad 1$$

$$y=\frac{3x}{2}-15 \quad - \; - \; - \quad 2$$

Substituting in equation 1 the value of y, given by equation 2,

$$\frac{x}{2}=\frac{1}{3}\left(\frac{3x}{2}-15\right)+5,$$

$$\text{or } \frac{x}{2}=\frac{x}{2}-5+5;$$

$$\therefore \frac{x}{2}-\frac{x}{2}=-5+5,$$

$$\text{or } 0=0.$$

Consequently the two equations are indeterminate (Art. 78).

This conclusion may be deduced from equations 1 and 2, by removing the denominators, and transposing both the unknown quantities to the first members of the equation.

For since $\frac{x}{2}=\frac{y}{3}+5$,

$$3x=2y+30,$$

and $3x-2y=30$.

Since $y=\frac{3x}{2}-15$,

$$2y=3x-30,$$

and $3x-2y=30$.

Whence equations 1, 2, being reducible to the single equation $3x-2y=30$, it follows that the two conditions indicated in the enunciation differ in appearance only.

The two equations, therefore, as well as the problem from which they are deduced, are indeterminate.

80. The preceding problems illustrate the cases of exception mentioned in Article 57.

In all cases of exception the results express either impossibility or indetermination.

To establish this proposition: it is evident that, from the processes of elimination employed, there can happen only one of the three contingencies:

1st. From any number of equations to the same number of unknown quantities, a single equation of the first degree containing only one unknown quantity is deduced; from this equation a single value of the remaining unknown quantity is obtained; then, by substituting this value in the equation which contains this and another unknown quantity, a single value of the second unknown quantity is obtained; and, in the same manner, by returning through the equations containing 3, 4, . . . unknown quantities, a single value of each unknown quantity is determined. This is the case of the Rule in Article 55.

Or, 2d. In the course of the operation or at its conclusion an equation is found in which the unknown terms destroy each other, and the remaining terms of the two members are given quantities, different from each other.

There exists in this case an evident absurdity, and the conclusion is, that whatever values are given to the unknown quantities, neither the primitive equations nor the question from which these equations are deduced can be satisfied. The problem of Article 77 falls under this case.

Or, 3d. If neither the first nor the second case occurs, one or more equations must be found in which the unknown quantities destroy one another, and the two members are equal to the same known quantity.

In this case any values may be given to the unknown quantities contained in the last equation, and then by means of these the values of the other unknown quantities may be found. The problem of Article 78 falls under this case.

In this case there exists a true indetermination in the primitive equations, and by consequence in the problem of which they express the conditions.

If in considering two equations to two unknown quantities an absurd equality is deduced, since it is evident that each equation taken individually is possible, it ought to be concluded that the two equations are contradictory or incompatible.

If the result is an identical equation, the conclusion is that one of the equations is always verified when the other is verified, or that the one equation is a consequence of the other.

81. The number of equations and unknown quantities being three, there will be impossibility if one equation is incompatible with either of the others, or with the equation deduced from them, without being so with either equation considered separately;

And there will be indetermination if one equation is a consequence of either of the others, or if two of the equations are consequences of the third, or if one equation is a consequence of the equation deduced from the other two without being a consequence of either of them taken separately.

When the number of equations is greater (but always equal to the number of unknown quantities) impossibility and indetermination arise from more varied causes; but in general, if it is required to find whether an equation is incompatible with or a consequence of several others, the values of the unknown quantities in the latter equations are to be found in terms of one unknown quantity and of the known quantities of the equations, and these values are to be substituted for the unknown quantities in the first equation (with the exception of the unknown quantity not eliminated from the other equations); then, if the result after reduction is an absurd equality, the first equation is incompatible with the system of the other equations; and if the result is an identity, the first is a consequence of the system of the other equations.

82. Besides the cases of impossibility presented by equations, there are others which may arise from the conditions of a question. For example, it may be required by the enunciation of a problem that the value of an unknown quantity shall be a whole number, and the value obtained from the equations of the problem may be a fraction; or it may be required that the value of the unknown quantity shall be less than a given number, and the result obtained may be greater than that number. . . . . . In general, the enunciation of a problem may impose upon one or upon several unknown quantities certain conditions which it is impossible to express by equations; and in these cases, after obtaining the values of the unknown quantities, it remains to try whether they satisfy all these conditions.

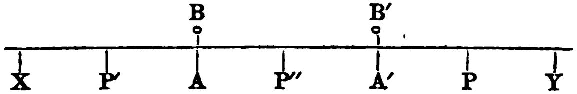
Example of the Discussion of a Problem of the First Degree.

83. If an equation of the first degree, in which the given quantities are represented by letters, is resolved, the value of the unknown quantity (assuming, for the sake of simplicity, that there is only one) is expressed by a formula which indicates the operations to be performed upon the given quantities.

Now, the given quantities (or the letters by which they are represented) may be supposed to receive all possible values, and it may be required to determine what, under the different hypotheses, are the values of the unknown quantity, x.

The determination of the different values of x, and the interpretation of the remarkable results which are obtained, constitute what is termed the Discussion of a problem.

To illustrate the manner of conducting the discussion of a problem, let two bodies, B, B′, move uniformly in the indefinite straight line XY with given velocities, v, v', and in the direction XY; also, let B reach the point A h hours before B′ reaches A′; and let the distance AA′ be d. It is required to find in what point of the line XY the bodies B, B′ are in conjunction.



Suppose that the point P between A′ and Y is the place in which B, B′ come together or are in conjunction: express A′P by x, and therefore AP by $d+x$.

The times taken by B, B′ respectively to describe the distances $d+x$ and x with the given velocities, v, v', are $\frac{d+x}{v}$ and $\frac{x}{v'}$;

And since B reaches A h hours before B′ reaches A′, the time $\frac{d+x}{v}$ must exceed the time $\frac{x}{v'}$ by h; consequently,

$$\frac{d+x}{v}-\frac{x}{v'}=h \quad \cdot \quad \cdot \quad \cdot \quad 1$$

$$\text{Therefore } v'd+v'x-vx=vv'h.$$

$$v'x-vx=vv'h-v'd.$$

$$x=\frac{v'(vh-d)}{v'-v},$$

$$\text{or } x=\frac{v'(d-vh)}{v-v'} \quad \cdot \quad \cdot \quad 2$$

In this solution it is assumed that B, B′ come together between A′ and Y; they may, however, come together either before or after reaching A′. Both cases are satisfied by considering x to be either additive or subtractive (Art. 76. 3); and, interpreted thus, equation 2 is perfectly general.

Discussion:

1st. The numerator and denominator of the value of x are both positive, when at the same time $v>v'$, $d>vh$.

Whence the value of x is positive, and, by consequence, B, B′ come together between A′ and Y, as has been assumed.

This conclusion is in conformity with that which is obtained from the question itself; for vh is the distance described by B during the time, h, which elapses between the arrival of B at A and that of B′ at A′.

Now, to suppose $v>v'$ and $d>vh$ is to admit that B moves faster than B′, and that B′ has reached A′ before B reaches A′; consequently B must overtake B′ on the Y side of A′.

2d. When at the same time $v<v'$ and $d<vh$, the value of x is positive; and consequently B, B′ still come together on the Y side of A′.

In effect it is evident in this case that B moves slower than B′, and that it has passed the point A′ before B′ has reached A′; therefore B must be overtaken by B′ on the Y side of A′.

3d. When at the same time $v<v'$ and $d>vh$,

or $v>v'$ and $d<vh$, the value of x is negative.

This negative value of x indicates that B, B′ come together on the X side of A′, and not on the Y side (as was assumed), and at a distance from A′ marked by the absolute value of x; that is, by the value of x considered without regard to its sign.

Returning, as before, to the question when $d>vh$, B′ reaches A′ before B reaches A′; and when $v'>v$, B′ moves faster than B. Consequently, since B′ reaches A′ before B, and also moves faster than B, the bodies B, B′ cannot be in conjunction on the Y side of A′.

Again, when $d<vh$, B has passed A′ before B′ has reached A′; and when $v>v'$, B moves faster than B′. Therefore, in this hypothesis also, the conjunction cannot take place on the Y side of A′.

Consequently, under both the hypotheses of case 3, the conjunction of B, B′ must take place on the X side of A′.

It has been supposed that B, B′ are in conjunction on the Y side of A′. Now the negative value of x proves that in the 3d case this supposition is inadmissible, but does not expressly point out whether the conjunction takes place on the X side of A, or between A and A′.

If the conjunction takes place in P′ on the X side of A, make $A'P'=x$; then the time taken by B to move from P′ to A is $\frac{x-d}{v}$, and by B′ to move from P′ to A′ is $\frac{x}{v'}$. The time elapsed between the arrival of B at the point A, and that of B′ at the point A′, is therefore expressed by

$$\frac{x}{v'}-\frac{x-d}{v};$$

and as the question requires that this time shall be equal to h,

$$\frac{x}{v'}-\frac{x-d}{v}=h \quad - \quad - \quad 3$$

If the conjunction takes place between A′ and A in the point P″, and the unknown quantity A′ P″ is denoted by x, the time taken by B to move from A to P″ is $\frac{d-x}{v}$, and the time taken by B′ to move from P″ to A′ is $\frac{x}{v'}$. Therefore the time elapsed between the arrival of B in A and that of B′ in A′ is $\frac{d-x}{v}+\frac{x'}{v'}$; and by the question,

$$\frac{d-x}{v}+\frac{x}{v'}=h \quad - \quad - \quad 4$$

But since equation 3, or

$$\frac{x}{v'}-\frac{x-d}{v}=\frac{vx}{vv'}-\frac{v'x}{vv'}+\frac{v'd}{vv'}=\frac{v'(d-x)}{vv'}+\frac{vx}{vv'}=\frac{d-x}{v}+\frac{x}{v'},$$

which is the left member of equation 4, it follows that equations 3, 4 are the same. Therefore equation 4 ought to determine the point of conjunction, wherever may be its position on the X side of A′. And since equation 1 becomes equation 4 by changing x into $-x$, it is to be concluded that every positive value which satisfies equation 4 will be given, but with the sign $-$ by equation 1.

Whence equation 1 is of itself sufficient to determine in all cases the point of conjunction of the bodies B, B′, care only being taken to carry the negative values of x to the left of A′.

4th. Let $v=v'$, with $d \lessgtr vh$.

Then formula 2, or $x=\frac{v'(d-vh)}{v-v'}$ becomes $x=\frac{v'(d-vh)}{0}$.

Or (to simplify the expression) making $v'(d-vh)=m$,

$$x=\frac{m}{0}.$$

To interpret this expression, without reference to Article 40, suppose that instead of zero there are given to m a series of divisors becoming less and less, for example,

$$1, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \ldots$$

the quotients are m, $10m$, $100m$, $1000m$ . . . .

And these quotients will continue to increase in such a manner that no quantity, how great soever, can be conceived which shall not be exceeded by the quotient, the divisor being taken sufficiently small.

This is expressed by the proposition: The quotient of a quantity by the divisor 0 is infinite.

The expression $\frac{m}{0}$, considered in itself, shows that there exists no distance sufficiently great to represent that distance at which the bodies B, B′ are in conjunction; which is the same as to say, that they never are in conjunction.

Referring to the question and figure, since $d \lessgtr vh$, B, B′ reach the point A′ at different times; and since $v=v'$, B, B′ have the same velocity. Whence they must always be at the same distance from each other, and the conjunction of B, B′ is impossible.

5th. Let $v=v'$ and $d=vh$.

The formula $x=\frac{v'(d-vh)}{v-v'}$ becomes $x=\frac{0}{0}$.

Since in every division the quotient multiplied by the divisor is equal to the dividend, it is necessary that in the case $\frac{0}{0}$ the quotient shall be such that its product by 0 shall be equal to 0. Now the product of any number multiplied by 0 is 0; therefore any number may be assumed at pleasure for the quotient. This is expressed otherwise, by saying that $\frac{0}{0}$ represents an indeterminate quantity.

From this explanation it is to be concluded that B, B′ are never separated.

In effect, since $v=v'$ and $d=vh$, B, B′ have the same velocity, and they reach A′ at the same time; therefore they must have been and must continue to be always together.

The principles explained with respect to the change of sign (Art. 76) give authority for the application of formula 2 to problems whose enunciations do not differ from that from which formula 2 has been deduced, except in respect that quantities which are considered additive in the one question are subtractive in the other, and conversely.

Whence it may be supposed that B′ moves in the direction from A′ to x; and to resolve this case, it will be sufficient to change v' into $-v'$ in formula 2.

If B changes the direction of its motion, v must be made $-v$.

If B, B′ both change the direction of their motion, v becomes $-v$; and v', $-v'$.

Lastly, if the arrival of B at the point A, instead of preceding that of B′ at A′, takes place after it, it becomes necessary to change h to $-h$.

OF THE SYMBOLS $\frac{m}{0}$ AND $\frac{0}{0}$.

84. When the given quantities of any equations are represented by letters, and formulæ are deduced expressing the values of the unknown quantities in terms of these letters; then, if hypotheses are made respecting the given letters, and these hypotheses are introduced into the formulæ, particular values corresponding to these hypotheses are obtained for the unknown quantities.

Now it may happen that these hypotheses make the equations fall upon a case of impossibility or indetermination, which happening, there is at least one of the formulæ which becomes $\frac{m}{0}$ or $\frac{0}{0}$.

Suppose that these hypotheses are introduced into the equations, and that the calculations which have been made to deduce the general formulæ are repeated.

If the equations lead to a case of impossibility, it must happen, in seeking the value of one of the unknown quantities, x for example, that the elimination must lead to an absurd equation, in which x is not found, and of which the two members are different quantities.

Suppose that before the hypotheses this equation in x was

$$Px=Q;$$

P and Q representing literal expressions composed of given quantities. The general value of x is,

$$x=\frac{Q}{P}.$$

Now, in the case of the hypotheses which lead to impossibility, x must disappear from the equation $Px=Q$, and the two members must be unequal quantities. Therefore P ought to be zero, without Q becoming so.

Whence, denoting what Q becomes by m, the general value of x ought to give $x=\frac{m}{0}$.

Next, suppose that the hypotheses lead to a case of indetermination. The process of elimination, instead of ending in an equation from which the value of the unknown quantity can be obtained, must give an identical equation.

Therefore, if before the hypotheses this equation is $Px=Q$, it must happen that after the hypotheses P and Q are both zero.

Therefore the general value is reduced to

$$x=\frac{0}{0}.$$

It may happen that the expressions of more than one of the unknown quantities become $\frac{m}{0}$ or $\frac{0}{0}$.

The preceding remarks show only that in the case of impossibility there must be at least one unknown quantity which becomes $\frac{m}{0}$, and in the case of indetermination one which becomes $\frac{0}{0}$.

85. In the expression $x=\frac{m}{0}$, the value of x is considered infinite (Art. 83. 4). This infinite value may be sometimes positive, sometimes negative, and sometimes either positive or negative.

1st. Let $x=\frac{m}{(n-z)^2}$, m, n being constants, positive and different from 0, and z a quantity which may have all possible values.

Making $z=n$, $x=\frac{m}{(n-z)^2}$ becomes $x=\frac{m}{0}$.

Now, although when $z<n$, $n-z$ is additive, and when $z>n$, $n-z$ is subtractive, still, as $(-r)\times(-r)=+r^2$, it follows that the sign of the denominator is always additive; hence in this case the expression $\frac{m}{0}$ is considered to express infinity, taken positively, and it is written,

$$x=+\infty$$

2d. Similarly - - if $x=\frac{-m}{(n-z)^2}$, when $z=n$,

$$x=-\infty.$$

3d. Let $x=\frac{m}{n-z}$; the hypothesis $z=n$ makes $x=\frac{m}{0}$; but in this case the symbol of infinity ought to be preceded by the sign $\pm$. For, supposing $z<n$, and making z to increase, the formula gives increasing values, which are positive; and supposing $z>n$, and making z to diminish so as to become equal to n, the formula gives increasing values which are all negative (Art. 3 a, and 4 d). Hence the infinite value corresponding to $z=n$ belongs equally to a series of positive increasing numbers, and to a series of negative increasing numbers; for this reason the hypothesis $z=n$ is considered as making the formula take two infinite values, one positive, the other negative.

This is indicated in an abridged form by writing thus,

$$x=\pm\infty$$

When a fractional formula involves a denominator which may become infinite without the numerator becoming so, it is evident that at this limit the numerical value of the fraction is less than any quantity, however small, which is tantamount to saying that it is equal to zero.

Denoting the numerator by m, $\frac{m}{\infty}=0$.

If a quantity decreases to zero without change of sign, the trace of the sign may be preserved by prefixing it to the zero thus,

$$\frac{m}{+0}=+\infty;\ \frac{m}{-0}=-\infty,$$

$$\frac{m}{+\infty}=+0;\ \frac{m}{-\infty}=-0.$$

86. With respect to the symbol $\frac{0}{0}$, it is proved, in Article 78, to be the expression of an indeterminate quantity; but indetermination may appear under other forms.

By the rules for the calculus of fractions,

$$P\times\frac{1}{Q}=\frac{P}{Q}, \text{ and}$$

$$\frac{\frac{1}{Q}}{\frac{1}{P}} \text{ or } \frac{1}{Q}\div\frac{1}{P}=\frac{P}{Q}.$$

Now, when $P=0$ and $Q=0$, $\frac{1}{P}=\infty$ and $\frac{1}{Q}=\infty$. Therefore the expressions, $P\times\frac{1}{Q}=0\times\infty$, $\frac{\frac{1}{Q}}{\frac{1}{P}}=\frac{\infty}{\infty}$ ought to receive the same interpretation as $\frac{0}{0}$; that is, they ought to be considered symbols of indetermination.

There are, however, cases in which the expression $\frac{0}{0}$ is considered under another point of view.

$$\text{Let } x=\frac{3(z^2-4)}{z(z-2)}.$$

If $z=2$, x becomes $\frac{0}{0}$, or indeterminate

But if, before making $z=0$, the fraction is simplified by taking away the common factor $z-2$, the result is, $x=\frac{3(z+2)}{z}$, which, for $z=2$, gives $x=6$. This is the true value of x for the hypothesis $z=2$.

In explanation of this result, instead of supposing $z=2$, let it be supposed that the difference $z-2$ is diminished gradually. The formula $x=\frac{3(z^2-4)}{z(z-2)}$ will give successively for x different values, all determinate and equal to those given by the formula $x=\frac{3(z+2)}{z}$. These values may, therefore, be regarded as approximating to the value $x=6$; and for this reason the value $x=6$ is considered as being that which truly corresponds to the hypothesis $z=2$; but as the indetermination which results from the first formula prevents the discovery of this value, recourse is necessarily had to the second.

Whence, if, for a particular hypothesis, the value of a fractional expression is $\frac{0}{0}$, it is necessary to try whether the fraction contains a factor common to both its terms, and which, by becoming zero, gives to the fraction the form $\frac{0}{0}$. The suppression of this common factor permits the true value of the fraction to be discovered.

The fractional expressions $\frac{3z^2-3}{2z-2}$, $\frac{z^2-2z+1}{3z-3}$, $\frac{3z^2-3}{z^2-2z+1}$, all become $\frac{0}{0}$, for the hypothesis $z=1$.

They are reducible to the forms,

$$\frac{3(z+1)\,(z-1)}{2(z-1)},\quad \frac{(z-1)\,(z-1)}{3(z-1)},\quad \frac{3(z+1)\,(z-1)}{(z-1)\,(z-1)},$$

in which, suppressing the common factor $z-1$, making $z=1$, and reducing, the results obtained are, 3, 0, $\pm\infty$.

87. Since a fraction may become equal to zero when its numerator is 0, or its denominator ∞, it follows that if an equation has denominators which contain x, and the equation is reduced to the form $\frac{m}{n}=0$, it becomes necessary, in order to resolve the equation completely, to seek not only the values of x, for which $m=0$, but also those for which $n=\infty$.

It is, however, to be inquired whether, for these values of m and n, the fraction does not become $\frac{0}{0}$ or $\frac{\infty}{\infty}$; for if it becomes either, the true value of the fraction may not be zero, and then the equation is not satisfied.

In the equation $\frac{x+3}{1-x^2}=0$, as the value $x=-3$ renders the numerator 0 without making the denominator 0, it may be concluded that the value $x=-3$ really satisfies the equation.

By making $x=+\infty$ or $x=-\infty$, the denominator $1-x^2$ becomes ∞; but these values render the numerator $x+3$ also ∞; whence the question is reduced to an apparent indetermination. To make this disappear the fraction $\frac{x+3}{1-x^2}$ must be reduced to a form under which the numerator no longer becomes ∞. This is done by dividing both terms of the fraction by x.

It thus becomes $\dfrac{1+\frac{3}{x}}{\frac{1}{x}-x}$

Now, observing that the values $x=\pm\infty$ render $\frac{3}{x}=0$ and $\frac{1}{x}=0$, it is evident the raction is reduced to $\frac{1}{-\infty}$ or $\frac{1}{+\infty}$, that is, to zero; whence the values $x=\pm\infty$ satisfy the proposed equation.

Most commonly, in removing denominators which contain x, infinite values of x are not taken into consideration, because these values rarely, if ever, satisfy the problems which the equations are employed to resolve; but this omission is always voluntary, and the preceding explanation shows how it can be repaired.

Of General Formulæ.

88. Every equation of the first degree with one unknown quantity can be reduced to the form $ax=n$, in which a denotes the algebraic sum of the quantities by which the unknown quantity is multiplied, and n the algebraic sum of all the given terms (Art. 49).

The value of x given by the general equation $ax=n$ is $x=\frac{n}{a}$.

Every equation of the first degree involving two unknown quantities can be reduced to the form $ax+by=n$.

For if the proposed equation contains denominators, these are taken away by Article 47. d; the terms in x and y are transferred to the first member, and the known quantities to the second member, by Article 47. b; then denoting the algebraic sum of the terms in x by ax, that of the terms in y by by, and that of the known terms by n, the equation $ax+by=n$ is obtained, in which a, b, n are known quantities, additive or subtractive, and having the integer form.

Two independent equations being required for the determination of the values of two unknown quantities (Art. 57), let

$$ax+by=n,$$
$$a'x+b'y=n',$$

be given. Then it is found (Art. 52) that

$$x=\frac{nb'-bn'}{ab'-ba'}$$
$$y=\frac{an'-na'}{ab'-ba'}.$$

In like manner, the number of unknown quantities being three, the general equations are,

$$ax+by+cz=n,$$
$$a'x+b'y+c'z=n',$$
$$a''x+b''y+c''z=n'';$$

And the values of x, y, z are given by the formulæ

$$x=\frac{nb'c''-nc'b''+cn'b''-bn'c''+bc'n''-cb'n''}{ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a''}$$
$$y=\frac{an'c''-ac'n''+ca'n''-na'c''+nc'a''-cn'a''}{ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a''};$$
$$z=\frac{ab'n''-an'b''+na'b''-ba'n''+bn'a''-nb'a''}{ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a''}.$$

The process of elimination, when the number of unknown quantities and equations is greater than three, may be conducted by any of the known methods, but most conveniently by that of reduction. The employment, however, of accented letters to represent the coefficients of the unknown quantities and the algebraic sums of the given quantities has led to the observation of a law, by means of which the formulæ which satisfy any number of equations may be obtained without the necessity of elimination.

89. Resuming the equation $ax=n$, or $x=\frac{n}{a}$, the numerator of the value of x is the algebraic sum of the terms which are entirely composed of known quantities, and the denominator is the coefficient of the unknown quantity.

Next, the two equations, $ax+by=n$, $a'x+b'y=n'$, give the results,

$$x=\frac{nb'-bn'}{ab'-ba'}, \quad y=\frac{an'-na'}{ab'-ba'}.$$

In these values of x, y the denominator, which is the same for both unknown quantities, is composed of a, a', b, b', the coefficients of the unknown quantities. It can be formed by writing the letter a before b, thus, ab; then after b, thus, ba; connecting these expressions by the sign —, and placing an accent over the second letter of each term. Thus the denominator of the values of x and y is $ab'-ba'$.

To form the numerator of the value of x, a, a' are changed, in the expression $ab'-ba'$, into n, n'; and to form the numerator of the value of y, b, b' are changed in the same expression into n, n'.

Thus, the numerator of x is $nb'-bn'$, and the numerator of y is $an'-na'$.

To obtain the denominator which is common to the values of three unknown quantities, assume the denominator $ab-ba$ of the preceding case (omitting the accents). Into each of the terms ab, ba, introduce the letter c at the end, in the middle, and at the beginning; make the signs alternately + and —, and place in each term one accent on the second letter and two accents on the third letter. The denominator thus formed is

$$ab'c''-ac'b''+ca'b''-ba'c''+bc'a''-cb'a''.$$

To form the numerator of the value of each unknown quantity, replace in the denominator the letter which represents the coefficient of that unknown quantity by the letter which represents the quantity wholly known, leaving the accents in the same place. Thus, for x change a into n, for y change b into n, and for z change c into n.

If the number of unknown quantities and equations is four, the coefficients being a, b, c, d, a', b', c', d', &c. &c., and the known quantities n, n', n'', n''', it is necessary to introduce the coefficient d into each of the six products $abc-acb+cab-bac+bca-cba$, and to make it occupy in succession all the places.

The product abc, for example, gives

$$abcd-abdc+adbc-dabc.$$

Proceeding in the same manner with the five remaining terms, the complete denominator will contain 24 terms, in each of which the second letter will have one accent, the third two, and the fourth three.

The denominator which is common to the values of all the unknown quantities being formed, the numerator of the value of each unknown quantity is obtained from it in the same manner as in the case of three unknown quantities, namely, for each unknown quantity by replacing in the denominator the letter which represents the coefficient of that unknown quantity by the letter which represents the quantity wholly known, leaving the accents as in the denominator.

This law can be extended to any number of equations. The demonstration of it is complicated, and by no means elementary.

The manner of applying these formulæ to the resolution of numerical equations of the first degree has been already explained (Art. 55).

DISCUSSION OF THE GENERAL FORMULÆ FOR THE RESOLUTION OF EQUATIONS OF THE FIRST DEGREE.

90. The general equations of the first degree involving two unknown quantities being

$$ax+by=n \quad - \quad - \quad 1$$
$$a'x+b'y=n' \quad - \quad - \quad 2$$

the value of $x=\frac{nb'-bn'}{ab'-ba'}$, and the value of $y=\frac{na'-an'}{ab'-ba'}$.

In these values of x, y, let the denominator $ab'-ba'=0$;

And, first, let neither of the numerators be equal to zero.

$$\text{Therefore } x=\frac{nb'-bn'}{0},\ y=\frac{na'-an'}{0}.$$

Whence in this case the values of x and y are infinite.

From the equality $ab'-ba'=0$ is obtained, by transposition and division,

$$a'=\frac{ab'}{b}.$$

Substituting this value of a' in equation 2, or

$$a'x+b'y=n',$$
$$\frac{ab'}{b}x+b'y=n',$$
$$\text{or } ab'x+bb'y=bn' \text{ or } b'(ax+by)=bn';$$

the first member of this result is the first member of equation 1 multiplied by b'; therefore n, the second member of equation 1, if multiplied by b', ought to be equal to bn'; that is, $b'n$ ought to be equal to bn', or nb' equal to bn'.

But if $nb'=bn'$, $nb'-bn'=0$, or the numerator of the value of x is equal to zero, which is contrary to the hypothesis.

In this manner the impossibility of finding values of x and y which at once satisfy the equations,

$$x=\frac{nb'-bn'}{0},\ y=\frac{na'-an'}{0},$$

is made evident.

But the impossibility is more distinctly marked by the infinite values which not only show its existence, but also that it arises from the circumstance that the values of the unknown quantities are greater than any which can be assigned.

If the value of the expression $ab'-ba'$ is very small, the values of x, y are very great, but such as satisfy the equations.

This being true, if, when $ab'-ba'=0$, the verification cannot be made, the reason why it cannot is, that then x, y are greater than any assignable magnitudes.

It ought to be observed, that values $=\infty$, deduced from equations, may give the true solution of the question the conditions of which are expressed by these equations. Of this the application of algebra to geometry affords many instances. To give one: when an angle is unknown, and an infinite value is found as the expression of its tangent, it is certain that the angle is a right angle.

Second, let the denominator $ab'-ba'=0$, and one of the numerators $nb'-bn'=0$; then shall the other numerator $an'-na'$ be also equal to zero.

$$\text{For since } ab'-ba'=0,\ a'=\frac{ab'}{b};$$
$$\text{and since } nb'-bn'=0,\ n'=\frac{nb'}{b}.$$

Substituting these values of a', n' in the expression $an'-na'$, it becomes

$$a\times\frac{nb'}{b}-n\times\frac{ab'}{b},$$
$$\text{or } \frac{anb'}{b}-\frac{anb'}{b}, \text{ which is equal to zero.}$$

For the hypothesis $ab'-ba'=0$, $an'-na'=0$, it can be in like manner proved that $nb'-bn'=0$.

Whence the common denominator, and one of the numerators of the values of x, y being zero, the other numerator is also zero; and the values of x, y are

$$x=\frac{0}{0},\ y=\frac{0}{0}.$$

These symbols indicate indeterminate quantities, and by returning to the equations it can be proved that this case is indeterminate.

The values of a', n', found above, are $\frac{ab'}{b}$, $\frac{nb'}{b}$.

If these values are substituted for a', n' in equation 2,

$$\frac{ab'}{b}x+b'y=\frac{nb'}{b};$$

$$\text{Whence } \frac{b'}{b}(ax+by)=\frac{b'}{b}n.$$

This equation can be obtained by multiplying both members of the equation $ax+by=n$ by $\frac{b'}{b}$. Therefore all the values of x, y which satisfy one equation, must also satisfy the other.

Now, if to x arbitrary values are given, and these values are substituted in the first equation, corresponding values of y can be obtained from it; thus are obtained solutions which satisfy the first equation. But these values must also satisfy the second equation; therefore it follows that the proposed equations admit an indefinite number of solutions, or, in other words, are indeterminate.

The indetermination does not, however, permit an arbitrary value of x to be taken, and also an arbitrary value of y; for the preceding explanation shows that the value of one of the unknown quantities ought always to be calculated by means of the arbitrary value of the other.

This case comprehends that in which $n=0$, $n'=0$, $ab'-ba'=0$, for by these conditions $x=\frac{0}{0}$ and $y=\frac{0}{0}$.

By the hypothesis $n=0$, $ax+by=n$ is reduced to $ax+by=0$.

$$\text{Therefore } y=-\frac{a}{b}x.$$

In like manner from $n'=0$ is obtained $y=-\frac{a'}{b'}x$;

And from $ab'-ba'=0$ is obtained $\frac{a}{b}=\frac{a'}{b'}$.

Whence the two values of y are equal for every value of x, and by consequence there is a case of indetermination.

It is proper to observe that the ratio of y to x in these expressions is determinate;

$$\text{for, since } y=-\frac{a}{b}x=-\frac{a'}{b'}x,$$

$$\frac{y}{x}=-\frac{a}{b}=-\frac{a'}{b'},$$

without the condition $\frac{a}{b}=\frac{a'}{b'}$, the values of y could not have been equal except for $x=0$. In this case y must also have been equal to zero, and the ratio $\frac{y}{x}$, $\frac{0}{0}$, that is, indeterminate.

In employing the expressions $a'=\frac{ab'}{b}$, $n'=\frac{nb'}{b}$, it has been assumed that the coefficients a, b are different from zero.

If at the same time $b=0$, $b'=0$, the two expressions $ab'-ba'$, $nb'-bn'$ must be equal to zero, and this without giving any determinate value of $an'-na'$.

In this case the values of x, y are $x=\frac{0}{0}$, $y=\frac{an'-na'}{0}$.

In like manner, if a, a' are at once equal to zero, the results for x, y are $x=\frac{nb'-bn'}{0}$, $y=\frac{0}{0}$.

This particular case is, however, hardly admissible, for by it the given equations are reduced to two equations to one unknown quantity, namely,

$$ax=n,\ a'x=n',\ \text{for } b \text{ and } b'=0,$$
$$by=n,\ b'y=n',\ \text{for } a \text{ and } a'=0,$$

while the subject under consideration is that of two equations to two unknown quantities.

91. The general formulæ relative to more than two equations are more complicated, and the discussion of the conditions of impossibility and indetermination more laborious and difficult.

Denoting the numerator of the value of x by N, and the denominator by D, if by the application of the general formulæ to a system of equations, a result $x=\frac{N}{0}$ is obtained, this result is a character of impossibility. But the result $x=\frac{N}{D}=\frac{0}{0}$, is sometimes a character of indetermination, sometimes of impossibility; and it sometimes indicates the existence of a factor common to N and D.

To ascertain the precise signification of these results, there is no simpler method than to determine the values of the unknown quantities from the system of equations by the method of Article 55.

The proposed equations being,

$$x+9y+6z=16 \quad - \quad 1$$
$$2x+3y+2z=7 \quad - \quad 2$$
$$3x+6y+4z=13 \quad - \quad 3$$

The general formulæ give $x=\frac{0}{0}$, $y=\frac{0}{0}$, $z=\frac{0}{0}$.

But if the second equation is multiplied by 3, and the first equation is subtracted from the product, that is, if from

$$6x+9y+6z=21$$
$$x+9y+6z=16 \text{ is subtracted,}$$
$$\text{the remainder } 5x \qquad =5,$$
$$\therefore x=1.$$

Substituting this value of x in the proposed equations, the results are

1st. $9y+6z=15$, or $3y+2z=5$.
2d. - - - $3y+2z=5$.
3d. $6y+4z=10$, or $3y+2z=5$.

The value is x, consequently determinate and equal to 1; but the values of y and z are indeterminate, since there is only one equation, $3y+2z=5$, to the two unknown quantities, y, z.

SECTION V.

OF THE INDETERMINATE ANALYSIS OF THE FIRST DEGREE.

92. In resolving an equation of the first degree involving two unknown quantities, if any value whatever is given to one of the unknown quantities, the equation gives a corresponding value of the other unknown quantity.

Considered in this manner, the equation admits of an infinite number of solutions. The number of solutions is more limited if it is required that the values of x, y shall be whole numbers; and still more if it is required that the values shall be both integer and positive. Conditions of this kind cannot be expressed by equations.

Any equation of the first degree to two unknown quantities can be reduced to the form $ax+by=c$, in which a, b, c are any whole numbers, positive or negative.

The factors common to a, b, c may be suppressed; and when it is required that the values of x, y shall be integer, it is necessary, after this suppression of the common factors, that a, b be prime to each other. For, supposing a and b both prime to c, but not to each other, and that any integer values of x, y are substituted in the equation $ax+by=c$, the first member of the resulting equality can be divided by the common factor of a, b; but c, which is prime with a, b, cannot be divided by it. Consequently, under this hypothesis, the equality is impossible.

93. Having given the equation $ax+by=c$ - - - - 1
in which a, b, c represent any whole numbers, positive or negative (with the limitation that a, b are prime to each other), the problem is, to find all the systems of integer values of x, y capable of satisfying equation 1.

If the coefficient of one of the unknown quantities, y for example, is 1, the equation becomes

$$ax+y=c.$$
$$\therefore y=c-ax,$$

a result in which, if any whole number is substituted for x, the corresponding value of y is also a whole number.

When, therefore, the coefficient of either of the unknown quantities is 1, the problem presents no difficulty.

Next, let $a>b$ and $b>1$,

the value of y given by equation 1 is $y=\frac{c-ax}{b}$.

Let a be divided by b, let $q=$ the quotient, and $r=$ the remainder.

$$\therefore a=bq+r,$$
$$\text{and } y=\frac{c-(bq+r)x}{b},$$
$$\text{or } y=-qx+\frac{c-rx}{b}.$$

If any whole number is substituted for x, the expression $-qx$ must be a whole number. But in order that y also may have an integer value, it is necessary that $\frac{c-rx}{b}$ shall be a whole number.

Let t denote a new indeterminate quantity, and make

$$\frac{c-rx}{b}=t, \text{ or } rx+bt=c \quad - \quad - \quad - \; 2$$

The question is thus reduced to the resolution of equation 2 for integer values of x and t.

If $r=1$, the investigation is ended. Suppose $r>1$.

Resolving equation 2, with respect to x (since r, the coefficient of x, is less than b, the coefficient of t),

$$x=\frac{c-bt}{r}.$$

Divide b by r; let q' represent the quotient, and r' the remainder.

$$\therefore b=rq'+r',$$
$$\text{and } x=\frac{c-(rq'+r')t}{r}=-q't+\frac{c-r't}{r}.$$

Since x and t are to be whole numbers, it is necessary to assume t such that $\frac{c-r't}{r}$ may be a whole number.

Make, therefore, as before,

$$\frac{c-r't}{r}=t' \text{ or } r't+rt'=c \quad - \quad - \quad 3$$

t' being a new indeterminate quantity.

The problem now is, to resolve equation 3 for integer values of t, t'. In this case also the investigation is ended if $r'=1$.

But if $r'>1$, equation 3 gives,

$$t=\frac{c-rt'}{r'}.$$

Making q'' the quotient of r by r', r'' the remainder, and substituting as before,

$$t=\frac{c-(r'q''+r'')t'}{r'}=-q''t'+\frac{c-r''t'}{r'}.$$

Let $\frac{c-r''t'}{r'}=t''$ or $r''t'+r't''=c \quad - \quad - \quad - \quad 4$

t'' being a new indeterminate quantity, and $r''t'+r't''=c$ a new equation, which is to be resolved for integer values of t', t''.

The process to be followed in the calculation is now sufficiently evident; and it is also evident that when an equation has been found in which the coefficient of one of the indeterminate quantities is 1, the question is resolved.

Now, r, r', r'', ... the coefficients of x, t, t, ... in equations 2, 3, 4, ... are the successive remainders obtained by dividing a by b, b by r, r by r' . . . . as in the process for finding the greatest common measure of a, b. Whence, since a, b are prime to each other, a remainder $=1$ must be at length obtained (Part I. Art. 109), that is, an equation must be found in which the coefficient of one of the indeterminate quantities is 1.

Assume, therefore, that $r''=1$.

Then equation 4 gives $t'=c-r't'' \quad - \quad - \quad - \quad 5$

The investigation shows that t, t', t'', ought to be whole numbers; and it is evident, from equation 5, that any integer number whatever being assumed as the value of t'', t' must be a whole number; and consequently that t, x, and y must also be whole numbers. Therefore the proposed equation admits an indefinite number of solutions in whole numbers.

The values of t', t, x, y, given by equations 5, 4, 3, 2, may be written thus,

$$\begin{aligned} t'&=-r't''+c, \\ t&=-q''t'+t'', \\ x&=-q't+t', \\ y&=-qx+t. \end{aligned}$$

Whence, to express the primitive indeterminate quantities x, y in terms of t'', it is necessary to substitute the value of t' in that of t; then the values of t, t' expressed in terms of t'' in that of x; and, lastly, the values of x, t, also expressed in terms of t'' in that of y.

94. The success of this method depends on the principle (Part I. Art. 109) that if the process for finding the greatest common measure is employed upon two numbers which are prime to each other, the last remainder is 1. In the preceding investigation the process is applied to the coefficients of the indeterminate quantities only; but there is no reason why the constant term c may not also be divided. If it is, the calculation must involve smaller numbers, which is an advantage.

As an example, let the equation $3x-8y=43$ be proposed.

Since the coefficient of x is less than that of y, the equation must be resolved with respect to x.

By transposition and division $x=\frac{8y+43}{3}$.

Dividing 8 by 3, the quotient is 2 and the remainder 2; and dividing 43 by 3, the quotient is 14 and the remainder 1.

$$\text{Therefore } x=2y+14+\frac{2y+1}{3}.$$

Making $\frac{2y+1}{3}=t$, and substituting in the last equation $x=2y+14+t$.

$$\text{Again, since } \frac{2y+1}{3}=t,\ y=\frac{3t-1}{2},$$

dividing $3t-1$ by 2, the expression of the quotient is $t+\frac{t-1}{2}$;

$$\text{whence } y=t+\frac{t-1}{2},$$

$$\text{making } \frac{t-1}{2}=t',\ y=t+t'.$$

But since $\frac{t-1}{2}=t'$, $t-1=2t'$, and $t=2t'+1$.

The coefficient t in the equation $t=2t'+1$ being 1, if to t' all possible integer values are given, the corresponding values of t are whole numbers.

Substituting the value of t in the equation whose first member is y,

$$y=t+t' \text{ becomes } y=2t'+1+t', \text{ or } y=3t'+1.$$

Substituting, next, these values of y and t in the equation whose first member is x,

$$x=2y+14+t \text{ becomes } x=2\times(3t'+1)+14+2t'+1,$$
$$\text{or } x=8t'+17.$$

Whence the formulæ which express x, y in terms of t' are

$$y=3t'+1,\quad x=8t'+17.$$

Giving to t' the values - -	0,	1,	2,	3,	4,	. . .
the corresponding values of y are	1,	4,	7,	10,	13,	. . .
the corresponding values of x are	17,	25,	33,	41,	49,	. . .
And giving to t' the values -		−1,	−2,	−3,	−4,	. . .
the corresponding values of y are		−2,	−5,	−8,	−11,	. . .
the corresponding values of x are		+9,	+1,	−7,	−15,	. . .

The additive values of y, viz. 1, 4, 7, 10, 13, . . . obtained by giving to t' the successive values 0, 1, 2, 3, 4, . . . form a series of numbers, each term of which exceeds that which precedes it by 3; and the corresponding additive values of x, which are 17, 25, 33, 41, 49, . . . form a second series, each term of which exceeds the preceding term by 8.

If the values of x, y, which are obtained by making t' successively −1, −2, −3, . . . are considered, they are found to form for y the series

$$-2,\ -5,\ -8,\ -11,\ \ldots$$

and for x the series $+9,\ +1,\ -7,\ -15,\ \ldots$

in the first of which each term is less by 3, and the second by 8, than the preceding term.

95. A series of numbers whose terms are increased or diminished by constant increments or decrements (as in the preceding instances) forms an arithmetical progression, or progression by difference. The constant increment or decrement is termed the ratio.

The values of x, y which satisfy the equation $3x-8y=43$ form, therefore, progressions by difference; the ratio of one progression being 3, and that of the other 8.

These numbers are the coefficients of x, y in the proposed equation; and it is to be observed that the ratio of the series of values of y is 3, the coefficient of x, and that the ratio of the series of values of x is 8, the coefficient of y, but with its sign changed.

96. The general equation $ax+by=c$ - - - 1
is capable of an indefinite number of integer solutions, whatever be the signs and values of a, b, provided they are prime to each other. Let the corresponding values given by one solution be

$$x=\mathrm{A},\ y=\mathrm{B}.$$

These numbers ought to satisfy equation 1; therefore,

$$a\mathrm{A}+b\mathrm{B}=c \qquad - \quad - \quad - \quad 2$$

Subtracting equation 2 from equation 1,

$$a(x-\mathrm{A})+b(y-\mathrm{B})=0,$$

$$\text{whence } y=\mathrm{B}+\frac{a(\mathrm{A}-x)}{b}.$$

The values of x ought to be integer, and such that y may also be a whole number; therefore $a(\mathrm{A}-x)$ must be divisible by b. Now, a is prime with b; therefore $\mathrm{A}-x$ must be a multiple of b (Part I. Art. 111).

Let $\mathrm{A}-x=bt$ (t being capable of expressing any whole number).

$$\text{Since } bt=\mathrm{A}-x \begin{cases} x=\mathrm{A}-bt, \\ y=\mathrm{B}+\dfrac{a\times bt}{b}=\mathrm{B}+at. \end{cases}$$

The formulæ $x=\mathrm{A}-bt$, $y=\mathrm{B}+at$, express the law of the values which are obtained for x, y, by giving to t the successive integer values 0, 1, 2, 3, 4, ...

For these values of t, $x=\mathrm{A}$, $\mathrm{A}-b$, $\mathrm{A}-2b$, $\mathrm{A}-3b$, $\mathrm{A}-4b$, ...
$y=\mathrm{B}$, $\mathrm{B}+a$, $\mathrm{B}+2a$, $\mathrm{B}+3a$, $\mathrm{B}+4a$, ...
And if t is made successively equal to -1, -2, -3, -4, ...
the corresponding values of x are $\mathrm{A}+b$, $\mathrm{A}+2b$, $\mathrm{A}+3b$, $\mathrm{A}+4b$, ...
the corresponding values of y are $\mathrm{B}-a$, $\mathrm{B}-2a$, $\mathrm{B}-3a$, $\mathrm{B}-4a$, ...

In general, when t is augmented by unity y is augmented by a, and x is augmented by $-b$; therefore the integer solutions of the equation $ax+by=c$ are the corresponding terms of two arithmetical progressions. In the progression which expresses the consecutive values of either of the indeterminate quantities x, y, the ratio is equal to the coefficient of the other indeterminate quantity; but it is necessary to take one only of the coefficients with the sign which it has in the equation and the other with a contrary sign.

It is indifferent whether it is the coefficient of x or of y which is taken with a contrary sign; for in the formulæ which express x, y, the signs of the terms $+bt$, $-at$, may be changed, since the indeterminate t may receive all possible values, positive and negative.

This proposition gives the means of obtaining all the integer solutions of an equation of the form $ax+by=c$ when one solution is known.

Thus, the equation $7x-5y=9$ being proposed, it is easily found by trials that the values $x=2$, $y=1$, satisfy it;

$$\text{for } 7\times2-5\times1=14-5=9.$$

Then, observing that $+7$ and -5 are the coefficients of x and of y in the proposed equation, the general formulæ

$$x=\mathrm{A}-bt,\quad y=\mathrm{B}+at, \text{ become}$$
$$x=2+5t,\quad y=1+7t.$$

Making t successively - - 0, 1, 2, 3, 4, ...
the corresponding values of x are 2, 7, 12, 17, 22, ...
the corresponding values of y are 1, 8, 15, 22, 29, ...
And making t successively - -1, -2, -3, -4, ...
the corresponding values of x are -3, -8, -13, -18, ...
the corresponding values of y are -6, -13, -20, -27, ...

97. In the equation $ax+by=c$, let $c=0$,

$$\text{then } ax+by=0.$$

The values $x=0$, $y=0$, verify this equation.

By the substitution of these values for A, B, the general formulæ are reduced to

$$x=-bt,\ y=+at,$$
$$\text{or to } x=+bt,\ y=-at.$$

These results are deducible from the equation $ax+by=0$.

For $x=-\frac{by}{a}$; and since b, a are prime to each other, the integer values of y, which render x a whole number, must be multiples of a. Whence, t representing any whole number whatever, the equation must be $x=-bt$, and consequently $y=+at$.

Example: Let $31x-22y=0$.

Therefore $x=22t$, $y=31t$.

a. Suppose that c is a multiple of one of the coefficients, b for example, and let $c=bd$, the equation to be resolved is $ax+by=bd$.

$$\text{Making } x=0,\quad y=\frac{bd}{b}=d,$$

therefore the equation $ax+by=bd$ is satisfied by the values $x=0$, $y=d$; and the general formulæ are reduced to

$$x=bt,\quad y=d-at.$$

$$\text{The equation } ax+by=bd \text{ gives } x=\frac{b(d-y)}{a};$$

Whence $d-y$ must be a multiple of a. Making $d-y=at$, the values obtained for x, y are $x=bt$, $y=d-at$.

Example. Given $5x-7y=21$, to find the integer values of x, y.

The proposed equation, $5x-7y=21$ may be put under the form, $5x-7\times y=7\times 3$.

$$\text{Making } x=0,\quad y=-\frac{7\times 3}{7}=-3,$$

therefore $x=0$, $y=-3$, are values of x, y.

Consequently, the general values are $x=7t$, $y=-3+5t$.

Making t successively - - 0, 1, 2, 3, 4, . . .
the corresponding values of x are 0, 7, 14, 21, 28, ...
the corresponding values of y are −3, 2, 7, 12, 17, ...

98. Two simplifications, introduced into the following example, are useful in similar instances.

Let the proposed equation be $80x-17y=39$;

$$\text{and therefore } y=\frac{80x-39}{17}.$$

1st. Dividing 80 by 17, the quotient is 4, and the remainder 12; this remainder exceeds half the divisor.

$$\text{But } 80=17\times 4+17-17+12=17(4+1)-(17-12),$$
$$\text{or } 80=17\times 5-5.$$

Augmenting, therefore, the quotient of $\frac{80}{17}$ by 1, the remainder, which is negative, is less than half the divisor. By taking $80=17\times 5-5$ instead of $80=17\times 4+12$, the reduction of the numbers is effected more rapidly.

The quotient of 39 by 17 being 2, and the remainder 5,

$$y=\frac{80x-39}{17}=5x-2-\frac{5x+5}{17}.$$

2d. The numerator $5x+5$ of the fraction $\frac{5x+5}{17}$ can be resolved into the factors 5 and $x+1$; and since the denominator 17 is prime with 5, it follows that to render the product $5(x+1)$ divisible by 17 it is necessary to take $x+1$, a multiple of 17.

The auxiliary equation may therefore be assumed,

$$\frac{x+1}{17}=t,\quad\text{or } x+1=17t.$$

$$\text{Consequently } x=17t-1;$$
$$y=5(17t-1)-2-5t=80t-7.$$

Making t successively - -	0,	1,	2,	3,	4,	. . . .
the corresponding values of x are	-1,	16,	33,	50,	67,	. . . .
the corresponding values of y are	-7,	73,	153,	233,	313,	. . . .

And making t successively - -	-1,	-2,	-3,	-4,	. . . .
the corresponding values of x are	-18,	-35,	-52,	-69,	. . . .
the corresponding values of y are	-87,	-167,	-247,	-327,	. . . .

RESOLUTION OF THE EQUATION $ax+by=c$ IN WHOLE POSITIVE NUMBERS.

99. When the equation $ax+by=c$ is to be resolved in whole positive numbers, the calculation is made as if the numbers were to be integer only.

The expressions which serve to satisfy this condition are,

$$x=\text{A}-bt,\quad y=\text{B}+at.$$

But, instead of giving to t all the integer values possible, it is necessary to select only those which render x, y positive; hence result certain limitations of the values which may be assigned to t.

1st. In the equation $ax+by=c$, let a, b be of the same sign; and if this sign is $-$, let the signs of all the terms be changed, to render a, b positive. Also, since the equation must be impossible in positive numbers if c were negative, let c be positive.

The general values of x, y being put under the forms

$$x=b\left(\frac{\text{A}}{b}-t\right),\quad y=a\left(t-\frac{-\text{B}}{a}\right),$$

it is evident that to render x positive it is necessary and sufficient to take $t<\frac{\text{A}}{b}$; and, similarly, that to render y positive, it is necessary and sufficient to take $t>\frac{-\text{B}}{a}$. Therefore, in order to have positive and integer solutions alone, it is necessary to assign to t only the integer values comprehended between the two limits, $t>\frac{-\text{B}}{a}$, $t<\frac{\text{A}}{b}$. It is, however, proper to remark, that if the first limit, for example, is a whole number n, t may be taken equal to n. The corresponding value of x is 0.

Since t must be a whole number chosen between two limits, it follows that the number of solutions of the equation must be limited. This is evident from the equation itself; for a, b being positive, if positive numbers are substituted for x, y, the two terms, ax, by, must be positive; and as their sum ought to be constantly equal to c, it is impossible that either of the terms can be augmented indefinitely.

It may happen that there is no whole number between the limits assigned to t. When this occurs the equation is impossible; as, for example, if the limits are $t>4\frac{1}{3}$, $t<4\frac{11}{12}$.

Besides, the limits must not imply contradiction, as in the instance $t>4\frac{1}{2}$, $t<3\frac{3}{4}$. For it would be necessary that $\frac{\text{A}}{b}<\frac{-\text{B}}{a}$, or, which is the same thing, $a\text{A}+b\text{B}<0$, while at the same time A and B ought to satisfy the equation, $a\text{A}+b\text{B}=c$, in which c is positive.

2d. Let a, b have contrary signs, and let $ax-by=c$, a and b representing positive numbers. In this case the general values of x, y are,

$$x=\text{A}-(-bt)\text{ or }x=\text{A}+bt;\ y=\text{B}+at.$$

These expressions may be put under the form,

$$x=b\left(t-\frac{-\text{A}}{b}\right);\ y=a\left(t-\frac{-\text{B}}{a}\right);$$

from which it follows, that to render x, y positive it is necessary to have

$$t > \frac{-A}{b},\quad t > \frac{-B}{a};$$

that is to say, t may have any integer value above the greatest of these limits (without excluding equality to the greatest, if that limit is a whole number).

From the preceding investigation, it appears that the equation $ax-by=c$ admits an infinite number of integer positive solutions, while the equation $ax+by=c$ can never have more than a limited number, and may sometimes have not one.

Examples:

1st. A number of men and women contributed 50 £. to a charity; each man gave 19 sh. and each woman 11 sh. How many men and women were contributors?

Let x denote the number of men;

Let y denote the number of women;

then $19x$ sh.=the sum contributed by x men,
and $11y$ sh.=the sum contributed by y women;
also 50 £.=1000 sh.=amount contributed.

The equation which is to be resolved in whole positive numbers is therefore this, $19x+11y=1000$.

Whence $y=\frac{1000-19x}{11}=91-2x+\frac{3x-1}{11}$, making $\frac{3x-1}{11}=t$,

$$y=91-2x+t.$$

$$\therefore \frac{3x-1}{11}=t,\ 3x-1=11t, \text{ and } x=\frac{11t+1}{3}=4t+\frac{1-t}{3}.$$

Let $\frac{1-t}{3}=t'$, the value of x becomes $x=4t+t'$.

$$\therefore \frac{1-t}{3}=t',\ 1-t=3t' \ \therefore t=1-3t'.$$

Substituting $1-3t'$ for t in the expression $x=4t+t'$,

$$x=4(1-3t')+t'=4-12t'+t'=4-11t';$$

and substituting $4-11t'$ for x, $1-3t'$ for t, in the expression

$$y=91-2x+t,$$

$$y=91-2(4-11t')+(1-3t')=91-8+22t'+1-3t'=84+19t'.$$

Consequently the formulæ by which x, y are expressed in terms of t' are,

$$x=4-11t';\ y=84+19t'.$$

Whence, in order that x may be positive, it is necessary that $11t'<4$ or $t'<\frac{4}{11}$; and in order that y may be positive, it is necessary that $19t'>-84$ or $t>-\frac{84}{19}$, or $t'>-4\frac{8}{19}$.

The integer values which t' can have between $+\frac{4}{11}$ and $-4\frac{8}{19}$ are $t'=0$, -1, -2, -3, -4.

To which values of t' correspond

$$x=4,\ 15,\ 26,\ 37,\ 48.$$
$$y=84,\ 65,\ 46,\ 27,\ 8.$$

The number of solutions is limited; and since the terms in x and y are of the same sign, it ought to be so.

The solutions are 1st. 4 men and 84 women.
2d. 15 - 65 -
3d. 26 - 46 -
4th. 37 - 27 -
5th. 48 - 8 -

The general values of x, y can be formed when a single solution of the equation $19x+11y=1000$ has been obtained (Art. 96). Therefore if, after obtaining $t=1-3t'$, t' is made equal to zero, and the corresponding values $t=1$, $x=4$, $y=84$ are computed, it is evident that the values $x=4$, $y=84$, form a solution. Whence the general equations are

$$x=4-11t';\ y=84+19t'.$$

Other examples:

Problem 2d. A farmer sold a part of his live stock consisting of oxen and horses; for each ox he received 8 £. and for each horse 27 £., and for all the oxen 97 £. more than for all the horses. How many oxen and how many horses did he sell?

Ans. The number of solutions is infinite. The corresponding solutions are 29 oxen and 5 horses, 56 oxen and 13 horses, 83 oxen and 21 horses, &c.

Problem 3d. It is required to find a number such that if it is divided by 11 the remainder is 3, and if it is divided by 17 the remainder is 10?

Ans. The number is $146+187t$; t being an indeterminate to which may be assigned any integer positive value, beginning with 0.

Problem 4th. It is required, with two measuring rods, the one 5 feet, the other 7 feet, in length, to measure a distance of 23 feet?

Ans. The equation which is to be resolved in whole positive numbers is $5x+7y=23$. The limits of the indeterminate t are found to be $t>\frac{1}{5}$ and $t<\frac{6}{7}$. Whence, there being no whole number between $\frac{1}{5}$ and $\frac{6}{7}$, the problem is impossible.

100. The determination of the limits in Article 99 leads to the inquiry, what are the values of the final indeterminate t which render positive expressions such as $A+bt$, or which are such that $A+bt>0$?

The term A may be transposed as in an equation. This gives $bt>-A$.

If b is positive both members of this inequality may be divided by b; the result is $t>-\frac{A}{b}$.

But if b is negative the division by b will change the signs of the two members of the inequality, and consequently, from the conventions of Article 3, the member which was the greater becomes the less. In this case the conclusion is $t<-\frac{A}{b}$.

Suppose, as a more general case, the inequality

$$at+b>ct+d.$$

Transposing, $(a-c)t>d-b$.

Then, according as $a-c$ is a positive or negative quantity, $t \gtrless \frac{d-b}{a-c}$.

The inequality, being reduced to this form, is said to be resolved.

The transformations which it may be requisite to make on inequalities are so analogous to those made on equations, that it is always easy to perceive in what manner they ought to be made.

Care must, however, be taken to avoid errors with regard to the signs; and, to avoid such errors, it is only necessary to attend to the conventions of Article 3.

Resolution, in whole numbers and in whole positive numbers, of two or more equations of the first degree, the number of equations being less than the number of unknown quantities.

101. Let it be required to resolve, in whole numbers, the equations

$$2x+14y-7z=341 \quad - \quad - \quad - \quad 1$$
$$10x+4y+9z=473 \quad - \quad - \quad - \quad 2$$

If the first equation is multiplied by 5, and the product is subtracted from the second, an equation is obtained in y, z,

$$\text{viz. } 66y-44z-1232,$$
$$\text{or } 3y-2z=56 \quad - \quad - \quad - \quad 3$$

The integer values of y, z, which satisfy the proposed equations, ought also to satisfy equation 3.

Applying to equation 3 the method of Article 93, it is found that

$$y=2t,\ z=3t-28.$$

The integer solutions of equation 3 are obtained by giving t all possible integer values. But all the integer values of y, z, thus determined, are not capable of verifying the proposed equations; for, that this may be accomplished, the values of y, z must be such that, being connected with certain integer values of x, one of the equations 1, 2 shall be verified.

Substituting the preceding values of y, z in equation 1, it is found that $2x+yt=145$.

From this equation are obtained $x=69+7t'$, $t=1-2t'$; t' denoting any whole number whatever.

If $1-2t'$ is substituted for t in the values of y, z, the unknown quantities x, y, z are found to be expressed in terms of t' by the equations following,

$$x=69+7t',\ y=2-4t',\ z=-25-6t';$$

these equations give all the integer values which satisfy equations 1, 2.

If it is required that the values shall be both integer and positive, it is necessary to choose t' in such a manner that

$$69+7t'>0,\text{ or }t'>-9\tfrac{6}{7},$$
$$2-4t'>0,\text{ or }t'<\tfrac{1}{2},$$
$$-25-6t'>0,\text{ or }t'<-4\tfrac{1}{6}.$$

Whence the only values which can be given to t' are

$$t'=-5,\ -6,\ -7,\ -8,\ -9.$$

Substituting these values of t' in the equations

$$x=69+7t',\ y=2-4t',\ z=-25-6t',$$

five integer positive values of x, y, z are obtained; viz.

$$x=34,\ 27,\ 20,\ 13,\ 6;$$
$$y=22,\ 26,\ 30,\ 34,\ 38;$$
$$z=5,\ 11,\ 17,\ 23,\ 29.$$

This example may suffice to indicate the method to be followed for the resolution, in whole positive numbers, of any number of equations of the first degree, which contain one unknown quantity more than there are equations.

102. When the number of unknown quantities exceeds the number of equations by two or more, the indetermination is still greater than in the preceding cases; but the condition that the values shall be integer and positive considerably limits the number of solutions.

To give an instance; let it be required to resolve, in whole positive numbers, the equation

$$10x+9y+7z=58.$$

Expressing z (which has the least coefficient) in terms of the other quantities,

$$z=\frac{58-9y-10x}{7}=8-y-x+\frac{2-2y-3x}{7}.$$

Since the numerator $2-2y-3x$ ought to be a whole number divisible by 7, let $2-2y-3x=7t$.

$$\text{Consequently } y=\frac{2-3x-7t}{2}=1-x-3t-\frac{x+t}{2}.$$

Again, since $x+t$ ought to be a whole number divisible by 2, let $x+t=2t'$.

$$\text{Therefore } x=-t+2t'.$$

Returning to y and z, these unknown quantities are to be expressed in terms of t, t'.

The formulæ which are obtained for x, y, z are

$$x=-t+2t',\ y=1-2t-3t',\ z=7+4t+t'.$$

To obtain all the integer positive solutions of the proposed equation it is necessary to give to t, t' all the integer values which at once satisfy the three conditions:

$$-t+2t'>0,\ 1-2t-3t'>0,\ 7+4t+t'>0.$$

Hence result for t, t' certain limitations which are discovered by performing with these inequalities operations analogous to those of elimination.

For the sake of exactness, let the signs $\lessgtr$ exclude equality, which amounts to the exclusion of the value zero, for any of the unknown quantities x, y, z.

Multiplying $-t+2t'>0$ by 3, $-3t+6t'>0$,

and multiplying $1-2t-3t'>0$ by 2, $2-4t-6t'>0$.

The two first members of these inequalities being each greater than zero, their sum must be greater than zero.

By the addition $6t'$ is destroyed, and the result is

$$2-7t>0, \text{ whence } t<\tfrac{2}{7}.$$

By a similar elimination between the second inequality and the third it is found that

$$22+10t>0, \text{ and therefore that } t>-2\tfrac{1}{5}.$$

The value of the indeterminate t is therefore comprehended between the limits $-2\frac{1}{5}$ and $+\frac{2}{7}$.

Consequently the only integer values which t can have are $t=-2, -1, 0$:

1st. By the substitution of -2 for t,

$$-t+2t'>0 \text{ becomes } 2+2t'>0; \quad \therefore t'>-1.$$
$$1-2t-3t'>0 \text{ becomes } 1+4-3t'>0; \quad \therefore t'>1\tfrac{2}{3}.$$
$$7+4t+t'>0 \text{ becomes } 7-8+t'>0; \quad \therefore t'>1.$$

Since there is no whole number between 1 and $1\frac{2}{3}$, it follows that the value $t=-2$ must be rejected.

2d. By the substitution of -1 for t,

$$-t+2t'>0 \text{ becomes } 1+2t'>0; \quad \therefore t'>-\tfrac{1}{2}.$$
$$1-2t-3t'>0 \text{ becomes } 1+2-3t'>0; \quad \therefore t'>1.$$
$$7+4t+t'>0 \text{ becomes } 7-4+t'>0; \quad \therefore t'>-3.$$

Since $-3<-\frac{1}{2}$ (Art. 3), the limits are $-\frac{1}{2}$ and $+1$.

The only expression between these limits which is not fractional is 0. This value is a consequence of $t=-1$. Therefore $t=-1$, $t'=0$, are corresponding values of t, t'.

3d. By the substitution of 0 for t, the results obtained are $t'>0, t'<\frac{1}{3}, t'>-7$. Whence, since there is no whole number between 0 and $\frac{1}{3}$, the value $t=0$ must be rejected.

The single values of t and t', to which correspond integer and positive values of x, y, z, are $t=-1$, $t'=0$.

By the substitution of these values,

$$x=-t+2t' \text{ becomes } x=1;$$
$$y=1-2t-3t' \text{ becomes } y=3;$$
$$z=7+4t+t' \text{ becomes } z=3;$$

and this is the only solution of which the proposed equation is susceptible.

As a second example,

$$\text{Let } 6x+7y+3z+2u=100 \quad - \quad 1$$
$$24x+12y+7z+3u=200 \quad - \quad 2$$

By eliminating u the two equations are reduced to this,

$$30x+3y+5z=100,$$

which, resolved with regard to z, gives

$$z=20-6x-\frac{3y}{5}.$$

Since y must be a multiple of 5, make $y=5t$ - 3

$$\therefore z=20-6x-3t \quad - \quad 4$$

Substituting these values of y, z in equation 1,

$$6x+35t+60-18x-9t+2u=100,$$
$$\text{or } -12x+26t+2u=40,$$
$$\text{and therefore } u=20+6x-13t \quad - \quad 5$$

In the equations 3, 4, 5 the three unknown quantities x, y, z are consequently expressed in terms of x and the indeterminate t.

Now, to resolve equations 1, 2 in positive numbers, it is evidently necessary to take x and t positive; for x is one of the unknown primitives, and $y=5t$.

It is also necessary to satisfy the inequalities,

$$20-6x-3t>0,\ 20+6x-13t>0.$$

Adding these inequalities together, x disappears, and the result is

$$40-16t>0;\ \text{whence } t<2\tfrac{1}{2}.$$

The only values, therefore, which t can have are

$$t=0,\ 1,\ 2.$$

1st. Making $t=0$,

$$y=0,\ z=20-6x,\ u=20+6x.$$

Also, since z must be integer and positive, $20-6x$ must be the expression of a whole positive number. It can be so for the values of $x=0, 1, 2, 3$.

From the systems of values, $t=0$, $x=0, 1, 2, 3$, result the four solutions following:

$t=0,\ x=0$	$t=0,\ x=1$	$t=0,\ x=2$	$t=0,\ x=3$
$x=0$	$x=1$	$x=2$	$x=3$
$y=0$	$y=0$	$y=0$	$y=0$
$z=20$	$z=14$	$z=8$	$z=2$
$u=20$	$u=26$	$u=32$	$u=38.$

2d. Making $t=1$,

$$y=5,\ z=17-6x,\ u=7+6x.$$

In this case the value of x is limited by the condition that $z=17-6x$ must be a whole positive number; z is integer and positive, for $x=0, 1, 2$.

From the systems of values, $t=1$, $x=0, 1, 2$, are obtained the three solutions,

$t=1,\ x=0$	$t=1,\ x=1$	$t=1,\ x=2$
$x=0$	$x=1$	$x=2$
$y=5$	$y=5$	$y=5$
$z=17$	$z=11$	$z=5$
$u=7$	$u=13$	$u=19.$

3d. Making $t=2$,

$$y=10,\ z=14-6x,\ u=-6+6x.$$

From $u=-6+6x$, x cannot be zero; for thus, $u=-6$.
From $z=14-6x$, x may be 0 (excluded by the value of u), 1, 2,
Therefore, with $t=2$, x may be equal to 1 or to 2.

From the systems of values, $t=2$, $x=1, 2$, are obtained two solutions; viz.

$t=2,\ x=1$	$t=2,\ x=2$
$x=1$	$x=2$
$y=10$	$y=10$
$z=8$	$z=2$
$u=0$	$u=6.$

The number of solutions is nine. Excluding those in which the values of one or more quantities are zero, the number of solutions is reduced to three; namely, those with $t=1$, $x=1$; $t=1$, $x=2$; $t=2$, $x=2$.

Additional problems:

1st. Find two numbers, such that, if the first is multiplied by 17, and the second by 26, the first product shall exceed the second by 7?

Ans. 5 and 3, 31 and 20, 57 and 37, ... $5+26t$, and $3+17t$.

2d. In a foundry two kinds of cannon are cast; each cannon of the first sort weighs 16 cwt., and each of the second 25 cwt., and yet for the second kind there are used 100 lbs. of metal less than for the first. How many cannons are there of each kind?

Ans. Of the first 11, and of the second 7; or, of the first 36 and of the second 23; . . . . or, of the first $11+25t$ and of the second $7+16t$.

3d. Divide the number 1591 into two parts, such that the one may be divisible by 23 and the other by 34?

Ans. The parts are 1081 and 510, or 299 and 1292.

4th. It is required to find two numbers whose sum and product are equal?

Ans. If x and y denote the two numbers, then x is arbitrary, and

$$y=\frac{x}{x-1}.$$

5th. Required three numbers, such that if the first is multiplied by 7, the second by 9, and the third by 11, the first product shall be 1 less than the second, and 2 greater than the third?

Ans. 5, 4, 3; or 104, 81, 66; or 203, 158, 129; ......

6th. A farmer buys 124 head of cattle, viz., pigs, goats, and sheep, for 400 £. Each pig costs 4 £. 10 sh.; each goat 3 £. 3 sh. 4 d.; and each sheep 1 £. 5 sh. How many are there of each kind?

Ans. 17, 99, 8; or 40, 60, 24; or 63, 21, 40.

7th. Required three numbers such that the sum of their products by the numbers 3, 5, 7, respectively, may be 560, and the sum of their products by the numbers 9, 25, 49, respectively, may be 2920?

Ans. 15, 82, 15; or 50, 40, 30.

8th. Find a number, N, which, being divided by 11, gives the remainder 3; divided by 19, gives the remainder 5; and divided by 29, gives the remainder 10?

Ans. $N=4128+6061t$; 4128 being the least number which satisfies the problem.

SECTION VI.

FORMATION OF THE SQUARE AND EXTRACTION OF THE SQUARE ROOT. CALCULUS OF RADICAL QUANTITIES OF THE SECOND DEGREE.

103. The quantity which, raised to the square, reproduces a given quantity, is termed the square root of that quantity. In arithmetic there are no negative quantities, and the absolute values of numbers are alone considered; a square root can therefore have only a single value. The square root of 4, for example, can be no number but 2. But in algebra, into the calculations of which negative as well as positive quantities enter, the case is otherwise; for it follows from Article 4, that not only $(+2)^2=4$ and $(+a)^2=a^2$, but also that $(-2)^2=4$ and $(-a)^2=a^2$. Therefore, if it is agreed to denote by $\sqrt{a}$ a certain quantity of which the square is a, this square root may be either $+\sqrt{a}$ or $-\sqrt{a}$. The two values are generally expressed thus, $\pm\sqrt{a}$.

$+\sqrt{a}$ and $-\sqrt{a}$ are the only square roots of the quantity a. To establish this proposition, let $x^2=a$. Then the different square roots of a are the values which satisfy the equation $x^2=a$ or $x^2-a=0$.

$$\text{Since } a=\sqrt{a^2},\ x^2-a=x^2-\sqrt{a^2}=0.$$

$$\text{But } \sqrt{a}\times\sqrt{a}=\sqrt{a^2} \text{ (Part I. Art. 257),}$$

$$\text{and } (x-\sqrt{a})\,(x+\sqrt{a})=x^2-\sqrt{a}\times\sqrt{a};$$

$$\therefore x^2-a=(x-\sqrt{a})\,(x+\sqrt{a}),$$

$$\text{and } \therefore (x-\sqrt{a})\,(x+\sqrt{a})=0.$$

But this product cannot be equal to zero, unless one or both its factors are equal to zero.

$$\text{From } x-\sqrt{a}=0 \text{ is obtained } x=\sqrt{a},$$

$$\text{and from } x+\sqrt{a}=0 \text{ is obtained } x=-\sqrt{a}.$$

Whence, since any value of x which is not either $+\sqrt{a}$ or $-\sqrt{a}$, cannot render either of the factors $x-\sqrt{a}$, $x+\sqrt{a}$, equal to zero, it cannot render $x^2-a=0$; therefore the quantity a has no square roots but $+\sqrt{a}, -\sqrt{a}$.

Consequently the square root of an algebraic quantity has two values which are equal and of contrary signs.

The expression $\sqrt{a}$ is taken to signify $+\sqrt{a}$. Therefore when a negative or subtractive root is meant, it is proper to write $-\sqrt{a}$.

104. The quantity placed under the radical sign may be negative, as in the instances $\sqrt{-4}$, $\sqrt{-a^2}$. Now the square of every quantity is positive; for by the rule of signs $(\pm a)\times(\pm a)=+a^2$. Therefore no quantity can have a negative square, and no negative quantity a square root.

The values of the roots of $\sqrt{-4}$, $\sqrt{-a^2}$ are termed imaginary quantities or imaginary roots. In strictness they are not quantities; yet, being operated upon by the rules of the algebraic calculus, this designation is applied to them.

To mark the difference between quantities which are positive or negative and expressions such as $\sqrt{-a}$, the former are termed real quantities.

Every imaginary expression of the form $\sqrt{-A}$ can be transformed into a product of two factors, the one real, the other equal to $\sqrt{-1}$.

For the quantity A, if taken additively, must have two square roots which are real. Let a denote one of these roots, that which is positive; then $\sqrt{-A}$ may be represented by ay, provided y is determined in such a manner that $(ay)^2$ may be equal to $-A$; or, which is the same thing, that $a^2y^2=-A$.

But since A is the square of a, $A=a^2$,

$$\text{therefore } Ay^2=a^2y^2,$$
$$Ay^2=-A,$$
$$y^2=-1,$$
$$\text{and } y=\pm\sqrt{-1};$$
$$\text{therefore } ay=\pm a\sqrt{-1};$$
$$\text{but } ay=\sqrt{-A} \;\therefore\; \sqrt{-A}=\pm a\sqrt{-1}.$$

105. Let $abc\ldots\ldots$ represent any product:

$$(abc\ldots.)^2=(abc\ldots.)\times(abc\ldots.)=aabbcc\ldots.=a^2b^2c^2\ldots\ldots$$

Therefore the square of a product is formed by raising each factor to the square.

The square of a factor a^n, which has an exponent n, is $(a^n)^2=a^n\times a^n=a^{2n}$.

Whence, the square of any monomial is obtained by squaring the coefficients and doubling the exponents:

$$\text{For example } \left(\frac{3}{5}a^2b^3c\right)^2=\frac{9}{25}a^4b^6c^2.$$

Reciprocally, from the preceding principles it is concluded that the square root of a product is obtained by extracting the square root of all the factors; and that the square root of a monomial is obtained by extracting the square root of the coefficient, and dividing the exponents by 2.

These rules give $\sqrt{abc}=\sqrt{a}\times\sqrt{b}\times\sqrt{c}$.

$$\sqrt{64a^4b^2}=\pm 8a^2b.$$

106. A quantity placed under the sign $\sqrt{}$ is termed a radical quantity, or, simply, a radical.

When all the factors of a monomial or of a product are not squares, the extraction of the square root is first indicated; and the radical is afterwards simplified by placing on the exterior of the radical sign all the factors which are squares.

For example, if it is required to extract the square root of $50a^5b^2c$;

$$\text{Since } 50a^5b^2c=2\times 25\times a^4\times a\times b^2\times c=25a^4b^2\times 2ac,$$
$$\text{since also } \sqrt{50a^5b^2c}=\sqrt{25a^4b^2}\times\sqrt{2ac},$$
$$\text{and } \sqrt{25a^4b^2}=5a^2b,$$
$$\text{therefore } \sqrt{50a^5b^2c}=5a^2b\sqrt{2ac}.$$

In general, a radical of the second degree is simplified by decomposing the quantity placed under the radical sign into two products, of which the one contains only square factors, and the other, factors which are not squares; and then extracting the square root of the first product, and indicating the extraction of the square root of the second.

Hence, if a denotes the square root of A,

$$\sqrt{-A}=\sqrt{a^2\times-1}=a\sqrt{-1}.$$

107. Let a denote the numerator and b the denominator of any fraction; then, by Article 34, $\left(\frac{a}{b}\right)^2=\frac{a}{b}\times\frac{a}{b}=\frac{a^2}{b^2}$.

Whence the square of an algebraic fraction is obtained by squaring the terms of the fraction.

Conversely, the square root of an algebraic fraction is obtained by extracting the square root of the terms of the fraction.

Example. Required the square root of $\frac{49a^4b^6}{16c^2d^4}$?

$$\sqrt{49a^4b^6}=7a^2b^3, \text{ and } \sqrt{16c^2d^4}=4cd^2.$$

$$\text{Therefore } \sqrt{\frac{49a^4b^6}{16c^2d^4}}=\frac{7a^2b^3}{4cd^2}.$$

108. When the terms of a fraction are not perfect squares, the root of each term may be indicated, and, if possible, simplified; but, usually, the denominator is rendered rational. This is accomplished by introducing into both terms of the fraction the factors which render the denominator a perfect square; then the square root of the denominator is extracted, and the radical remains only in the numerator.

Example. Extract the square root of $\frac{3a^4b}{50c^3}$?

The number 50 becomes a square if multiplied by 2; and the quantity c^3, if multiplied by c.

$$\frac{3a^4b}{50c^3}=\frac{3a^4b}{50c^3}\times\frac{2c}{2c}=\frac{6a^4bc}{100c^4}$$

$$\sqrt{6a^4bc}=a^2\sqrt{6bc}, \text{ and } \sqrt{100c^4}=10c^2,$$

$$\text{therefore } \sqrt{\frac{3a^4b}{50c^3}}=\frac{a^2\sqrt{6bc}}{10c^2}.$$

109. Let $a+b+c+d$ be a polynomial, in which a, b, c, d represent any quantities whatever.

If all the terms, excepting the last, are considered as one term, and the square of the polynomial is formed as if it were a binomial, $m+d$,

$$\text{then, since } (m+d)^2=m^2+2md+d^2,$$

$$\text{in like manner } \{(a+b+c)+d\}^2=(a+b+c)^2+2(a+b+c)d+d^2,$$

$$\{(a+b)+c\}^2=(a+b)^2+2(a+b)c+c^2,$$

$$(a+b)^2=a^2+2ab+b^2;$$

$$\therefore (a+b+c+d)^2=a^2+2ab+b^2+2(a+b)c+c^2+2(a+b+c)d+d^2.$$

As this method may be extended to include polynomials composed of any number of terms, it follows, from the result obtained by it, that the square of a polynomial consisting of any number of terms is composed of the square of the first term, plus the double product of the first term by the second term, plus the square of the second term, plus the double product of the sum of the first and second terms by the third term, plus the square of the third, &c. &c.

If the products $2(a+b)c$, $2(a+b+c)d$, &c.,
or $2ac+2bc$, $2ad+2bd+2cd$,
are formed, it appears that the square of a polynomial is composed of the

squares of all its terms, plus the double product of all its terms, taken two by two.

110. Let it be required to extract the square root of any polynomial expression, P.

The terms of the proposed polynomial can be arranged in such a manner that the exponents of the same letter (x) are diminished from left to right.

Let the terms be thus arranged, and suppose

$$P=A+B+C+\ldots$$

Also, represent by $a+b+c+\ldots$ the square root of P, arranged in the same manner.

The square of $a+b+c+\ldots$ must reproduce P; whence, from the order observed in arranging the terms, and the principle of Article 12′, it follows that among the terms which compose the square of $a+b+c+\ldots$ that term in which x has the highest exponent is a^2; therefore A is the square of a; consequently the first term of the root is obtained by extracting the square root of the first term of the given polynomial.

Subtracting the square of the first term of the root from P, and denoting the remainder by R,

$$R=B+C+\ldots.$$

R contains the double product of the first term of the root by the second term, plus the square of the second term of the root, &c.

Now it is evident that the double product of the first term of the root by the second term must contain x with an exponent higher than in the other terms of R.

Therefore B is this double product; therefore the second term of the root is found by dividing the first term of R by the double of the first term of the root.

Two terms, a, b, of the root being found, if b is added to $2a$ and the sum $2a+(+b)$ is multiplied by b, the product, $(2a+b)b=2ab+b^2$, is equal to twice the product of the first term of the root by the second term plus the square of the second term.

The product $(2a+b)b$ being subtracted from R, a second remainder, R′, is obtained, which contains the double product of the sum of the first and second terms of the root by the third term plus the square of the third term, &c.

The term in x of this remainder which has the highest exponent must be the double product of the first term of the root by the third; whence it follows that the third term of the root is obtained by dividing the first term of R′ by the double of the first term of the root, that is, by $2a$.

Let the third term of the root be c; the terms a, b, c of the root, and R″, the remainder left by the subtraction of $(2a+2b+c)c$ from R′, afford the means of obtaining the fourth term of the root, in the same manner as a, b, the first and second terms of the root, and R′, the second remainder, have led to the discovery of the third term of the root c.

When P is a square all the terms of the root must be obtained by continuing this series of operations, for each division gives one term.

111. If the process for the extraction of the square root of a number is compared with that for the extraction of the square root of an algebraic quantity, the periods of figures of the number being considered the analogues of the terms, arranged according to the descending powers of x in the algebraic quantity, it is found that the principal difference arises from the distinction between arithmetical and algebraic quantities; namely, that in the former the operations are executed, and in the latter only indicated.

Arranging the power, root, and divisors as in extraction of the square root of a number, and proceeding in the manner pointed out by the investigation contained in the last Article, the square root of any algebraic quantity may be found, as in the following example:

Example. Required the square root of $4x^4+12x^3+5x^2-6x+1$.

Calculation:

$$\begin{array}{rl} & 4x^4+12x^3+5x^2-6x+1\,(2x^2+3x-1 \\ & 4x^4 \\ \hline 4x^2+3x) & 12x^3+5x^2 \\ & 12x^3+9x^2 \\ \hline 4x^2+6x-1) & -4x^2-6x+1 \\ & -4x^2-6x+1 \\ \hline \end{array}$$

Detail: The first term of P is $4x^4$.

The square root of $4x^4$ (by the rule for monomials) is $2x^2$; the first term of the square root of P is therefore $2x^2$.

Subtracting the square of $2x^2$ from P, the first term of R is $12x^3$.

The double of the first term of the root is $4x^2$.

$12x \div {}^3 4x^2 = 3x$; therefore $3x$ is the second term of the root.

$(4x^2+3x)3x = 12x^3+9x^2 =$ the double product of the first term of the root by the second term plus the square of the second term.

Subtracting $12x^3+9x^2$ from R, the first term of R′ is $-4x^2$.

$-4x^2 \div 4x^2$ (the double of the first term of the root) $=-1$; therefore -1 is the third term of the root.

$(4x^2+6x-1)\times(-1) = -4x^2-6x+1 =$ the double product of the sum of the first and second terms of the root by the third term plus the square of the third term.

Subtracting $-4x^2-6x+1$ from R′, it is found that $R''=0$; therefore $4x^4+12x^3+5x^2-6x+1$ is a perfect square, and its square root is $2x^2+3x-1$.

112. When a remainder $=0$ is obtained, it is to be concluded that the proposed polynomial is a square, and its square root complete; for, from the manner in which the calculation is made, each remainder which is successively obtained is the given polynomial, diminished by all the parts which compose the square of those terms of the root which have been found by the part of the calculation already made; therefore when the root is complete the remainder must be equal to zero; and, reciprocally, when a remainder equal to zero is found, it is evident that the terms written in the root compose the exact square root of the polynomial proposed.

On the other hand, when P is not a square, the calculation shows that it is not; for, supposing always that the terms of P are arranged according to the descending powers of x, and observing that at each successive subtraction the first term of P, R, R′ . . . . is destroyed, it must happen that the exponent of x is diminished in the first term of each successive remainder, and by consequence in the terms of the root.

This admitted, let K be the last term of P, that is, the term in which x has the least exponent, and let the square root of K be k. Now it is obvious that if P is a square, k must be the last term of its square root; consequently in the course of the calculation k ought to be found; if it is not the calculation must, by the diminution of the exponents, as above mentioned, give a term of the root of a degree inferior to k, in which case it is evident that P is not a square. The conclusion is the same if the calculation gives the term k with a remainder not equal to zero.

113. The preceding explanations are accommodated to the case in which the terms of P are arranged according to the descending powers of x.

But when the contrary order is adopted they still subsist, it being only necessary to make a slight modification in the test by which it is sought to ascertain whether the polynomial is a square.

First, it is obvious that the same process of calculation gives all the terms of the root, and that when the last remainder is 0, P is a square, and the terms found are those of its square root.

Second. To modify the test by which it is ascertained whether P is a square, it is sufficient to observe that the exponent of the letter according to the powers of which P is arranged increases in the successive terms of the root; and that in the case in which P is a square the calculation ought to give a last term equal to the square root of the last term of P; therefore if such a term is found and the remainder is not zero, or if a higher power of x is found, P is not a square.

It can sometimes be perceived from inspection merely that a polynomial is not a square; for if the terms containing the highest and lowest exponents of any of the letters which enter into P are not squares P cannot be a square. This impossibility may appear at once, or it may be encountered in the course of the calculation, as when the first term of a remainder is not divisible by the double of the first term of the root.

A binomial cannot be a perfect square; for the square of the most simple polynomial (namely, a binomial) contains three distinct parts or terms which cannot be reduced with each other.

When the terms of a polynomial have been arranged according to the powers of x, it may happen that there are several terms in which x has the same exponent. In this case the arrangements prescribed in the analogous case of division (Art. 20) are to be adopted. Then it is evident that the investigations and rules which are applicable to the terms in x must apply to the polynomial coefficient of any power of x.

114. To form the square of any integer expression,

Rule. Multiply the expression by itself; or, if the expression is a monomial, square the coefficient and double the exponents of all the literal factors; if a polynomial, form the square of the first term, the double product of the first term by the second, the square of the second term, the double product of the algebraic sum of the first and second terms by the third term, the square of the third term, &c. &c., and take the algebraic sum of these results.

To form the square of a fractional expression,

Divide the square of its numerator by the square of its denominator.

To extract the square root of a monomial,

Extract the square root of the numeral coefficient, and divide the exponent of every literal factor by 2.

To extract the square root of a polynomial,

Arrange the terms of the polynomial according to the descending powers of the same letter. Find the square root of the first term, and subtract its square from the polynomial; divide the first term of the remainder by double the first term of the root; the quotient is the second term of the root. Multiply the algebraic sum of double the first and once the second term of the root by the second term of the root, and subtract the product from the first remainder; divide the first term of the second remainder by double the first term of the root; the quotient is the third term of the root. Multiply the algebraic sum of double the first and second and once the third terms of the root by the third term of the root; subtract the product from the second remainder, and proceed in this manner till a remainder equal to zero, or a remainder whose first term is not divisible by double the first term of the root, is found.

To extract the square root of a fractional expression,

Take the square root of the numerator of the proposed fraction for the numerator, and the square root of its denominator for the denominator, of the square root required.

a. It is required to form the squares of the following expressions:

1st. $5a^2b^3c$?............ *Ans.* $25a^4b^6c^2$.

2d. $\frac{3}{4}b^4c^5d^2$?............ *Ans.* $\frac{9}{16}b^8c^{10}d^4$.

3d. Cx^my^nz ?............ *Ans.* $C^2x^{2m}y^{2n}z^2$.

4th. $\dfrac{2ab^2d^3}{5c^2e^4}$?............ *Ans.* $\dfrac{4a^2b^4d^6}{25c^4e^8}$.

5th. $3-2x+x^2$?............ *Ans.* $9-12x+10x^2-4x^3+x^4$.

6th. $3ax+2by+cz$?

Ans. $9a^2x^2+12abxy+4b^2y^2+6acxz+4bcyz+c^2z^2$.

7th. $\dfrac{a}{b}+\dfrac{b}{a}$?............ *Ans.* $\dfrac{a^2}{b^2}+2+\dfrac{b^2}{a^2}$.

8th. $\dfrac{a-b}{x^2+2a}$?............ *Ans.* $\dfrac{a^2-2ab+b^2}{x^4+4ax^2+4a^2}$.

9th. $3x-5a-\dfrac{a^2}{2}$?...... *Ans.* $9a^2-30ax-3a^2x+25a^2+5a^3+\dfrac{a^4}{4}$.

10th. $\dfrac{2x}{3y}-\dfrac{3y}{2x}$?............ *Ans.* $\dfrac{4x^2}{9y^2}-2+\dfrac{9y^2}{4x^2}$.

b. Extract the square roots of the following expressions:

1st. $4a^2b^4c^6$?............ *Ans.* $2ab^2c^3$.

2d. $36m^4n^6r^2$?............ *Ans.* $6m^2n^3r$.

3d. $C^2x^{4m}y^{2m}z^{6m}$?............ *Ans.* $Cx^{2m}y^mz^{3m}$.

4th. $\frac{4}{25}a^2b^6c^{2x}$?............ *Ans.* $\frac{2}{5}ab^3c^x$.

5th. $\dfrac{64a^4b^6c^2}{25x^8u^4z^2}$?............ *Ans.* $\dfrac{8a^2b^3c}{5x^4u^2z}$.

6th. $\dfrac{81a^6c^8d^{12}}{256b^{10}g^{14}}$?............ *Ans.* $\dfrac{9a^3c^4d^6}{16b^5g^7}$.

7th. $a^2-2ab+b^2$?............ *Ans.* $a-b$.

8th. $a^2-ab+\dfrac{b^2}{4}$?............ *Ans.* $a-\dfrac{b}{2}$.

9th. a^2+2a+1 ?............ *Ans.* $a+1$.

10th. $m^6+6m^3n^4+9n^8$?............ *Ans.* m^3+3n^4.

11th. $x^4-ax^3+\frac{1}{4}a^2x^2$?............ *Ans.* $x^2-\frac{1}{2}ax$.

12th. $\dfrac{9a^8}{4}+2a^4n^3+\dfrac{4n^6}{9}$?............ *Ans.* $\dfrac{3a^4}{2}+\dfrac{2n^3}{3}$.

13th. $\frac{9}{4}+6x-17x^2-28x^3+49x^4$?............ *Ans.* $\frac{3}{2}+2x-7x^2$.

14th. $\dfrac{a^2x^2+2ab^2x^3+b^4x^4}{a^{2m}+2a^mx^n+x^{2n}}$?............ *Ans.* $\dfrac{ax+b^2x^2}{a^m+x^n}$.

15th. a^2-x^2 ?............ *Ans.* $a-\dfrac{x^2}{2a}-\dfrac{x^4}{8a^3}-\dfrac{x^6}{16a^5}-$, &c.

16th. a^2+x^2 ?............ *Ans.* $a+\dfrac{x^2}{2a}-\dfrac{x^4}{8a^3}+\dfrac{x^6}{16a^5}-$, &c.

17th. $1-x$?............ *Ans.* $1-\dfrac{x}{2}-\dfrac{x^2}{8}-\dfrac{x^3}{16}-$, &c.

18th. $1+x$?............ *Ans.* $1+\dfrac{x}{2}-\dfrac{x^2}{8}+\dfrac{x^3}{16}-$, &c.

19th. $1-6x+15x^2-20x^3+15x^4-6x^5+x^6$?

Ans. $1-3x+3x^2-x^3$.

20th. $\frac{x^4}{y^4}-\frac{8x^2}{y^2}+24-\frac{32y^2}{x^2}+\frac{16y^4}{x^4}$?................*Ans.* $\frac{x^2}{y^2}-4+\frac{4y^2}{x^2}$.

115. It happens most frequently that roots, of whatever order, cannot be expressed exactly without the help of radicals, or quantities under the radical sign. Consequently all the calculations into which these roots enter are more or less encumbered with radicals.

The present purpose is to explain certain transformations and reductions which can be made on radicals of the second degree.

1st. Since the square root of a product is obtained by extracting the square root of every factor of that product,

$$\sqrt{a^2b}=\sqrt{a^2}\times\sqrt{b}=a\sqrt{b};$$

and, reciprocally, $a\sqrt{b}=\sqrt{a^2}\times\sqrt{b}=\sqrt{a^2b}$.

Hence the method of removing a factor from under the radical sign, or of introducing a factor under it.

2d. Since $\frac{a}{\sqrt{b}}=\frac{a}{\sqrt{b}}\times\frac{\sqrt{b}}{\sqrt{b}}=\frac{a\sqrt{b}}{\sqrt{b^2}}=\frac{a\sqrt{b}}{b}$;

$$\therefore \frac{a}{\sqrt{b}}=\frac{a\sqrt{b}}{b}.$$

$$\frac{a}{\sqrt{b}+\sqrt{c}}=\frac{a(\sqrt{b}-\sqrt{c})}{(\sqrt{b}+\sqrt{c})(\sqrt{b}-\sqrt{c})}=\frac{a(\sqrt{b}-\sqrt{c})}{b-c}.$$

$$\frac{a}{\sqrt{b}-\sqrt{c}}=\frac{a(\sqrt{b}+\sqrt{c})}{(\sqrt{b}-\sqrt{c})(\sqrt{b}+\sqrt{c})}=\frac{a(\sqrt{b}+\sqrt{c})}{b-c}.$$

$3a^2+5\sqrt{a^2b^3}-2\sqrt{\frac{a^2}{b}}$ may be taken as another example in which the radical can be made to disappear from the denominator.

For $5\sqrt{a^2b^3}=5\sqrt{a^2}\sqrt{b^2}\sqrt{b}=5ab\sqrt{b}$;

and $2\sqrt{\frac{a^2}{b}}=2\sqrt{\frac{a^2\times b}{b^2}}=\frac{2a}{b}\sqrt{b}$;

$$\therefore 3a^2+5\sqrt{a^2b^3}-2\sqrt{\frac{a^2}{b}}=3a^2+\left(5ab-\frac{2a}{b}\right)\sqrt{b}.$$

These examples show how in certain cases radicals of the second degree may be made to disappear from the denominator of a fraction.

Since the square root of a product is obtained by extraction of the square root of every factor,

$$\sqrt{ab}=\sqrt{a}\times\sqrt{b};$$

and $\sqrt{a}\times\sqrt{b}=\sqrt{ab}$.

Also, since the square root of a fraction is obtained by dividing the square root of the numerator by the square root of the denominator,

$$\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}};$$

or $\frac{\sqrt{a}}{\sqrt{b}}=\sqrt{\frac{a}{b}}$.

The multiplication and division of radicals of the second degree are effected in this manner; namely, the quantities which are under the radical sign are multiplied together, or divided one by the other, and the result is placed under the radical sign.

If the unknown quantity in an equation is a radical of the second degree, by disengaging the radical from the other quantities by the rule for the resolution of equations of the first degree, the equation can be reduced to the form $\sqrt{x}=n$.

$$\text{Whence } \sqrt{x}\times\sqrt{x}=n\times n;$$
$$\text{or } x=n^2.$$

Wherefore, to resolve an equation in which the unknown quantity is a radical of the second degree (any reduction which may be necessary being previously made), transpose the quantities so that the radical may form one member of the equation, the given quantities the other, and square both members of this equation.

1st. $\sqrt{3x+7}=10$ to find x?..*Ans.* $x=31$.

2d. $\sqrt{4x+2}=\sqrt{4x}+5$?..*Ans.* $x=1\frac{129}{400}$.

3d. $\sqrt{x-32}=16-\sqrt{x}$?..*Ans.* $x=81$.

4th. $\dfrac{\sqrt{x}+28}{\sqrt{x}+4}=\dfrac{\sqrt{x}+38}{\sqrt{x}+6}$?..*Ans.* $x=4$.

SECTION VII.

OF EQUATIONS OF THE SECOND DEGREE.

116. An equation of the second degree to one unknown quantity may consist of two terms only; the one involving the square of the unknown quantity, x^2, and the other composed of quantities which are known; or it may consist of three terms, the first involving the square of the unknown quantity, x^2; the second involving the first power of the unknown quantity, x; and the third composed of quantities which are known.

The equation consisting of two terms is called an Incomplete equation of the second degree, or a Pure quadratic equation.

The equation consisting of three terms is called a Complete equation of the second degree, or an Adfected quadratic equation.

117. The equation which contains only x^2 and known quantities can always be reduced to the form, $x^2=A$ (Art. 49). Then, since the square of x ought to reproduce A, it follows that x is the square root of A. Now A has two square roots, and no more; which square roots are represented by $\pm\sqrt{A}$. Whence all the solutions of the equation $x^2=A$ are contained in the two values,

$$x=\pm\sqrt{A}.$$

Example 1. Given $\dfrac{2x^2}{a}-5b=a+\dfrac{3x^2}{b}$, to find the value of x.

Taking away the denominators and transposing, $(2b-3a)x^2=5ab^2+a^2b$.

$$\therefore x^2=\frac{5ab^2+a^2b}{2b-3a},$$

$$\text{and } x=\pm\sqrt{\frac{5ab^2+a^2b}{2b-3a}}.$$

Example 2d. Given $\dfrac{9x-18}{5x}=\dfrac{x}{x+2}$, to find the value of x.

Taking away denominators and transposing, $4x^2=36$.

$$\therefore x^2=9,$$

$$\text{and } x=\pm\sqrt{9}=\pm3.$$

118. The most general equation of the second degree contains three sorts of terms; viz. terms which involve x^2, terms which involve x, and terms composed of quantities entirely known.

After transposing all the terms into the first member, those which involve x^2 being combined into one term, those which involve x into another term, and the terms composed of known quantities into a third, the equation takes the form,

$$ax^2+bx+c=0 \quad - \quad - \quad 1$$

To disengage x^2 from its coefficient, let all the terms of equation 1 be divided by a; then

$$x^2+\frac{b}{a}x+\frac{c}{a}=0.$$

If the term in x^2 has the sign $-$, the signs of all the terms are changed, so that x^2 may always have the sign $+$.

Then, to abridge the expression, making $\frac{b}{a}=p, \frac{c}{a}=q,$

$$x^2+px+q=0 \quad - \quad - \quad 2$$

Such is the general form to which equations of the second degree are reduced; p and q being known quantities, either additive or subtractive.

As an example of the reduction of an equation to this form,

$$\text{Let } \frac{3x}{2}-\frac{x^2}{3}=10-4x.$$

Taking away the denominators, $9x-2x^2=60-24x$;

transposing, $9x-2x^2-60+24x=0$,

or $-2x^2+33x-60=0$;

changing signs, $2x^2-33x+60=0$;

dividing by 2, $x^2-\frac{33}{2}x+30=0$.

In this instance $p=-\frac{33}{2}$, $q=30$.

119. If the general equation $x^2+px+q=0$ could be reduced to the form $(x+m)^2=n$ (m and n being known quantities), its resolution would be easy.

For if $(x+m)^2=n$, $x+m$ must be a square root of n.

But n has only two square roots, denoted by the expression $\pm\sqrt{n}$.

Therefore, since $(x+m)^2=n$, $x+m=\pm\sqrt{n}$, and $x=-m\pm\sqrt{n}$.

For the purpose of reducing the equation $x^2+px+q=0$ to the form $(x+m)^2=n$, let q be transposed to the second member;

$$\text{then } x^2+px=-q.$$

Since the square of a binomial is composed of the square of the first term, plus the double product of the first term by the second, plus the square of the second term, it follows that x^2+px may be regarded as the first and second terms of $(x+\frac{1}{2}p)^2$; for x^2 is the square of x, and px is the double product of x by $\frac{1}{2}p$.

The first member can therefore be rendered the square of $x+\frac{1}{2}p$ by adding to it the square of $\frac{1}{2}p$, or $\frac{1}{4}p^2$.

Now $\frac{1}{4}p^2$ may be added to the first member of the equation, provided it is also added to the second member.

In this manner the equation becomes

$$(x+\tfrac{1}{2}p)^2=\tfrac{1}{4}p^2-q.$$

Whence, as in the case of the equation $(x+m)^2=n$, it follows

$$\text{that } x+\tfrac{1}{2}p=\pm\sqrt{\tfrac{1}{4}p^2-q} \quad - \quad - \quad 3$$

$$\text{and that } x=-\tfrac{1}{2}p\pm\sqrt{\tfrac{1}{4}p^2-q} \quad - \quad - \quad 4$$

Two values of x are thus obtained. It cannot have more than two, for in order that $(x+\frac{1}{2}p)^2$ may be equal to $\frac{1}{4}p^2-q$, it is necessary that $x+\frac{1}{2}p$ be equal to the square root of $\frac{1}{4}p^2-q$; and this root has only the two values expressed by $\pm\sqrt{\frac{1}{4}p^2-q}$. Therefore all the values of x are given by equation 4.

All the values of the unknown quantity which satisfy an equation are said to be roots of that equation. The values of x in equation 4 are roots of equation 2.

To verify these values, it is sufficient to substitute them by turns in equation 2.

Substituting the first, in which the radical has the sign +,

$$(-\tfrac{1}{2}p+\sqrt{\tfrac{1}{4}p^2-q})^2+p(-\tfrac{1}{2}p+\sqrt{\tfrac{1}{4}p^2-q})+q=0,$$

$$\text{or } \tfrac{1}{4}p^2-p\sqrt{\tfrac{1}{4}p^2-q}+\tfrac{1}{4}p^2-q-\tfrac{1}{2}p^2+p\sqrt{\tfrac{1}{4}p^2-q}+q=0;$$

Reducing the like terms, the first member of the equation becomes 0, and consequently the equation is satisfied by the value

$$x=-\tfrac{1}{2}p+\sqrt{\tfrac{1}{4}p^2-q}.$$

The second value of x is verified in the same manner, and with no change except that of the sign − for + before the radical $\sqrt{\tfrac{1}{4}p^2-q}$.

120. In the equation $x^2+px+q=0$, p is the coefficient of x and q the term wholly known, transposed to the first member; the solution, $x=-\tfrac{1}{2}p\pm\sqrt{\tfrac{1}{4}p^2-q}$, may therefore be expressed in words as follows:

When an equation of the second degree is reduced to the form $x^2+px+q=0$, the unknown quantity is equal to half the coefficient of x taken with a contrary sign, plus or minus the square root of the sum formed by adding to the square of this half the term which is wholly known, this term being also taken with a contrary sign.

To apply this rule to examples:

1st. Let $x^2-10x+9=0$; required the values of x?

This example has the form required by the rule. The coefficient of x is -10; therefore half the coefficient of x taken with a contrary sign is $+5$; the term wholly known is 9.

Therefore the rule gives this result,

$$x=5\pm\sqrt{25-9}=5\pm\sqrt{16}=5\pm4;$$

And the two values of x are consequently $x=5+4=9$,

and $x=5-4=1$.

2d. Given $\dfrac{x}{2}+\dfrac{3}{x}=\dfrac{x+13}{3x}$; required the values of x?

To reduce this equation to the form $x^2+px+q=0$, it is necessary to remove the denominators; thus, $3x^2+18=2x+26$; next, to transpose the terms to the left member;

$$\text{whence } 3x^2-2x+18-26=0,$$

$$\text{or } 3x^2-2x-8=0;$$

and lastly, to divide all the terms by 3; making the division,

$$x^2-\tfrac{2}{3}x-\tfrac{8}{3}=0.$$

$$\text{Whence } \tfrac{1}{2}p=+\tfrac{1}{3},\ q=+\tfrac{8}{3},$$

$$\text{and } x=\tfrac{1}{3}\pm\sqrt{\tfrac{1}{9}+\tfrac{8}{3}}=\tfrac{1}{3}\pm\sqrt{\tfrac{1}{9}+\tfrac{24}{9}},$$

$$\text{or } x=\tfrac{1}{3}\pm\sqrt{\tfrac{25}{9}}=\tfrac{1}{3}\pm\tfrac{5}{3}.$$

Consequently the values of x are $x=\tfrac{1}{3}+\tfrac{5}{3}=\tfrac{6}{3}=2$,

$$x=\tfrac{1}{3}-\tfrac{5}{3}=-\tfrac{4}{3}.$$

3d. Given $x-2=\dfrac{4x-9}{x}$, required the values of x?

Clearing fractions, $x^2-2x=4x-9$.

Transposing, $x^2-6x+9=0$.

Whence $x=+3\pm\sqrt{3^2-9}=3\pm\sqrt{0}$,

or $x=3$.

In this example the two values of x are equal.

4th. Given $100x^2-100x+41=0$; required the values of x?

Dividing the terms by 100, $x^2-x+\tfrac{41}{100}=0$.

$$\text{Therefore } x=\tfrac{1}{2}\pm\sqrt{\tfrac{1}{4}-\tfrac{41}{100}},$$

$$\text{or } x=\tfrac{1}{2}\pm\sqrt{\tfrac{25}{100}-\tfrac{41}{100}},$$

$$\text{or } x=\tfrac{1}{2}\pm\sqrt{-\tfrac{16}{100}},$$

$$\text{or } x=\tfrac{1}{2}\pm\tfrac{2}{5}\sqrt{-1}.$$

The values of x in this example are imaginary. The expression $\sqrt{-\tfrac{16}{100}}$ is reduced to $\tfrac{2}{5}\sqrt{-1}$, by Article 104.

121. In a preceding article (Art. 103) the first member of the equation, $x^2-A=0$, is decomposed into the two factors $x-\sqrt{A}$ and $x+\sqrt{A}$. A like decomposition can be made in the general equation,

$$x^2+px+q=0.$$

Replacing x^2+px by the equivalent expression $(x+\frac{1}{2}p)^2-\frac{1}{4}p^2$, the result is, $x^2+px+q=(x+\frac{1}{2}p)^2-(\frac{1}{4}p^2-q)$; and since $\frac{1}{4}p^2-q$ can be changed into $\sqrt{(\frac{1}{4}p^2-q)^2}$, the difference $(x+\frac{1}{2}p)^2-\sqrt{(\frac{1}{4}p^2-q)^2}$, becomes

$$(x+\tfrac{1}{2}p-\sqrt{\tfrac{1}{4}p^2-q})\times(x+\tfrac{1}{2}p+\sqrt{\tfrac{1}{4}p^2-q}),$$

which corresponds with the result obtained in the case $x^2-A=0$.

Since the values of x ought to render the last product equal to zero, this decomposition furnishes a new process for the resolution of equation 2; for it is evident that a product is equal to zero if either of its factors is equal to zero, but that it cannot be zero if neither of its factors is zero. Therefore the values of x which satisfy equation 2, ought to be given by the equations

$$\left.\begin{array}{l} x+\frac{1}{2}p-\sqrt{\frac{1}{4}p^2-q}=0 \\ x+\frac{1}{2}p+\sqrt{\frac{1}{4}p^2-q}=0 \end{array}\right\} \quad . \quad . \quad \text{A.}$$

These equations give the roots already obtained, viz.

$$\left.\begin{array}{l} x=-\frac{1}{2}p+\sqrt{\frac{1}{4}p^2-q} \\ x=-\frac{1}{2}p-\sqrt{\frac{1}{4}p^2-q} \end{array}\right\} \quad . \quad . \quad \text{B.}$$

From these results it is concluded, that the two factors of the first member of the equation $x^2+px+q=0$ are the differences which are obtained by subtracting the roots severally from x, so that, expressing the two roots by the characters x', x'' respectively,

$$x^2+px+q=(x-x')(x-x'').$$

If the multiplication indicated in the second member of the last equation is performed, $x^2+px+q=x^2-(x'+x'')x+x'x''$; and since the two members of this equation must be equal, term by term, it follows that

$$p=-x'-x''$$
$$q=x'x''.$$

These relations, which can be verified directly by means of the values (Equation 4) are enunciated in words thus:

In every equation of the second degree reduced to the form $x^2+px+q=0$, the coefficient p of the second term is equal to the sum of the two roots, taken with contrary signs; and the term entirely known, q, is equal to the product of the roots.

When it is known that the two roots of an equation of the second degree are real, the preceding relations immediately make known the nature of the roots. For example, admitting that the roots of the equation $x^2-2x-7=0$ are real, it is inferred that they are of different signs, for their product is equal to -7, the term which is entirely known; also, that the greater is positive, for their sum is equal to $+2$, the coefficient of x taken with a contrary sign.

122. The general equation of the second degree being $x^2+px+q=0$, the general values of the roots are $-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}-q}$, in which the term $\frac{1}{4}p^2$, being a square, is positive, whatever be the sign of p.

a. If q is negative in the general equation, the quantity $\frac{1}{4}p^2-q$, or, in this case, $\frac{1}{4}p^2+q$, is positive and greater than $\frac{1}{4}p^2$; therefore $\sqrt{\frac{1}{4}p^2+q}$ is a real quantity, and greater in absolute value than $\frac{1}{2}p$; therefore $-\frac{1}{2}p$ placed before the radical does not change the signs of the two values of the radical; therefore the two values of x have contrary signs.

It is, besides, evident that the greater is of the same sign as $-\frac{1}{2}p$, and consequently of the sign contrary to p.

b. If $q=0$, the two values of x are,

$$x=-\tfrac{1}{2}p+\sqrt{\frac{p^2}{4}}=-\tfrac{1}{2}p+\tfrac{1}{2}p=0,$$

$$x=-\tfrac{1}{2}p-\sqrt{\frac{p^2}{4}}=-\tfrac{1}{2}p-\tfrac{1}{2}p=-p.$$

In this case the general equation is reduced to $x^2+px=0$, or $x(x+p)=0$, and it is evident that its roots are $x=0$ and $x=-p$.

c. If q is positive and less than $\frac{1}{4}p^2$, the radical $\sqrt{\frac{1}{4}p^2-q}$ is real, but less than $\frac{1}{2}p$; therefore the two values of x are of the same sign as the term $-\frac{1}{2}p$ placed before the radical, that is to say, they are both positive when p is negative, and both negative when p is positive.

d. If $q=\frac{1}{4}p^2$, the two values of x are reduced to one, viz., $x=-\frac{1}{2}p$.

In this case, since $q=\frac{1}{4}p^2$,

$$x^2+px+\tfrac{1}{4}p^2=0,\text{ or }(x+\tfrac{1}{2}p)^2=0,\text{ or }(x+\tfrac{1}{2}p)(x+\tfrac{1}{2}p)=0;$$

the last of which equations, it is evident, can be verified by the value $x=-\frac{1}{2}p$, and by no other.

e. If q is positive, and greater than $\frac{1}{4}p^2$, the quantity $\frac{1}{4}p^2-q$ under the radical sign is negative, and the two values of x are imaginary.

Changing the signs of the quantity $\frac{1}{4}p^2-q$, and making r the square root of $q-\frac{1}{4}p^2$, $\frac{1}{4}p^2-q=-r^2$;

$$\text{Therefore }\pm\sqrt{\tfrac{1}{4}p^2-q}=\pm\sqrt{-r^2}=\pm r\sqrt{-1},$$

$$\text{and }x=-\tfrac{1}{2}p\pm r\sqrt{-1}.$$

$$\text{From }\tfrac{1}{4}p^2-q=-r^2\text{ is obtained }q=\tfrac{1}{4}p^2+r^2.$$

Whence the general equation becomes

$$x^2+px+\tfrac{1}{4}p^2+r^2=0,\text{ or }(x+\tfrac{1}{2}p)^2+r^2=0.$$

The first member of this equation is the sum of two squares. Under this form the reason wherefore the two values of x are imaginary is obvious; namely, that there can exist no value, whether positive or negative, which being substituted for x can render the sum of these two squares equal to zero.

123. To resolve, in the ordinary manner, the equation

$$ax^2+bx+c=0,$$

all the terms are divided by a, the coefficient of x;

$$\text{then }x^2+\frac{b}{a}x+\frac{c}{a}=0.$$

$$\text{Whence, }x=-\frac{b}{2a}\pm\sqrt{\frac{b^2}{4a^2}-\frac{c}{a}};$$

$$\text{or, }x=-\frac{b}{2a}\pm\sqrt{\frac{b^2-4ac}{4a^2}};$$

$$\text{or, }x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\text{Which may be written }x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

In this expression, according as $b^2-4ac>0$, or $=0$, or <0, the values of x are real and unequal, real and equal, or imaginary.

a. Suppose that a, the coefficient of x^2, is diminished until it becomes equal to zero, the two values of x, taken separately, are

$$x=\frac{-b+\sqrt{b^2}}{0}=\frac{-b+b}{0}=\frac{0}{0},$$

$$x=\frac{-b-\sqrt{b^2}}{0}=\frac{-b-b}{0}=\frac{-2b}{0}.$$

The first value has the form $\frac{0}{0}$; but it does not from this certainly follow that x is indeterminate (Art. 38). The second value is infinite.

Resuming the general value of the first root,

$$x=\frac{-b+\sqrt{b^2-4ac}}{2a}.$$

If a were a common factor of the numerator and denominator of this expression, by suppressing the common factor, and then making $a=0$, the true value of x would be obtained; this, in the present instance, cannot be accomplished, but the difficulty can be avoided. Multiplying the terms of the fractional value of x by $-b-\sqrt{b^2-4ac}$, the result is

$$x=\frac{(-b+\sqrt{b^2-4ac})\ (-b-\sqrt{b^2-4ac})}{-2a(b+\sqrt{b^2-4ac})}.$$

The numerator being now the product of the sum and difference of the quantities $-b$ and $\sqrt{b^2-4ac}$, it is equal to the difference of the squares of these quantities; that is, to

$$b^2-(b^2-4ac),\text{ or to }4ac;$$

and the value of x is reduced to

$$x=\frac{4ac}{-2a(b+\sqrt{b^2-4ac})}.$$

$2a$, which is a common factor of both terms of this value of x, being suppressed, $x=\dfrac{-2c}{b+\sqrt{b^2-4ac}}$.

Making $a=0$ in this expression,

$$x=\frac{-2c}{b+\sqrt{b^2}}=-\frac{2c}{2b}=-\frac{c}{b};$$

and therefore $-\dfrac{c}{b}$ is the true value of the root which at first appeared under the form $\dfrac{0}{0}$.

With respect to the value, $x=\dfrac{-2b}{0}$, it is to be observed that the divisor, 0, may be considered in this case as the limit of decreasing magnitudes, whether positive or negative; and that, for this reason, the infinite value ought to have the ambiguous sign $\pm$. Whence, the values of x, deduced from the equation $ax^2+bx+c=0$ in the hypotheses $a=0$, are

$$x=\frac{-c}{b},\quad x=+\infty,\quad x=-\infty.$$

It deserves notice that in this particular case x has three values while in the general case it has only two.

To show that these values satisfy the equation $ax^2+bx+c=0$ when $a=0$, let the equation be reduced to the form $\dfrac{-bx-c}{x^2}=0$.

The problem consequently is, when $a=0$, to find values which render the fractional expression $\dfrac{-bx-c}{x^2}=0$.

Now it is evident that the value $x=-\dfrac{c}{b}$ renders the numerator $+\dfrac{bc}{b}-c=0$, and, by consequence, the expression $\dfrac{-bx-b}{x^2}=\dfrac{0}{x^2}=0$.

Again, $\dfrac{-bx-c}{x^2}$ can be put under the form $-\dfrac{b}{x}-\dfrac{c}{x^2}$; and it is evident that these expressions become zero, for $x=\pm\infty$ since $-\dfrac{b}{\infty}=0$ and $-\dfrac{c}{\infty}=0$.

b. Let $a=0$, $b=0$.

In this particular case the two general values of x become $\frac{0}{0}$.

It has been already shown that the first general value of x can be changed into $x=\frac{-2c}{b+\sqrt{b^2-4ac}}$.

Transforming the second value of x in the same manner it becomes first

$$x=\frac{(-b-\sqrt{b^2-4ac})\ (-b+\sqrt{b^2-4ac})}{2a(-b+\sqrt{b^2-4ac})};$$

$$\text{and, by reduction, } x=\frac{2c}{-b+\sqrt{b^2-4ac}}.$$

Making $a=0$, $b=0$, both the values of x, thus transformed, give $x=\infty$; and in this case (as in the preceding) the symbol ∞ ought to have the sign $\pm$ prefixed to it.

124. An equation which contains only two powers of the unknown quantity, the exponent of the one power being double that of the other, may be put under the form

$$x^{2m}+px^m+q=0 \quad - \ - \ - \ - \quad 1.$$

Considering x^m as the unknown quantity, x^{2m} is the square of the unknown quantity; consequently the equation can be resolved like an equation of the 2d degree.

$$\text{Whence } x^m=-\tfrac{1}{2}p\pm\sqrt{\tfrac{1}{4}p^2-q}.$$

Denoting $-\frac{1}{2}p+\sqrt{\frac{1}{4}p^2-q}$ by a, and $-\frac{1}{2}p-\sqrt{\frac{1}{4}p^2-q}$ by b, the two values of x^m are $x^m=a$, $x^m=b$.

When these equations can be resolved it is evident that all the values of x are known.

The only case which shall be considered at present is that of $m=2$. Equation 1, in this case, becomes

$$x^4+px^2+q=0 \quad - \ - \ - \ - \quad 2.$$

Making x^2 at first the unknown quantity,

$$x^2=-\tfrac{1}{2}p+\sqrt{\tfrac{1}{4}p^2-q},\quad x^2=-\tfrac{1}{2}p-\sqrt{\tfrac{1}{4}p^2-q}.$$

Since each of the values of x^2 gives two values of x, equation 4 has four solutions; viz.

$$x=\pm\sqrt{-\tfrac{1}{2}p+\sqrt{\tfrac{1}{4}p^2-q}},\quad x=\pm\sqrt{-\tfrac{1}{2}p-\sqrt{\tfrac{1}{4}p^2-q}}.$$

Assuming the particular instance, $x^4-12x^2-64=0$, it is necessary, in the preceding formula, to replace p by -12 and q by -64; then

$$x=\pm\sqrt{6+\sqrt{36+64}}=\pm\sqrt{6+\sqrt{100}}=\pm\sqrt{6+10}=\pm\sqrt{16}=\pm 4,$$

$$x=\pm\sqrt{6-\sqrt{36+64}}=\pm\sqrt{6-\sqrt{100}}=\pm\sqrt{6-10}=\pm\sqrt{-4}=2\sqrt{-1}.$$

Whence the proposed equation has four roots; two real, which are $x=+4$, $x=-4$, and two imaginary, $x=+2\sqrt{-1}$, $x=-2\sqrt{-1}$.

125. The principles by means of which equations of the first degree containing two or more unknown quantities are resolved receive their application in the resolution of equations of the second degree also.

Example. Let $x+y=a$, $x^2+y^2=b^2$; required the values of x, y?

From the 1st equation, $y=a-x$ $\therefore$ $y^2=(a-x)^2=a^2-2ax+x^2$.

Substituting this value, instead of y^2, in the 2d equation,

$$x^2+a^2-2ax+x^2=b^2,$$

$$\text{or, } 2x^2-2ax=b^2-a^2;$$

$$\therefore x^2-ax=\frac{b^2-a^2}{2}.$$

$$\therefore \text{by the formula, } x=\frac{a\pm\sqrt{2b^2-a^2}}{2}.$$

Substituting this value of x in the equation $y=a-x$, the values of y are,

$$y=a-\frac{a\pm\sqrt{2b^2-a^2}}{2}=\frac{a\mp\sqrt{2b^2-a^2}}{2}.$$

x and y have each two values; in employing these for any purpose it is necessary to take together the roots which have the two upper signs and also the roots which have the two lower signs.

The two solutions which are thus obtained differ from each other only by the change of x into y and y into x, a result which was to be anticipated, since this change makes none in the equations $x+y=a$, $x^2+y^2=b^2$.

The same observation applies to the following example, in which $x^2+y^2=a^2$, and $xy=b^2$, are given to find the values of x and y.

Since $xy=b^2$, $y=\frac{b^2}{x}$, and $y^2=\frac{b^4}{x^2}$.

Substituting this value of y^2 in the equation $x^2+y^2=a^2$,

$$x^2+\frac{b^4}{x^2}=a^2,$$

$$\therefore x^4+b^4=a^2x^2,$$

$$\text{and } x^4-a^2x^2+b^4=0.$$

Considering x^2 as the unknown quantity,

$$x^2=\tfrac{1}{2}a^2\pm\tfrac{1}{2}\sqrt{a^4-4b^4},$$

$$\therefore x=\pm\sqrt{\tfrac{1}{2}a^2\pm\tfrac{1}{2}\sqrt{a^4-4b^4}}.$$

This expression gives four roots of x, substituting these values of x in the equation $y=\frac{b^2}{x}$ the four corresponding values of y are,

$$y=\frac{b^2}{\pm\sqrt{\tfrac{1}{2}a^2\pm\tfrac{1}{2}\sqrt{a^4-4b^4}}}.$$

To simplify this expression, let both terms of the fractional value of y be multiplied by $\pm\sqrt{\tfrac{1}{2}a^2\mp\tfrac{1}{2}\sqrt{a^4-4b^4}}$, then

$$y=\frac{\pm b^2\sqrt{\tfrac{1}{2}a^2\mp\tfrac{1}{2}\sqrt{a^4-4b^4}}}{\pm\sqrt{\tfrac{1}{2}a^2\pm\tfrac{1}{2}\sqrt{a^4-4b^4}}\times\pm\sqrt{\tfrac{1}{2}a^2\mp\sqrt{a^4-4b^4}}}.$$

The denominator of this value of y becomes $\pm\sqrt{\tfrac{1}{4}a^4-\tfrac{1}{4}(a^4-4b^4)}=\pm b^2$.

$$\therefore y=\frac{\pm b^2\sqrt{\tfrac{1}{2}a^2\mp\tfrac{1}{2}\sqrt{a^4-4b^4}}}{\pm b^2}=\pm\sqrt{\tfrac{1}{2}a^2\mp\tfrac{1}{2}\sqrt{a^4-4b^4}}.$$

The equations $x^2+y^2=a^2$; $xy=b^2$, may be resolved more simply as follows:

To the first equation $x^2+y^2=a^2$,
add twice the second $2xy=2b^2$;
then $x^2+2xy+y^2$ or $(x+y)^2=a^2+2b^2$,

$$\therefore x+y=\pm\sqrt{a^2+2b^2}.$$

Subtracting twice the second equation from the first, $x^2-2xy+y^2$, or $(x-y)^2=a^2-2b^2$;

$$\therefore x-y=\pm\sqrt{a^2-2b^2},$$

then since $x+y=\pm\sqrt{a^2+2b^2}$,
and $x-y=\pm\sqrt{a^2-2b^2}$,
by addition $2x=\pm\sqrt{a^2+2b^2}\pm\sqrt{a^2-2b^2}$,

$$\therefore x=\pm\tfrac{1}{2}(\sqrt{a^2+2b^2}\pm\sqrt{a^2-2b^2}),$$

and by subtraction $2y=\pm\sqrt{a^2+2b^2}\mp\sqrt{a^2-2b^2}$,
and $y=\tfrac{1}{2}(\pm\sqrt{a^2+2b^2}\mp\sqrt{a^2-2b^2})$.

126. The four solutions obtained in this manner differ from the four obtained by the preceding method. It can, however, be shown that the solutions are identical.

For this purpose it is necessary to investigate a method for the extraction of the square root of a quantity which is partly commensurable and partly incommensurable.

a, b are rational quantities; $\sqrt{a}$, $\sqrt{b}$ their square roots; and if the of $\sqrt{a}+\sqrt{b}$ is formed, the result, which is $a+b+2\sqrt{ab}$, may be d by the general expression $A+\sqrt{B}$, in which A, B are rational

t is concluded, conversely, that the square root of an expression $\sqrt{A+\sqrt{B}}$ may, in certain cases, be reduced to the form required to effect this transformation when it is possible.

ary, therefore, to consider A, B as rational quantities, $\sqrt{B}$ as onal quantity, and to determine for a, b certain rational values hat,

$$(\sqrt{a}+\sqrt{b})^2=A+\sqrt{B},$$

$$\text{or } a+b+\sqrt{4ab}=A+\sqrt{B}.$$

In the first member of this equation $a+b$ is by hypothesis a rational number and only the part $\sqrt{4ab}$ irrational.

A being rational and $\sqrt{B}$ irrational,

$$\text{it is to be proved that } a+b=A \quad - \quad - \quad - \quad 1,$$

$$\text{and that } \sqrt{4ab}=\sqrt{B} \text{ or } 4ab=B \quad - \quad - \quad 2.$$

For suppose that these equalities do not subsist, and transpose the terms of the equation,

$$a+b+\sqrt{4ab}=A+\sqrt{B}.$$

So that $\sqrt{4ab}=A-a-b+\sqrt{B}$, where $a+b$ is by supposition not equal to A, or $A-a-b$ is not equal to zero. To abridge this expression make $A-a-b=k$; then,

$$\sqrt{4ab}=k+\sqrt{B}.$$

Squaring both members, $4ab=k^2+B+2k\sqrt{B}$, and transposing, $4ab-k^2-B=2k\sqrt{B}$, an absurd equality; for a rational cannot be equal to an irrational quantity.

The absurdity can be removed by alone making $k=0$, or $a+b=a$. Then the last equality is reduced to $4ab-B=0$, or $4ab=B$.

$$\text{Since } a+b=A,\ (a+b)^2=A^2, \text{ or } a^2+2ab+b^2=A^2.$$

$$\text{Subtracting } 4ab=B,$$

$$\text{therefore } a^2-2ab+b^2 \text{ or } (a-b)^2=A^2-B,$$

$$\text{and } a-b=\sqrt{A^2-B}^{*} \;.\;.\;.\; 3.$$

$$\text{But } a+b=A,$$

$$\text{therefore } a=\tfrac{1}{2}A+\tfrac{1}{2}\sqrt{A^2-B},$$

$$b=\tfrac{1}{2}A-\tfrac{1}{2}\sqrt{A^2-B}.$$

The question requires that the values of a and b shall be rational. Now in order that this condition may be fulfilled, it is necessary that A^2-B shall be a square. Let $A^2-B=C^2$.

$$\text{Then } a=\frac{A+C}{2};\ b=\frac{A-C}{2};$$

$$\therefore \sqrt{a}=\sqrt{\frac{A+C}{2}};\ \sqrt{b}=\sqrt{\frac{A-C}{2}}.$$

$$\text{But } \sqrt{a}+\sqrt{b}=\sqrt{A+\sqrt{B}};$$

$$\therefore \sqrt{A+\sqrt{B}}=\sqrt{\frac{A+C}{2}}+\sqrt{\frac{A-C}{2}} \;.\;.\; 4.$$

Since $\sqrt{B}$ has been tacitly supposed positive, the radicals of the second member ought both to be taken with the sign + or both with the sign − ; otherwise, if this member is raised to the second power, $-\sqrt{B}$ and not $+\sqrt{B}$ is reproduced.

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[*] It is assumed that a is greater than b, and for this reason the radical $\sqrt{A^2-B}$ is taken with the sign +.

This explanation of itself proves that if the expression is $\sqrt{A-\sqrt{B}}$, it is necessary to take the two radicals with contrary signs;

and consequently that $\sqrt{A-\sqrt{B}}=\sqrt{\frac{A+C}{2}}-\sqrt{\frac{A-C}{2}}$ . . 5.

As a particular case, let $\sqrt{A+\sqrt{B}}=\sqrt{12+\sqrt{140}}$;
then $A=12$, $B=140$, $A^2-B=144-140=4=C^2$.
Therefore $C=\sqrt{4}=2$.

$$\therefore \sqrt{12+\sqrt{140}}=\sqrt{\frac{12+2}{2}}+\sqrt{\frac{12-2}{2}}=\sqrt{7}+\sqrt{5}.$$

As a second particular case, let $\sqrt{A-\sqrt{B}}=\sqrt{1-2x\sqrt{1-x^2}}$.
In this case, $A=1$, $B=4x^2-4x^4$.

$$A^2-B=1-4x^2+4x^4=(1-2x^2)^2=C^2, \text{ and } \therefore C=1-2x^2.$$

$$\text{Whence } \sqrt{1-2x\sqrt{1-x^2}}=\sqrt{\frac{1+(1-2x^2)}{2}}-\sqrt{\frac{1-(1-2x^2)}{2}},$$

$$\text{or } \sqrt{1-2x\sqrt{1-x^2}}=\sqrt{1-x^2}-x.$$

When A^2-B is not a square, the values of a, b are irrational; yet by substituting them in $\sqrt{a}+\sqrt{b}$, the result is still an expression equivalent to $\sqrt{A+\sqrt{B}}$.

When $\sqrt{B}$ is imaginary, if B is changed into $-B^2$, $A+\sqrt{B}$ becomes $A+B\sqrt{-1}$. It deserves notice that the square root of $A+B\sqrt{-1}$ is reducible to an expression] of the form $\alpha+\beta\sqrt{-1}$, in which α, β are real quantities which, as well as A, B, need not be rational. This proposition is rendered evident by the substitution of $-B^2$ for B in the investigation contained in this article.

By this change C becomes equal to $\sqrt{A^2+B^2}$ instead of $\sqrt{A^2-B}$; therefore $a=\frac{A+\sqrt{A^2+B^2}}{2}$; $b=\frac{A-\sqrt{A^2+B^2}}{2}$. The radical $\sqrt{A^2+B^2}$ being greater than A, the value of a is positive and that of b negative.

Denoting these values by α^2 and $-\beta^2$, $\sqrt{a}=\pm\alpha$, $\sqrt{b}=\pm\beta\sqrt{-1}$; and consequently $\sqrt{A+B\sqrt{-1}}=\pm(\alpha+\beta\sqrt{-1})$, ... 6; in which expression α, β are both taken with the sign +, or both with the sign −; because the product $2\alpha\beta$ ought to be equal to B which is assumed to be positive.

If the sign− precedes $B\sqrt{-1}$, it is evident the expression must be,

$$\sqrt{A-B\sqrt{-1}}=\pm(\alpha-\beta\sqrt{-1}) \; . \; . \; . \; . \; . \; 7.$$

If the expression $\sqrt{\pm\sqrt{-1}}$ is compared with $\sqrt{A\pm B\sqrt{-1}}$,

$$A=0,\ B=1,\ a=\frac{A+\sqrt{A^2+B^2}}{2}=\frac{\sqrt{1}}{2}=\frac{1}{2},$$

$$b=\frac{A-\sqrt{A^2+B^2}}{2}=-\frac{\sqrt{1}}{2}=-\frac{1}{2};$$

$$\alpha=\sqrt{\tfrac{1}{2}},\ \beta=\sqrt{\tfrac{1}{2}},$$

$$\therefore \sqrt{\pm\sqrt{-1}}=\pm(\sqrt{\tfrac{1}{2}}\pm\sqrt{\tfrac{1}{2}}\sqrt{-1}).$$

If formulæ 6 and 7 are added together (taking only the upper sign where ± occurs) the imaginary parts destroy each other, and

$$\sqrt{A+B\sqrt{-1}}+\sqrt{A-B\sqrt{-1}}=2\alpha.$$

$$\text{But } \alpha^2=a=\frac{A+\sqrt{A^2+B^2}}{2}; \quad \therefore\ 2\alpha=\sqrt{4a}=\sqrt{2A+2\sqrt{A^2+B^2}};$$

$$\therefore \sqrt{A+B\sqrt{-1}}+\sqrt{A-B\sqrt{-1}}=\sqrt{2A+2\sqrt{A^2+B^2}}.$$

Whence, in the particular case, $A=0$, $B=1$.

$$\sqrt{+\sqrt{-1}}+\sqrt{-\sqrt{-1}}=\sqrt{2\sqrt{1}}=\sqrt{2}.$$

127. As a third example, let it be required to find two numbers, such that the sum of their products by the numbers a, b respectively may be equal to $2s$, and the products of the numbers to p.

Let x, y denote the two numbers.

Then, by the question, $ax+by=2s$ - - - 1,

$xy=p$ - - - 2.

From the 1st equation $y=\frac{2s-ax}{b}$.

Substituting this value of y in the 2d equation $\frac{x(2s-ax)}{b}=p$.

$$\text{Whence } ax^2-2sx=-bp,$$

$$x^2-\frac{2s}{a}x=-\frac{bp}{a},$$

$$x=\frac{s}{a}\pm\frac{1}{a}\sqrt{s^2-abp},$$

$$y=\frac{s}{b}\mp\frac{1}{b}\sqrt{s^2-abp}.$$

Since s is greater than $\sqrt{s^2-abp}$ x has two values affected with the same sign; for the same reason y has two values also affected with the same sign. When $s^2<abp$ these values are imaginary.

If $a=b=1$, $x=s\pm\sqrt{s^2-p}$, $y=s\mp\sqrt{s^2-p}$.

In this case the first value of x is equal to the second value of y, and the second value of x to the first value of y.

The equations from which these values are deduced are

$$x+y=2s, \quad xy=p.$$

If $x+y$ or $2s$ is considered the sum, and xy or p the product of the roots of a quadratic equation (Art. 121), two corresponding values of x and y are given by the equation

$$x^2-2sx+p=0;$$

and these values are, as above,

$$x=s\pm\sqrt{s^2-p}.$$

4th problem. Given $2e$, the sum of the extreme terms of four proportional numbers, $2m$, the sum of the mean terms, and $4s^2$, the sum of the squares, to find the four proportional numbers.

Let v, x, y, z represent the four proportionals.

Then, by the question, $v+z=2e$, $x+y=2m$, $v^2+x^2+y^2+z^2=4s^2$; and by Article 332, Part I, $vz=xy$.

To simplify the solution, let vz or $xy=u$ (u being a quantity which is unknown for the present), then the equations

$$\left.\begin{array}{r} v+z=2e \\ vz=u \end{array}\right\}\text{give, as in the last problem,}\left\{\begin{array}{l} v=e+\sqrt{e^2-u} \\ z=e-\sqrt{e^2-u}. \end{array}\right.$$

And in like manner the equations

$$\left.\begin{array}{r} x+y=2m \\ xy=u \end{array}\right\}\text{give}\left\{\begin{array}{l} x=m+\sqrt{m^2-u} \\ y=m-\sqrt{m^2-u}. \end{array}\right.$$

Substituting these values of v, x, y, z in the equation $v^2+x^2+y^2+z^2=4s^2$,

$$(e+\sqrt{e^2-u})^2+(e-\sqrt{e^2-u})^2+(m+\sqrt{m^2-u})^2+(m-\sqrt{m^2-n})^2=4s^2.$$

Squaring the terms, and reducing

$$4e^2+4m^2-4u=4s^2,$$

$$\therefore\ u=e^2+m^2-s^2.$$

Introducing this value of u into the expressions of the values of v, x, y, z

$$v=e+\sqrt{s^2-m^2}, \quad x=m+\sqrt{s^2-e^2},$$

$$z=e-\sqrt{s^2-m^2}, \quad y=m-\sqrt{s^2-e^2}.$$

Verification. $vz=(e+\sqrt{s^2-m^2})\times(e-\sqrt{s^2-m^2})=e^2-s^2+m^2$,

$xy=(m+\sqrt{s^2-e^2})\times(m-\sqrt{s^2-e^2})=m^2-s^2+e^2$.

The solution of this problem is much facilitated by the employment of the unknown auxiliary u.

128. The most general equation of the second degree containing one unknown quantity has been reduced to the form

$$ax^2+bx+c=0.$$

In like manner the most general equation of the second degree containing two unknown quantities is reducible to the form

$$ax^2+byx+cy^2+dx+ey+f=0,$$

in which a, b, c, d, e, f, are known quantities, numerical or algebraic.

Since two equations are required for the determination of two unknown quantities, let two proposed equations of the second degree in x, y be

$$ax^2+byx+cy^2+dx+ey+f=0 \quad - \quad - \quad - \quad - \quad 1,$$
$$a'x+b'yx+c'y^2+d'x+e'y+f'=0 \quad - \quad - \quad - \quad 2;$$

and from these equations let it be required to determine the values of x, y.

Arranging the terms of equations 1, 2 according to the powers of x,

$$ax^2+(by+d)x+cy^2+ey+f=0 \quad - \quad - \quad - \quad - \quad 3,$$
$$a'x^2+(b'y+d')x+c'y^2+e'y+f'=0 \quad - \quad - \quad - \quad 4.$$

Next, multiplying the terms of equation 3 by a', and the terms of equation 4 by a, and subtracting the second product from the first,

$$aa'x^2+(by+d)a'x+ca'y^2+ea'y+fa'=0,$$
$$aa'x^2+(b'y+d')ax+ac'y^2+ae'y+af'=0;$$
$$\big((ba'-ab')y+(da'-ad')\big)x+(ca'-ac')y^2+(ea'-ae')y+fa'-af'=0;$$
$$\therefore\ x=\frac{(ac'-ca')\,y^2+(ae'-ea')\,y+af'-fa'}{(ba'-ab')y+da'-ad'}.$$

If this expression of x is substituted in either of the equations 1, 2, the result is an equation involving only the unknown quantity y; but it is evident, without making the substitution, that since the numerator of the value of x is of the form my^2+ny+p, the expression of x^2 must be of the fourth degree in y. Now this expression of x^2 forms a part of the result of this substitution.

Whence in general the resolution of two equations of the second degree involving two unknown quantities depends on the resolution of one equation of the fourth degree containing one unknown quantity.

129. The analysis of the second degree in those cases in which the number of unknown quantities exceeds the number of equations is not sufficiently elementary to be considered in this treatise. The subject is discussed in the "Théorie des nombres" of Legendre.

130. To resolve an equation of the second degree.

Rule. Reduce the equation to the form $x^2+px=q$; add to both members the square of half the coefficient of x, $\left(\frac{p^2}{4}\right)$; extract the square root of both members, prefixing to the second the sign $\pm$, and deduce the value of x from the new equation.

Additional equations and problems of the second degree:

1st. $ax^2=b$?..*Ans.* $x=\pm\sqrt{\frac{b}{a}}$.

2d. $x^2+6x=27$?..................................*Ans.* $x=3$; $x=-9$.

3d. $3x-x^2=2$?....................................*Ans.* $x=2$; $x=1$.

4th. $x^2-7x+3\frac{1}{4}=0$?..............................*Ans.* $x=6\frac{1}{2}$; $x=\frac{1}{2}$.

5th. $3x^2-2x=65$?..............................*Ans.* $x=5$; $x=-4\frac{1}{3}$.

6th. $x^2+\frac{10x}{3}=19$?..............................*Ans.* $x=3$; $x=-6\frac{1}{3}$.

7th. $x=\frac{5}{3}+\frac{x^2}{12}$?......................................*Ans.* $x=10$; $x=2$.

8th. $\frac{7x^2}{11}-\frac{2x}{3}=\frac{11x+18}{33}$?.........................*Ans.* $x=2$; $x=-\frac{3}{7}$.

9th. $\frac{x+22}{3}-\frac{4}{x}=\frac{9x-6}{2}$?……………………*Ans.* $x=2$; $x=\frac{12}{25}$.

10th. $\frac{x}{x+1}+\frac{x+1}{x}=\frac{13}{6}$?……………………*Ans.* $x=2$; $x=-3$.

11th. $\frac{x}{x+60}=\frac{7}{3x-5}$?……………………*Ans.* $x=14$; $x=-10$.

12th. $\frac{8x}{x+2}-6=\frac{20}{3x}$?……………………*Ans.* $x=10$; $x=-\frac{2}{3}$.

13th. $\frac{48}{x+3}=\frac{165}{x+10}-5$?……………………*Ans.* $x=5\frac{2}{5}$; $x=5$.

14th. $x^2-8x=14$?…………*Ans.* $x=9{\cdot}4772+$; $x=-1{\cdot}4772+$.

15th. $3x^2+x=7$?…………*Ans.* $x=1{\cdot}3699+$; $x=-1{\cdot}7032+$.

16th. $6x-30=3x^2$?……………………*Ans.* $x=1\pm3\sqrt{-1}$.

17th. $cx+\frac{ac}{a+b}=(a+b)x^2$?……………*Ans.* $x=\frac{c\pm\sqrt{c^2+4ac}}{2(a+b)}$.

18th. $\left.\begin{matrix}x^2+y^2=a\\x^2-y^2=b\end{matrix}\right\}$?………*Ans.* $x=\pm\sqrt{\frac{a+b}{2}}$; $y=\pm\sqrt{\frac{a-b}{2}}$.

19th. $\left.\begin{matrix}x+y=a\\x^2+y^2=b\end{matrix}\right\}$?……*Ans.* $x=\frac{a\pm\sqrt{2b-a^2}}{2}$; $y=\frac{a\mp\sqrt{2b-a}}{2}$.

20th. $\left.\begin{matrix}xy=a\\x^2+y^2=b\end{matrix}\right\}$?

$$Ans.\begin{cases}x=\pm\sqrt{b+2a}\pm\sqrt{b-2a}\text{ or }x=\pm\sqrt{\frac{b\pm\sqrt{b^2-4a^2}}{2}}\\y=\pm\sqrt{b+2a}\mp\sqrt{b-2a}\text{ or }y=\pm\sqrt{\frac{b\mp\sqrt{b^2-4a^2}}{2}}.\end{cases}$$

21st. $\left.\begin{matrix}x+y=a\\x^3+y^3=b\end{matrix}\right\}$? *Ans.* $x=\frac{a}{2}\pm\sqrt{\frac{4b-a^3}{12a}}$; $y=\frac{a}{2}\mp\sqrt{\frac{4b-a^3}{12a}}$.

22d. $x^4-13x^2+36=0$?……………………*Ans.* $x=\pm3$; $x=\pm2$.

23d. $(x^2-2)^2=\frac{1}{4}(x^2+12)$?………………*Ans.* $x=\pm2$; $x=\pm\frac{1}{2}$.

24th. $(x^2-1)(x^2-2)+(x^2-3)(x^2-4)=x^4+5$?
Ans. $x=\pm1$; $x=\pm3$.

25th. It is required to find a number such that, if its half is multiplied by its third part, the product shall be 864?
Ans. 72.

26th. Find two numbers whose product is 750, and the quotient obtained by dividing the greater number by the less $3\frac{1}{3}$?……………………*Ans.* 50 and 15.

27th. The product of two numbers $=a$, their quotient $=b$; required the expressions of the numbers in terms of a, b?……………………*Ans.* $\sqrt{ab}$ and $\sqrt{\frac{a}{b}}$.

28th. The sum of the squares of two numbers is 13001, and the difference of their squares is 1449; required the numbers?……………………*Ans.* 85 and 76.

29th. The sum of the squares of two numbers being expressed by a, and the difference of their squares by b, it is required to express the two numbers in terms of a, b?

Ans. $\sqrt{\frac{a+b}{2}}$; $\sqrt{\frac{a-b}{2}}$.

30th. If two numbers are to each other as 3 to 4, and the sum of their squares is 324900, what are the numbers?

Ans. 342 and 456.

31st. The sum of the squares of two numbers is b, and the first number is to the second as m is to n; what are the numbers?........................*Ans.* $\frac{m\sqrt{b}}{\sqrt{m^2+n^2}}$; $\frac{n\sqrt{b}}{\sqrt{m^2+n^2}}$.

32d. The difference of two numbers is 8 and their product 240; required the numbers?...............*Ans.* 12 and 20.

33d. The difference of two numbers is a, and their product b; required the numbers?

Ans. $\frac{a+\sqrt{a^2+4b}}{2}$; $\frac{a-\sqrt{a^2+4b}}{2}$.

34th. Find a number whose square exceeds the number itself by 306?..*Ans.* 18.

35th. A charitable person distributes 36 £. in equal shares amongst the poor of a village; six of those whom he intended to relieve needing no assistance, each of the remaining paupers receives 10 sh. 8 d. more than he otherwise would have had; how many paupers were there at first?..*Ans.* 54.

36th. A horse-dealer pays a certain sum for a horse, which he afterwards sells for 144 £., and gains exactly as much per cent. as the horse cost him; how much did the horse cost?.......................................*Ans.* 80 £.

37th. The sum of two numbers being 41, and the sum of their squares 901; what are the numbers?....*Ans.* 15 and 26.

38th. Divide the number 16 into two parts, such that the squares of the two parts augmented by their product shall be 208?...........................*Ans.* The parts are 4 and 12.

39th. A grazier bought a certain number of oxen for 240 £.; after losing three he sold the remainder for 8 £. a head more than they cost him, and thus gained 59 £. by the bargain; what number did he buy?...............*Ans.* 16.

40th. It is required to find two numbers such, that if they are multiplied respectively by the given numbers a, b, the sum of the products shall be a given number, p; and if their squares are multiplied respectively by the same numbers a, b, the sum of the new products shall be equal to another given number, q.

Ans. $x=\frac{ap\pm\sqrt{ab[(a+b)q-p^2]}}{a(a+b)}$ $y=\frac{bp\mp\sqrt{ab[(a+b)q-p^2]}}{b(a+b)}$.

SECTION VII

OF THE FORMATION OF POWERS AND EXTRACTION OF ROOTS.

131. Let $abc \ldots\ldots$ be a product, and n any positive whole number.

$(abc\ldots\ldots)^n = abc\ldots \times abc\ldots\ldots \times abc\ldots\ldots \times$ &c. to n repetitions of the quantity $abc\ldots\ldots$

$= aaa\ldots$ to n repetitions of the factor $a \times bbb\ldots\ldots$ to n repetitions of the factor $b \times ccc\ldots$ to n repetitions of the factor $c \times \ldots\ldots = a^n b^n c^n \ldots\ldots$

Whence a product is raised to any power by raising each of its factors to that power.

Let it be required to raise a^m to the power n, m and n being whole and positive numbers.

$(a^m)^n = a^m \times a^m \times a^m \ldots$ to n repetitions of the quantity $a^m = a^{m+m+m\ldots} = a^{mn}$.

Consequently a quantity which is affected with an exponent is raised to a power by multiplying its exponent by the degree of that power.

If it is required to raise any monomial to a power, regarding the monomial as a product, all its factors must be raised to the required power. This is accomplished by raising the coefficient to the required power, and multiplying all the exponents by the degree of that power.

When a quantity is positive, all its powers are positive; when negative it must have + before the second power, − before the third, + before the fourth, &c. (Art. 4. c). Whence, in general, whatever be the sign of a quantity, the even powers of that quantity have the sign +, and the odd powers have the same sign as the quantity.

$$\text{Hence } (\pm 2a^3b^5c)^4 = +16a^{12}b^{20}c^4.$$
$$(\pm 2a^3b^5c)^5 = \pm 32a^{15}b^{25}c^5.$$

132. By reversing these rules the following rules for the extraction of roots are obtained:

1st. The root of a product is obtained by extracting the required root of each factor of that product.

2d. The root of a quantity, which has an exponent, is obtained by dividing that exponent by the exponent or index of the root.

3d. The root of any monomial is obtained by extracting the root of the coefficient, and dividing the exponent of each letter by the index of the root.

4th. When the index of the root extracted is even, the root ought to have the ambiguous sign, ±; and when the index of the root is odd, the root has the same sign as the power.

$$\text{Hence } \sqrt[4]{81a^{16}b^{12}} = \pm 3a^4b^3.$$
$$\sqrt[5]{32a^{10}b^5} = 2a^2b.$$
$$\sqrt[5]{-32a^5b^{15}} = -2ab^3.$$

133. When the coefficient of a monomial is not an exact power of the order marked by the index of the root which is to be extracted, or when the exponents of the different factors are not divisible by that index, the root of the monomial is first indicated by means of the sign $\sqrt{}$; then the radical is simplified, as has been already done in the case of extraction of the square root.

As an illustration let it be required to extract the 5th root of $96a^7b^5c^{12}$.

$$96 = 2^5 \times 3;\ a^7 = a^5 \times a^2;\ c^{12} = c^{10} \times c^2.$$
$$\therefore \sqrt[5]{96a^7b^5c^{12}} = \sqrt[5]{2^5a^5b^5c^{10} \times 3a^2c^2} = 2abc^2\sqrt[5]{3a^2c^2}.$$

Whence to simplify any radical it is necessary to decompose the quantity under the radical sign into two factors, of which the one involves exact powers of the same order as the root which is to be extracted, and the other involves no exact powers of that order; then to extract the root of the first factor, and to indicate the root of the second.

134. No real quantity, whether positive or negative, when raised to an even power, can give a negative result. Consequently every expression composed of a radical of an even degree placed over a negative quantity represents an imaginary quantity.

Supposing a, b, positive magnitudes, the expressions $\sqrt[4]{-a}$, $\sqrt[6]{-a^5}$, $\sqrt[8]{-2ab^3b^4}$, are imaginary.

But expressions such as these may, like imaginary radicals of the second degree, be transformed into products in which the only imaginary quantity is a root of -1.

For example, $\sqrt[4]{-a}=\sqrt[4]{a\times-1}=\sqrt[4]{a}\times\sqrt[4]{-1}$.

135. In the same manner as division has led to the use of negative exponents, the extraction of roots leads to the use of fractional exponents.

By the preceding rule the 5th root of a^{10} is obtained by dividing the exponent of a by 5, and the root required is a^2. But the division may be indicated only, and the expression written either thus, $\sqrt[5]{a^{10}}$, or thus, $a^{\frac{10}{5}}$.

If the exponent of a is not divisible by 5, the root cannot be extracted, but nothing prevents the indication of the division of the exponent, which in such a case obtains for denominator the index of the root. This convention has been adopted by algebraists, and in consequence the two expressions $\sqrt[n]{a^m}$ and $a^{\frac{m}{n}}$ are considered equivalent.

THEORY OF ARRANGEMENTS, PERMUTATIONS, AND COMBINATIONS.

136. To form any power whatever of a binomial, it is incidentally found to be necessary to investigate the theory of arrangements, permutations, and combinations.

Many letters, a, b, c, d, e...being given, suppose them taken two by two in all manners possible, in such sort that each assemblage of two letters differs from the others either by the letters which compose it, or by the order in which they are written; there are thus formed what are named the arrangements, two by two, of the given letters. When the letters are, in like manner, taken three by three, four by four, &c., the arrangements three by three, four by four, &c. are obtained.

It is easy to form these arrangements and to determine their number. In forming them it is convenient to follow an order by which no one of the terms will be repeated.

Thus in arranging the terms two by two it is sufficient to annex in turn to each letter all the other letters.

In this manner are obtained arrangements such as

$$\begin{array}{l} ab,\ ac,\ ad,\ ae,\ af,\ \ldots\ldots \\ ba,\ bc,\ bd,\ be,\ bf,\ \ldots\ldots \\ ca,\ cb,\ cd,\ ce,\ cf,\ \ldots\ldots \end{array}$$

and it is manifest, if the number of letters a, b, c, d..... is m, and the number of arrangements two by two is A_2, that each horizontal row contains one arrangement fewer than there are letters (or $m-1$), and that there are as many horizontal rows as there are letters (m); and therefore that the whole number of arrangements, or $A_2=m(m-1)$.

The arrangements three by three of the letters $a, b, c, d, e, \ldots$. can be formed from the arrangements of the letters, taken two by two, by annexing to each of the $m(m-1)$ arrangements in turn each of the $m-2$ letters which do not enter into the arrangement.

Thus from ab are formed abc, abd, abe, abf, ...
from ac - acb, acd, ace, acf, ...
. . . .
from ba - bac, bad, bae, baf, ...
. . . .
from ca - cab, cad, cae, caf, ...
. . . .

If these arrangements are represented at length in a tabular form, it is evident that there are A_2 horizontal rows, and that the number of arrangements in each row is $m-2$. Therefore, making A_3 the expression of the whole number of arrangements of m letters, taken three by three,

$$A_3 = A_2 \times (m-2) = m(m-1)(m-2).$$

The arrangements four by four of the letters a, b, c, d, e.... are formed from the arrangements of the same letters taken three by three, by annexing to each of the $m(m-1)(m-2)$ arrangements in turn each of the $m-3$ letters which do not enter into the arrangement.

Denoting by A_4 the whole number of arrangements of m letters, taken four by four,

$$A_4 = A_3 \times (m-3) = m(m-1)(m-2)(m-3).$$

For every additional letter introduced into the arrangements, it is evident that another factor must be introduced into the formula, and that this factor must be less by 1 than the factor which precedes it.

Whence the general expression of the number of arrangements of m letters, taken n by n, must involve n factors, taken consecutively in the series m, $m-1$, $m-2$, &c., the last factor being $m-(n-1)$ or $m-n+1$.

Consequently, denoting by A_n the number of arrangements of m letters, taken n by n,

$$A_n = m(m-1)(m-2)(m-3) \ldots\ldots. (m-n+1) \quad - \quad 1$$

If n is made successively 2, 3, 4,...the last factor $m-n+1$ becomes $m-1$ in the first instance, $m-2$ in the second, $m-3$ in the third;...... and the values of A_2, A_3, A_4, ... are recovered.

137. If any letters, *abcde*....., are written consecutively, and the order of these letters is changed in every possible manner, their permutations are obtained.

For example, the permutations of the three letters *abc* are,

abc, *acb*, *bac*, *bca*, *cab*, *cba*.

The permutations of n letters are consequently the arrangements of these n letters taken n by n.

Whence, if m is replaced by n, and the number of permutations of n letters is denoted by P_n, formula 1 gives

$$P_n = n(n-1)(n-2)(n-3) \ldots\ldots. (n-n+1),$$

$$\text{or } P_n = n(n-1)(n-2)(n-3) \ldots\ldots.. \times 1;$$

or, by reversing the order of the factors,

$$P_n = 1 \times 2 \times 3 \times \ldots\ldots \times n \quad - \quad - \quad 2$$

Formula 2 can be obtained as follows, without deduction from formula 1:

With two letters only two permutations, *ab*, *ba*, can be formed. The permutations of three letters are obtained by writing successively after each of the three letters the permutations of the two others; this gives 3×2 or 2×3 as the number of permutations which can be formed with three letters.

Similarly for 4 letters the number of permutations is $2 \times 3 \times 4$, and for n letters it is $1 \times 2 \times 3 \times \ldots\ldots \times n$.

138. Of the arrangements of m letters, taken n by n, some differ from others only in respect that the same letters are taken in different orders, as *abc*, *acb*, *bac*, *bca* .....; while others differ from each other in respect that, although the number of letters is the same, one or more of these letters are different, as *abc*, *abd*, *acd*, *bcd*....

The latter arrangements are distinguished by the name of Combinations or Products.

It is to be observed that the six arrangements, *abc*, *acb*, *bac*, *bca*, *cab*, *cba*, form only one combination or product (Part I. Art. 73).

Among the arrangements of m letters n by n, it is evident that each combination of n letters must be repeated as many times as there are permutations of the n letters of that combination. Therefore, if the total number of arrangements which can be formed of m letters, taken n by n, is divided by the number of permutations which can be formed with n letters, the number of combinations of m letters, taken n by n, must be obtained. Denoting this number of combinations by C_n,

$$C_n=\frac{A_n}{P_n}=\frac{m(m-1)(m-2)\ldots\ldots(m-n+1)}{1\times2\times3\times\ldots\ldots\times n} \quad . \quad . \quad 3$$

If the number of combinations of m letters, taken 2 by 2, 3 by 3, ... is required, it is necessary to make $n=2$, 3, ... in formula 3,

$$\text{then } C_2=\frac{m(m-1)}{1,2},$$

$$C_3=\frac{m(m-1)(m-2)}{1,2,3.}.$$

The hypothesis $n=1$ gives $C_1=m$, and it is evident that the combinations of m letters, taken 1 by 1, can only be the m letters taken individually.

The number of combinations which can be made with any given letters being necessarily integer, it follows that the division indicated in formula 3 is exact.

Whence a product of n consecutive whole numbers is divisible by the product $1\times2\times3\times\ldots\times n$.

Binomial Theorem; the Exponent being a Whole Positive Number.

139. Every binomial may be represented by $x+a$; and the problem is to find a general formula by means of which any power, m, of $x+a$ may be obtained without the necessity of forming all the $m-1$ preceding powers.

The investigation is usually made by forming several consecutive powers of $x+a$ by actual multiplication, and inferring from the results a law which gives the means of forming any power whatever without forming the inferior powers.

$$\begin{array}{rl}
 & x+a \\
 & x+a \\ \cline{2-2}
 & x^2+ax \\
 & \qquad ax+a^2 \\ \cline{2-2}
(x+a)^2= & x^2+2ax+a^2 \\
 & x+a \\ \cline{2-2}
 & x^3+2ax^2+a^2x \\
 & \qquad ax^2+2a^2x+a^3 \\ \cline{2-2}
(x+a)^3= & x^3+3ax^2+3a^2x+a^3 \\
 & x+a \\ \cline{2-2}
 & x^4+3ax^3+3a^2x^2+a^3x \\
 & \qquad ax^3+3a^2x^2+3a^3x+a^4 \\ \cline{2-2}
(x+a)^4= & x^4+4ax^3+6a^2x^2+4a^3x+a^4 \\
 & x+a \\ \cline{2-2}
 & x^5+4ax^4+6a^2x^3+4a^3x^2+a^4x \\
 & \qquad ax^4+4a^2x^3+6a^3x^2+4a^4x+a^5 \\ \cline{2-2}
(x+a)^5= & x^5+5ax^4+10a^2x^3+10a^3x^2+5a^4x+a^5 \\
 & . \quad . \quad . \quad . \quad . \quad . \quad . \quad .
\end{array}$$

The general law of the exponents is evident; it is also apparent that the coefficients of the first and last terms are equal to 1; that the coefficients of the second term and the last but one are equal to the exponent of the binomial; and, with respect to the intermediate terms, that the coefficients equally distant from the extremes are equal; but their composition is not distinctly seen.

These coefficients arise from the reduction of the like terms.

Now if binomials, such as $x+a$, $x+b$, $x+c$, ... whose second terms are different, are multiplied together, these reductions of like terms cannot be made, and the composition of the terms of the product may then be perceived.

Forming the successive products of these binomials,

$$\begin{array}{r|l|l|l}
(x+a)(x+b)\quad - \quad - =x^2+a & x+ab & & \\
+b & & & \\
(x+a)(x+b)(x+c)\quad - =x^3+a & x^2+ab & x+abc & \\
+b & +ac & & \\
+c & +bc & & \\
(x+a)(x+b)(x+c)(x+d)=x^4+a & x^3+ab & x^2+abc & x+abcd \\
+b & +ac & +abd & \\
+c & +bc & +acd & \\
+d & +ad & +bcd & \\
 & +bd & & \\
 & +cd & &
\end{array}$$

Considering as one all the terms which involve the same power of x, the same law is observed to hold good for the exponents of x in the products $(x+a)(x+b)$, $(x+a)(x+b)(x+c)$, ... as in the powers of $x+a$. It is this: in the first term the exponent of x is equal to the number of binomial factors; and it is diminished progressively from term to term by unity to the last term, in which it is 0.

The law of composition of the multipliers of the descending powers of x is equally obvious; viz. the highest power is multiplied by 1; the power whose exponent is less by 1 is multiplied by the sum of the second terms of the binomials; that whose exponent is less by 2 is multiplied by the sum of the different products obtained by combining the second terms two by two; that whose exponent is less by 3 is multiplied by the sum of the different products obtained by combining the second terms three by three; and so on to the last term (in which the exponent of x is zero) which is equal to the product of all the second terms of the binomials.

These laws of the composition of the exponents and coefficients of the powers of a binomial are derived by induction from a certain and not very great number of multiplications.

To prove them true for any number of binomials, or power of a binomial, let it be assumed that they are true for the product of a certain number, m, of binomials.

Then, if it can be proved that they are true for the number $m+1$, the general proposition will be established.

Let therefore $x^m+Ax^{m-1}+Bx^{m-2}+Cx^{m-3}+ \ldots\ldots +Y$, be the product of m binomials, $x+a$, $x+b$, $x+c$, ...

And let this product be multiplied by another binomial, $x+l$; the result is

$$\begin{array}{r|r|r|l}
x^{m+1}+A & x^m+B & x^{m-1}+C & x^{m-2}+ \ldots\ldots +lY. \\
+l & +lA & +lB &
\end{array}$$

In this result the law of the exponents is evidently the same; for the exponent of x in the first term is equal to the number of binomial factors; and it is diminished successively by 1 in each term down to the last term which does not contain x.

The law of the coefficients also holds good; for,

1st. The coefficient of the first term is 1.

2d. In the second term, in which the exponent of x is less by 1 than in the first term, the coefficient is $A+l$.

Now A being the sum of the second terms of the m binomials $x+a$, $x+b$, $x+c$, ... it follows that $A+l$ is the sum of the second terms of the $m+1$ binomials $x+a$, $x+b$, $x+c$, ... $x+l$.

3d. In the third term, in which the exponent of x is less by 2 than in the first term, the coefficient is $B+lA$.

Now B expresses the sum of the products of the m second terms taken two by two, and lA is the sum of the products of these m terms by l; therefore

$B+lA$ is the sum of all the products which can be formed with the $m+1$ second terms $a, b, c, \ldots . l$, taken two by two.

4th. In the fourth term, in which the exponent of x is less by 3 than in the first term, the coefficient is $C+lB$.

Now C expresses the sum of the products which can be formed with the m quantities $a, b, c, \ldots .$ taken three by three, and B is the sum of the products of these m quantities taken two by two; therefore lB is the sum of these products multiplied by l.

Whence $C+lB$ is the sum of all the products which can be formed with the $m+1$ second terms $a, b, c, \ldots . l$, taken three by three.

This reasoning may be extended to all the other terms, including the last, lY, which is evidently the product of the $m+1$ second terms $a, b, c, \ldots .. l$, since Y expresses the product of the m quantities $a, b, c, \ldots ..$

Therefore, the law of composition supposed true for a number of binomials, m, is also true for a number $m+1$.

Now by actual multiplication this law is proved to be true in the case of two binomial factors, therefore it is true for $2+1$ or 3. Being true for 3, it is also true for $3+1$ or 4, and so on, whatever the number of factors.

Therefore the law of composition of the exponents and coefficients is general.

To return from the product of m binomial factors $x+a, x+b, x+c, \ldots ..$ to the mth power of the binomial $x+a$, it is sufficient to make each of the second terms $a, b, c, \ldots ..$ equal to a. Then,

1st. It is evident that the powers of x contained in the successive terms are $x^m, x^{m-1}, x^{m-2}, \ldots \ldots x^0$, and that the first term is simply x^m.

2d. In the second term of the product of the m binomials the multiplier of x^{m-1} is the sum of the m second terms of the binomials; therefore, in the development of $(x+a)^m$, this multiplier must be ma, and consequently the second term is max^{m-1}.

3d. In the third term of the product of the m binomials the different products, two by two, by which x^{m-2} is multiplied, become each equal to a^2. Their sum is equal to as many repetitions of a^2 as it is possible to form combinations of m letters taken two by two. This number is $\frac{m(m-1)}{1.2}$ (Art. 138).

Whence the third term of the development is

$$\frac{m(m-1)}{1.2}a^2x^{m-2}.$$

4th. In the fourth term of the product of the m binomials, the multiplier of x^{m-3} is the sum of the combinations of m quantities taken three by three. When all the second terms $b, c, d, \ldots .$ are equal to a, each of the combinations is equal to a^3; and the number of combinations of m quantities taken three by three is $\frac{m(m-1)(m-2)}{1.2.3}$; therefore the fourth term of $(x+a)^m$ is $\frac{m(m-1)(m-2)}{1.2.3}a^3x^{m-3}$.

5th. In like manner, in the fifth term, the different products by which, taken four by four, x^{m-4} is multiplied, become each equal to a^4; and their number is $\frac{m(m-1)(m-2)(m-3)}{1.2.3.4}$. Whence the fifth term is

$$\frac{m(m-1)(m-2)(m-3)}{1.2.3.4}a^4x^{m-4}.$$

The sixth, seventh, $\ldots ..$ $m-1$th $\ldots .$ terms are formed in the same manner. Whence

$$(x+a)^m=x^m+\frac{m}{1}ax^{m-1}+\frac{m(m-1)}{1.2}a^2x^{m-2}+\frac{m(m-1)(m-2)}{1.2.3}a^3x^{m-3}+\ldots ..+a^m.$$

140. The law of the terms is so evident that any term may be written at pleasure; for,

1st. Each coefficient has for denominator the series 1, 2, 3, ... continued to that number which marks the number of preceding terms, and for numerator the product of the decreasing factors m, $m-1$, $m-2$, ... continued to that factor in which m is diminished by the last factor but one of the denominator.

2d. The exponent of a in any term is always equal to that number which marks the number of the preceding terms, and that of x is equal to m diminished by the same number; so that the sum of the exponents of x and a is constantly equal to m.

Denoting by T_n the term whose rank is n, the term T_{n+1}, or that which has n terms before it, is

$$T_{n+1}=\frac{m(m-1)(m-2)\ \ldots\ldots\ (m-n+1)}{1.2.3\ \ \ldots\ldots\ \ n}a^n x^{m-n}.$$

This expression is called the general term of the formula. It is so named because, by making successively $n=1, 2, 3, \ldots.$, all the terms of the formula, beginning with the second, may be deduced from it. To find the sixth term, for example,

$$\text{Make } n=5\text{; then } T_6=\frac{m(m-1)(m-2)(m-3)(m-4)}{1.2.3.4.5}a^5 x^{m-5}.$$

To find the last term it is to be kept in mind that the formula contains $m+1$ terms, and that for the last term it is necessary to put $n=m$. Whence

$$T_{m+1}=\frac{m(m-1)(m-2)\ \ldots\ldots\ldots\ (m-m+1)}{1.2.3\ \ \ldots\ldots\ldots\ \ m}a^m x^{m-m}.$$

Now the numerator of this expression contains, in a contrary order, the same factors as the denominator, and $x^{m-m}=x^0=1$. Therefore the expression is reduced (as it ought to be) to a^m.

If n were taken greater than m, there would be among the factors of the general term one equal to $m-m$ or 0, which would serve to intimate that all the terms beyond $m+1$ are equal to zero; in other words, that the formula of the binomial $(x+a)^m$ contains no more than $m+1$ terms.

141. In the formation of any particular power of a binomial, it is convenient to employ successively each term in calculating the term which follows it.

The development of $(x+a)^m$ gives this general rule for passing from any term to the following term:

Multiply the numerator and denominator of the coefficient, respectively, by the exponent of x, and that number which marks the rank of the term; add 1 to the exponent of a, and subtract 1 from the exponent of x.

This rule may be obtained by changing n into $n+1$ in the expression of the general term; thus the term of the rank $n+2$ is

$$T_{n+2}=\frac{m(m-1)(m-2)\ \ldots\ldots\ (m-n+1)(m-n)}{1.2.3\ \ \ldots\ldots\ \ n\ \ \ (n+1)}a^{n+1}x^{m-n-1}.$$

And if this expression is compared with that of T_{n+1}, it becomes evident that T_{n+2} is formed from T_{n+1} by the rule which has been given.

The first term of any power of $x+a$ is known when the power is given, and the second term can be formed from the first, the third term from the second, ... by the preceding rule. For example, if it is required to raise $x+a$ to the fifth power,

x^5 is the first term of the development;

$\frac{5}{1}ax^{5-1}$ or $5ax^4$ is the second term;

$\frac{5\times 4}{2}a^{1+1}x^{4-1}$ or $10a^2x^3$ is the third term;

$$\frac{10\times 3}{3}a^{2+1}x^{3-1} \text{ or } 10a^3x^2 \text{ is the fourth term;}$$

$$\frac{10\times 2}{4}a^{3+1}x^{2-1} \text{ or } 5a^4x \text{ is the fifth term;}$$

$$\frac{5\times 1}{5}a^{4+1}x^{1-1} \text{ or } a^5 \text{ is the sixth term.}$$

The exponent of x in the sixth term being 0, $(=1-1)$, the coefficient of the seventh term involves a factor equal to zero; the sixth is consequently the last term, and the development of $(x+a)^5$ is

$$x^5+5ax^4+10a^2x^3+10a^3x^2+5a^4x+a^5.$$

a. In this example the coefficients of the terms which are equally distant from the extremes are equal. This must always be the case; for, the expression $(x+a)^m$ being composed in the same manner of a and x, the development of $(x+a)^m$ must be composed in the same manner. Therefore if the development contains a term of the form ka^nx^{m-n}, it must contain another equal to kx^na^{m-n}, or $ka^{m-n}x^n$.

These two terms are evidently equally distant from the two extremes of the development; for the number of terms which precede any term whatever being marked by the exponent of a in that term, it follows that the term ka^nx^{m-n} has n terms before it, and that the term $ka^{m-n}x^n$ has $m-n$ terms before it, and consequently n terms after it, since the whole number of terms is $m+1$. Therefore in the development of any power of a binomial the coefficients at equal distances from the two extremes are equal to each other.*

b. If the binomial is $(x-a)^m$, it becomes necessary to replace $+a$ by $-a$ in all the terms of the development of $(x+a)^m$. This amounts to the prefixing of the sign $-$ before all the terms which contain odd powers of a; that is, before the second, fourth, sixth, and in general all the terms of an even order.

142. In the developments $(x+a)^m$, $(x-a)^m$, let $a=1$ and $x=1$

$$\therefore 2^m=1+\frac{m}{1}+\frac{m(m-1)}{1.2}+\frac{m(m-1)(m-2)}{1.2.3}+, \text{\&c.}$$

$$0=1-\frac{m}{1}+\frac{m(m-1)}{1.2}-\frac{m(m-1)(m-2)}{1.2.3}+, \text{\&c.}$$

Whence, in every power of a binomial, the sum of the coefficients (including those of the extreme terms) is equal to a power of 2, indicated by the exponent m, of the binomial; and the sum of the coefficients of the even terms is equal to the sum of the coefficients of the odd terms.

143. As an example of the application of the binomial theorem, let it be required to expand the expression $(2a^3-3bx^2)^6$.

The most convenient method is, first, to expand $(x-a^6)$, by Articles 139-141, and then to replace x by $2a^3$ and a by $3bx^2$.

$$(x-a)^6=x^6-6ax^5+15a^2x^4-20a^3x^3+15a^4x^2-6a^5x+a^6.$$

Changing x into $2a^3$, and a into $3bx^2$,

$(2a^3-3bx^2)^6=(2a^3)^6-6(3bx^2)(2a^3)^5+15(3bx^2)^2(2a^3)^4-20(3bx^2)^3(2a^3)^3+15(3bx^2)^4(2a^3)^2-6(3bx^2)^5(2a^3)+(3bx^2)^6.$

Making the calculations,

$(2a^3-3bx^2)^6=64a^{18}-576a^{15}bx^2+2160a^{12}b^2x^4-4320a^9b^3x^6+4860a^6b^4x^8-2916a^3b^5x^{10}+729b^6x^{12}.$

---

* In the terms ka^nx^{m-n}, $ka^{m-n}x^n$, the two coefficients express the number of different combinations, n by n and $m-n$ by $m-n$, which can be formed with m quantities; therefore it is to be concluded that the number of combinations of m quantities, taken n by n, is equal to the number of combinations of m quantities taken $m-n$ by $m-n$.

144. The development of the powers of any binomial can be reduced to the development of the powers of a binomial whose first term is 1,

$$\text{for } x+a=x\left(1+\frac{a}{x}\right);$$

$$\text{making } \frac{a}{x}=z,\ x+a=x(1+z),$$

$$\therefore (x+a)^m=x^m(1+z)^m.$$

$$\text{Now } (1+z)^m=1+\frac{m}{1}z+\frac{m(m-1)}{1.2}z^2+\frac{m(m-1)(m-2)}{1.2.3}z^3+\ \ldots\ldots\ldots$$

$$\therefore (x+a)^m=x^m\left(1+\frac{m}{1}\cdot\frac{a}{x}+\frac{m(m-1)}{1.2}\cdot\left(\frac{a}{x}\right)^2+\frac{m(m-1)(m-2)}{1.2.3}\left(\frac{a}{x}\right)^3+\ldots\right)$$

This formula gives the following rule for the expansion of expressions such as $(1+z)^m$. Having formed with the exponent m the fractions $\frac{m}{1}, \frac{m-1}{2}, \frac{m-2}{3}$, &c., take one for the first term of the development; multiply 1 by the first fraction and also by z; multiply the result by the second fraction and again by z; multiply the new result by the third fraction and again by z ....; lastly, add all these results to the first term, 1. The result is the development of $(1+z)^m$.

145. If two terms of a trinomial are considered to form only one term, the formation of the powers of the trinomial is reduced to the formation of the powers of a binomial.

Thus $(a+b+c)^m=(a+(b+c))^m=a^m+\frac{m}{1}(b+c)a^{m-1}+\frac{m(m-1)}{1.2}(b+c)^2a^{m-2}$ +, &c.

Making $m=3$,

$$(a+b+c)^3=a^3+3(b+c)a^2+3(b+c)^2a+(b+c)^3.$$

$$=a^3+3a^2b+3ab^2+b^3+3a^2c+6abc+3b^2c+3ac^2+3bc^2+c^3.$$

The powers of polynomials which contain more terms than three can be formed in the same manner.

Applications of the binomial theorem:

1st. $(a\pm b)^6$?..........*Ans.* $a^6\pm 6a^5b+15a^4b^2\pm 20a^3b^3+15a^2b^4\pm 6ab^5+b^6$.

2d. $(5-4x)^4$?..................*Ans.* $625-2000x+2400x^2-1280x^3+256x^4$.

3d. $(\frac{1}{2}x+2y)^7$?......*Ans.* $\frac{1}{128}x^7+\frac{7}{32}x^6y+\frac{21}{8}x^5y^2+\frac{35}{2}x^4y^3+70x^3y^4+168x^2y^5+224xy^6+128y^7$.

4th. $(\sqrt{a}\pm\sqrt{b})^4$?........................*Ans.* $a^2+6ab+b^2\pm(4a+4b)\sqrt{ab}$.

5th. $(a+b)^n+(a-b)^n$?........................*Ans.* $2\left\{a^n+\frac{n(n-1)}{1.2}a^{n-2}b^2+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}a^{n-4}b^4+\frac{n(n-1)\ldots\ldots(n-5)}{1.2\ldots 6}a^{n-6}b^6+\ldots\ldots\right\}$

6th. $(a+b)^n-(a-b)^n$?............*Ans.* $2\left\{\frac{n}{1}a^{n-1}b+\frac{n(n-1)(n-2)}{1.2.3}a^{n-3}b^3+\frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}a^{n-5}b^5+, \&c.\right\}$

7th. Required the third term of $(a+b)^{15}$?......................*Ans.* $105a^{13}b^2$.

8th. Required the fifth term of $(a^2-b^2)^{12}$?....................*Ans.* $495a^{16}b^8$.

9th. $(a+b+c)^3$?...*Ans.* $a^3+3a^2b+3a^2c+3ab^2+6abc+3ac^2+b^3+3b^2c+3bc^2+c^3$.

10th. $(a+b+c+d)^2$?...*Ans.* $a^2+2ab+2ac+2ad+b^2+2bc+2bd+c^2+2cd+d^2$.

EXTRACTION OF ROOTS.

146. In Part I. Section VIII. are given rules for the extraction of the square and cube roots of numbers. It may be useful, as a supplement to these rules, to generalise the process of extraction of the roots of numbers.

With this view let it be required to find the nth root of a whole number, N.

If N contains n figures, its root contains one figure which can be obtained by forming the nth powers of the numbers 1, 2, 3 . . . . . . 9; for, since 10^n contains $n+1$ figures, it is evident that $N<10^n$ and that $\sqrt[n]{N}$ cannot be expressed by a whole number greater than 9.

If N contains more than n figures its nth root contains more than one figure. This root may, therefore, be regarded as composed of tens and units. Denoting the tens by a, the units by b, and employing the binomial theorem, it is found that

$$N=(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{1.2}a^{n-2}b^2+, \&c.,$$

or that the given number N is composed of the nth power of the tens contained in it, plus n times the product of the $(n-1)$th power of the tens by the units, plus, &c.

Now the nth power of the tens cannot give units of an order inferior to 1 followed by n zeros. Therefore the n last figures of the number can make no part of the nth power of the tens. It is necessary, therefore, to point off n figures from the right of the number, and to extract the root of the greatest nth power contained in the number expressed by the remaining figures of N, and so on.

The number N being in this manner separated into periods of n figures each (but the extreme period on the left containing any number of figures between 1 and n inclusive), the root of the greatest nth power contained in the first period on the left is extracted. This gives the figure which expresses the units of the highest order contained in the root, or the figure which expresses the tens of the nth root of the number formed by the first and second periods of N.

Subtracting the nth power of this figure from the first period of N, a remainder is obtained which, with the second period of N, contains n times the product of the $(n-1)$th power of the first figure of the root by the second figure, plus, &c.

Since the $(n-1)$th power of the first figure cannot give units of an order inferior to 10^{n-1}, the $n-1$ last figures of the second period cannot make any part of the $(n-1)$th power of the first figure. It is therefore sufficient to annex the first figure of the second period to the first remainder; then forming n times the $(n-1)$th power of the first figure of the root and dividing by it the number composed of the first remainder and the first figure of the second period, the quotient is the second figure of the root, or a figure greater than the second. To make trial, the second figure is annexed to the first; the number thus formed is raised to the nth power, and the result is subtracted from the number composed of the first and second periods of N.

If the result cannot be subtracted, the second figure is too great, and the next less must be tried; if the nth power of the number composed of the first figure and this second figure can be subtracted from the first and second periods of N, the figures obtained are the first and second of the required root.

Next, the first figure of the third period of N is annexed to the second remainder, and the number thus formed is divided by n times the $(n-1)$th power of the number composed of the first and second figures of the root; the result is the third figure of the root, or a figure greater than the third . . . . . .

This series of operations is continued till all the periods of N are brought down.

147. When the index of the root which is to be extracted is a number which can be resolved into prime factors, the required root may be obtained by the successive extraction of roots of the degrees of which these prime factors are the indices.

For $\sqrt[m]{a^{mnp}}=a^{np}$; $\sqrt[n]{a^{np}}=a^{p}$; $\sqrt[p]{a^{p}}=a$ (Art. 132).

Now, if it were required to extract the *mnp*th root of a^{mnp}, this root is evidently a; that is, it is the same as that which is found by first extracting the *m*th root of a^{mnp}, then the *n*th root of a^{np}, and lastly the *p*th root of a^{p};

$$\text{Consequently } \sqrt[mnp]{a^{mnp}}=\sqrt[m]{\sqrt[n]{\sqrt[p]{a^{mnp}}}}.$$

The fourth root of a number can hence be obtained by two extractions of the square root, the eighth root by means of three, the sixth root by extraction of the square and then of the cube root, &c.

Although the successive roots may be extracted in any order, it is preferable to extract first the root whose index is least, because then the extraction of a root whose index is greater (and which is a more complicated operation) is performed with a smaller number of figures than the number proposed.

By means of these principles it is found that

$$\sqrt[4]{256}=\sqrt{\sqrt{256}}=\sqrt{16}=4,$$

$$\sqrt[6]{2985984}=\sqrt[3]{\sqrt{2985984}}=\sqrt[3]{1728}=12,$$

$$\sqrt[8]{1679616}=\sqrt[4]{1296}=\sqrt{36}=6.$$

In practice it is convenient to extract roots of a high degree by means of logarithms.

148. When the whole number whose *n*th root is required is not an *n*th power, the process indicated in the last Article gives the root to the nearest whole number only.

The fraction which ought to complete the root cannot be exactly obtained, for the *n*th power of a fraction $\frac{a}{b}$, expressed in the lowest terms, being $\frac{a^n}{b^n}$, this new number cannot be reduced to a whole number.

The root can, however, be determined to any degree of approximation required.

Let it be required to extract the *n*th root of the whole number a to within the $\frac{1}{p}$ part of 1, that is, to such a degree of accuracy that the error committed may be less than $\frac{1}{p}$.

Putting a under the form $\frac{a}{1}$, and multiplying both terms of this fractional expression by p^n, $a=\frac{a}{1}=\frac{ap^n}{p^n}$.

Let $r=\sqrt[n]{ap^n}$ to the nearest unit, then $\frac{ap^n}{p^n}$ or a is comprehended between $\frac{r^n}{p^n}$ and $\frac{(r+1)^n}{p^n}$.

$$\text{But } \sqrt[n]{\frac{r^n}{p^n}}=\frac{r}{p} \text{ and } \sqrt[n]{\frac{(r+1)^n}{p^n}}=\frac{r+1}{p}.$$

Therefore $\sqrt[n]{a}$ is comprehended between $\frac{r}{p}$ and $\frac{r+1}{p}$, and consequently $\frac{r}{p}$ is the *n*th root of a to the degree of approximation required.

Whence, to extract the *n*th root of a whole number to within the $\frac{1}{p}$ part of 1,

Rule. Multiply the number by p^n, extract the *n*th root of the product to the nearest unit, and divide the result by p.

149. Next, let $\frac{a}{b}$ be a fractional number of which it is required to extract the nth root.

If both terms of the fraction are multiplied by b^{n-1},

$$\frac{a}{b}=\frac{a}{b}\times\frac{b^{n-1}}{b^{n-1}}=\frac{ab^{n-1}}{b^n}.$$

Let r be the nth root of ab^{n-1} to the nearest whole number; then $\frac{a}{b}$ or $\frac{ab^{n-1}}{b^n}$ is comprehended between $\frac{r^n}{b^n}$ and $\frac{(r+1)^n}{b^n}$; therefore $\sqrt[n]{\frac{a}{b}}$ is comprehended between $\frac{r}{b}$ and $\frac{(r+1)}{b}$.

Whence, to extract the nth root of a fractional expression,

Rule. Multiply the numerator of the fraction by the $(n-1)$th power of the denominator, extract the nth root of the product to the nearest whole number, and divide this root by the denominator.

150. If a closer approximation than that indicated by the fraction $\frac{1}{b}$ is required, the nth root of the whole number ab^{n-1} is extracted to the required degree of approximation, $\frac{1}{p}$ (Art. 148).

Let $r'+\frac{m}{p}$ be that root; then $\frac{r'+\frac{m}{p}}{b}$ is the expression of the nth root of $\frac{a}{b}$ to a degree of approximation marked by $\frac{1}{pb}$.

The general principles of the extraction of roots by approximation have been already applied to the extraction of the square and cube roots of whole numbers and fractional expressions (Part I. Sect. VIII.).

As an instance of the application of these principles to the extraction of a root of a higher degree, let it be required to extract the sixth root of 23 to a degree of approximation marked by the fraction $\frac{1}{100}$ or ·01.

$$23=\frac{23000000000000}{1000000000000}=\frac{23\times100^6}{100^6}.$$

$$\therefore\ \sqrt[6]{23}=\sqrt[6]{\frac{23\times100^6}{100^6}}=\sqrt[3]{\sqrt{\frac{23000000000000}{100^6}}}$$

$$=\sqrt[3]{\frac{4795381}{100^3}}\ldots\ldots\mathit{Ans.}\ \frac{168}{100}=1{\cdot}68.$$

Other examples:

1st. $\sqrt[3]{473}$ to $\frac{1}{20}$? .... *Ans.* $\frac{155}{20}=7{\cdot}75$.
2d. $\sqrt[3]{79}$ to 0·001? .... *Ans.* 4·2908.
3d. $\sqrt[6]{13}$ to 0·01? .... *Ans.* 1·53.
4th. $\sqrt[7]{21035{\cdot}8}$? .... *Ans.* 4·14539+.
5th. $\sqrt[8]{2}$? .... *Ans.* 1·090508+.
6th. $\sqrt[9]{21035{\cdot}8}$? .... *Ans.* 3·022239+.
7th. $\sqrt[5]{39}$? .... *Ans.* 2·0807.
8th. $\sqrt[7]{108}$? .... *Ans.* 1·95204.

EXTRACTION OF ROOTS OF POLYNOMIAL EXPRESSIONS.

151. Let a given polynomial, P, be the mth power of an unknown polynomial, p; it is required to find p.

Regarding the polynomials P, p as arranged according to the decreasing powers of the same letter, x, and denoting by $a, b, c, \ldots$ the unknown terms of the root p, these quantities ought to be such that if the root $a+b+c+\ldots$ is raised to the power m, the terms of P are reproduced.

Now, if the mth power of $a+b+c+\ldots.$ is formed by successive multiplications, it is evident that in the result the term in x which has the highest exponent must be the mth power of a.

Therefore the first term of the root p is obtained by extracting the mth root of the first term of the given polynomial P.

Knowing the first term of the root, it is not difficult to obtain the second; but, instead of finding the terms consecutively, it seems preferable to show in what manner, when many successive terms of the root, beginning with the first, are known, that term which immediately follows the last known term can be obtained. For this purpose let k denote the sum of the terms which are known, and u the sum of the terms which are unknown; then $P=(k+u)^m$.

$$\text{Developing, } P=k^m+mk^{m-1}u+\frac{m(m-1)}{1.2}k^{m-2}u^2+\ldots.$$

$$\text{or (putting } m', m''\ldots \text{ respectively for } \frac{m(m-1)}{1.2}, \frac{m(m-1)(m-2)}{1.2.3},\ldots)$$

$$P=k^m+mk^{m-1}u+m'k^{m-2}u^2+m''k^{m-3}u^3+\ldots$$

Transposing the term km,

$$P-k^m=mk^{m-1}u+m'k^{m-2}u^2+m''k^{m-3}u^3+\ldots.$$

Now the first member of this equation, $P-k^m$, is a quantity which can be obtained by forming the mth power of the known quantity k, and subtracting it from P, and the second member is a sum of products by means of which it is easy to assign the composition of the first term of the remainder $P-k^m$, and by consequence to discover the first term of the unknown part of the root u.

For if k^{m-1} is developed, it follows from the rules of multiplication, that the first term of the development, or the term which involves the highest power of x, must be a^{m-1}; therefore denoting the first term of u by f, the first term of the product $mk^{m-1}\times u$ must be $ma^{m-1}f$.

By similar reasoning it is found that the first terms in the developments of the other products are, respectively, $m'a^{m-2}f^2$, $m''a^{m-3}f^3$, ... These terms (making abstraction of the coefficients which have no influence on the degree of x) may be deduced from the term $ma^{m-1}f$, by suppressing in it one or more factors equal to a, and replacing them by as many factors each equal to f.

The degree of x involved in f being inferior to the degree of x involved in a, these changes can only give terms in x of a degree inferior to $ma^{m-1}f$. Therefore after the subtraction of the mth power of k (the part of the root already found) from the given polynomial P, the first term of the remainder is equal to the product of m times the $(m-1)$th power of a, the first term of the root, by the first of the terms which remain to be found.

Therefore, in conclusion, if the first term of the remainder is divided by m times the $(m-1)$th power of the first term of the root, the quotient is a new term of the root.

This conclusion gives the means of finding successively all the terms of the root when the first term is known. To obtain the second term, b, the mth power of the first term of the root is subtracted from P, and the first term of the remainder is divided by ma^{m-1}.

To obtain the third term, c, the mth power of $(a+b)$ is subtracted from P, and the first term of the second remainder is divided by ma^{m-1}, ...

152. This process shows whether the given polynomial is an exact power of the degree m; for the first term of each remainder contains x with a less exponent than the first term of the preceding remainder; therefore, at last either a remainder $=0$ is found (in which case P is the mth power of the quantity obtained, as $\sqrt[m]{P}$), or a remainder is found whose first term is not divisible by ma^{m-1}, in which case it is certain that P is not an exact power of the degree m.

Likewise, if P is a power of the degree m, the mth root of its last term ought to be the last term of $\sqrt[m]{P}$. Therefore, if the calculation leads to the placing in the root of a term of an inferior degree to this, it is again certain that P is not an exact power of the degree m.

If the terms of P, p are arranged according to the ascending powers of x, the preceding reasonings are equally applicable. In this order the exponent of x in the first term of the successive remainders goes on increasing. This cannot be a hindrance to the effecting of the divisions which give the terms of the root. But as the root of the last term of P ought always to be the last term of the root p, it follows that the calculation cannot give to p a term of a higher degree in x, except in the case in which P is not an exact power of the degree m.

Examples of the extraction of roots of polynomial expressions:

1st. $\sqrt[3]{(8a^3-84a^2x+294ax^2-343x^3)}$?........................*Ans.* $2a-7x$.

2d. $\sqrt[3]{(x^6-6cx^5+12c^2x^4-8c^3x^3)}$?..............................*Ans.* x^2-2cx.

3d. $\sqrt[3]{(a^{3m}-6a^{2m+1}x^n+12a^{m+2}x^{2n}-8a^3x^{3n})}$?...............*Ans.* a^m-2ax^n.

4th. $\sqrt[4]{(16a^4-96a^3b+216a^2b^2-216ab^3+81b^4)}$?..............*Ans.* $2a-3b$.

5th. $\sqrt[5]{(243a^5c^5-810a^4c^4bd+1080a^3c^3b^2d^2-720a^2c^2b^3d^3+240acb^4d^4-32b^5d^5)}$?..*Ans.* $3ac-2bd$.

6th. $\sqrt[6]{(729-2916x^2+4860x^4-4320x^6+2160x^8-576x^{10}+64x^{12})}$?
Ans. $3-2x^2$.

CALCULUS OF RADICAL EXPRESSIONS.

153. When the quantity, monomial or polynomial, whose mth root is required, is not a power of m, the operation is indicated by writing the radical sign on the left of the quantity, and connecting with the sign $\sqrt{\ }$ that number which marks the degree of the root to be extracted; thus the mth root of a is written $\sqrt[m]{a}$.

The number m is called the index of the radical.

When the positive values alone of radical quantities are considered, the radicals are termed arithmetical; and when their values are considered as being positive, negative, or imaginary, the radicals are termed algebraic.

It is proved (Art. 131 and 132) that a product is raised to a power by raising each of its factors to that power; and that the root of a product is extracted by extracting the root of each factor of that product.

Whence $\sqrt[m]{a^mb}=\sqrt[m]{a^m}\times\sqrt[m]{b}=a\sqrt[m]{b}$.

Wherefore, if the exponent of any factor of the quantity under the radical sign is equal to or a multiple of the index of the radical, the root of that factor may be made a coefficient of the radical; and, conversely, the coefficient of a radical may be introduced under the radical sign by raising the coefficient to the power marked by the index of the radical.

154. Radicals which have the same index and the same quantity under the radical sign are termed like or similar.

a. An expression composed of like radicals connected by the signs +, —, can be simplified.

For example, the radicals $a\sqrt[3]{b^2c}$, $b\sqrt[3]{b^2c}$, being like, if connected by the sign +, their sum is equal to a repetitions of $\sqrt[3]{b^2c}$, plus b repetitions of $\sqrt[3]{b^2c}$; that is, to $a+b$ repetitions of the common quantity $\sqrt[3]{b^2c}$; and if connected by the sign —, the result is equal to a repetitions of $\sqrt[3]{b^2c}$, minus b repetitions of $\sqrt[3]{b^2c}$, that is, to $a-b$ repetitions of the common quantity $\sqrt[3]{b^2c}$.

Wherefore $a\sqrt[3]{b^2c}\pm b\sqrt[3]{b^2c}=(a\pm b)\sqrt{b^2c}$.

b. If the proposed radicals are not like, but admit of reduction to equivalent radicals which are like, the same simplification can be made.

Let the expression $3\sqrt[5]{2a^{13}b}-5\sqrt[5]{64a^8b^6}-7\sqrt[5]{2a^3b^{11}}$, be proposed.

$$\text{Since } 3\sqrt[5]{2a^{13}b}=3\sqrt[5]{a^{10}\times2a^3b}=3a^2\sqrt[5]{2a^3b},$$
$$5\sqrt[5]{64a^8b^6}=5\sqrt[5]{2^5a^5b^5\times2a^3b}=10ab\sqrt[5]{2a^3b},$$
$$7\sqrt[5]{2a^3b^{11}}=7\sqrt[5]{b^{10}\times2a^3b}=7b^2\sqrt[5]{2a^3b},$$

the proposed expression becomes $3a^2\sqrt[5]{2a^3b}-10ab\sqrt[5]{2a^3b}-7b^2\sqrt[5]{2a^3b}$,

$$\text{or } (3a^2-10ab-7b^2)\sqrt[5]{2a^3b}.$$

Whence, to find the sum or the difference of any like radical quantities, it is necessary to form the algebraic sum or difference of their coefficients, and to make this sum or difference the coefficient of the common radical quantity.

c. The addition or subtraction of unlike radicals is indicated by means of the proper signs.

155. Let the radicals $\sqrt[m]{a}$, $\sqrt[m]{b}$, which have the same index, be proposed, and let it be required to multiply or divide $\sqrt[m]{a}$ by $\sqrt[m]{b}$.

The product $\sqrt[m]{a}\times\sqrt[m]{b}=\sqrt[m]{ab}$; for, if $\sqrt[m]{a}\times\sqrt[m]{b}$ and $\sqrt[m]{ab}$ are raised to the mth power, the products are equal, both being ab.

$$\text{In like manner } \sqrt[m]{a}\div\sqrt[m]{b} \text{ or } \frac{\sqrt[m]{a}}{\sqrt[m]{b}}=\sqrt[m]{\frac{a}{b}},$$
$$\text{for } \left(\frac{\sqrt[m]{a}}{\sqrt[m]{b}}\right)^m=\frac{a}{b} \text{ and } \left(\sqrt[m]{\frac{a}{b}}\right)^m=\frac{a}{b}.$$

Whence, to find the product or the quotient of two radicals which have the same index, it is necessary to form the product or quotient of the quantities under the sign, and to place it under the sign of the common radical.

If there are coefficients, these must be multiplied or divided separately.

Examples:

1st. $\sqrt{8}+\sqrt{50}-\sqrt{18}$?........ *Ans.* $4\sqrt{2}$.

2d. $b\sqrt[m]{a}+c\sqrt[m]{a}-d\sqrt[m]{a}$?........ *Ans.* $(b+c-d)\sqrt[m]{a}$.

3d. $2\sqrt{\frac{5}{3}}+\sqrt{60}-\sqrt{15}+\sqrt{\frac{3}{5}}$?........ *Ans.* $\frac{28}{15}\sqrt{15}$.

4th. $4a\sqrt[3]{a^3b^4}+b\sqrt[3]{8a^6b}-\sqrt[3]{125a^6b^4}$?........ *Ans.* $a^2b\sqrt[3]{b}$.

5th. $\sqrt{(3a^2c+6abc+3b^2c)}$?........ *Ans.* $(a+b)\sqrt{3c}$.

6th. $\sqrt{45c^3}-\sqrt{80c^3}+\sqrt{5a^2c}$?........ *Ans.* $(a-c)\sqrt{5c}$.

7th. $\sqrt[m]{a}\times\sqrt[m]{b}\times\sqrt[m]{c}$?........ *Ans.* $\sqrt[m]{abc}$.

8th. $a\sqrt[n]{x}\times b\sqrt[n]{y}\times c\sqrt[n]{z}$?........ *Ans.* $abc\sqrt[n]{xyz}$.

9th. $4\times2\sqrt[4]{8}\times\sqrt[4]{72}$?........ *Ans.* $8\sqrt{6}$.

10th. $c\sqrt{a}\times d\sqrt{a}$?........ *Ans.* acd.

11th. $\sqrt[m]{a}\div\sqrt[m]{b}$?........ *Ans.* $\sqrt[m]{\frac{a}{b}}$.

12th. $c\sqrt[m]{a}\div d\sqrt[m]{b}$?........ *Ans.* $\frac{c}{d}\sqrt[m]{\frac{a}{b}}$.

13th. $a\div\sqrt[m]{b}$?........ *Ans.* $\sqrt[m]{\frac{a^m}{b}}$.

14th. $a\div\sqrt{a}$?........ *Ans.* $\sqrt{a}$.

15th. $\sqrt[5]{64}\div2$?........ *Ans.* $\sqrt[5]{2}$.

16th. $c\sqrt{a^2-x^2}\div\sqrt{a+x}$?........ *Ans.* $c\sqrt{a-x}$.

17th. $\sqrt{a^2-x^2}\div a-x$?........ *Ans.* $\sqrt{\frac{a+x}{a-x}}$.

18th. $\sqrt{ab^2-b^2c}\div\sqrt{a-c}$?........ *Ans.* b.

155. If it is required to form the nth power of $\sqrt[m]{a}$,

$$(\sqrt[m]{a})^n=\sqrt[m]{a}\times\sqrt[m]{a}\times\sqrt[m]{a}\ldots\ldots \text{ to } n \text{ factors},$$

$$\sqrt[m]{a}\times\sqrt[m]{a}\times\sqrt[m]{a}\ldots \text{ to } n \text{ factors}=\sqrt[m]{aaa\ldots \text{ to } n \text{ factors}},$$

$$\text{and } \sqrt[m]{aaa\ldots\ldots \text{ to } n \text{ factors}}=\sqrt[m]{a^n}.$$

$$\text{Wherefore } (\sqrt[m]{a})^n=\sqrt[m]{a^n}.$$

Whence a radical is raised to a power by raising the quantity under the radical sign to that power.

a. When the index of the radical is a multiple of the power the expression can be simplified.

For example, if it is required to raise the radical $\sqrt[mn]{a}$ to the power n, $(\sqrt[mn]{a})^n=\sqrt[m]{a}$.

To prove this: since $(\sqrt[mn]{a})^{mn}=a$, it follows that $\sqrt[mn]{a}$ is a quantity which is mn times a factor in a. Now the whole number mn of factors may be arranged in a tabular form, thus,

$$\sqrt[mn]{a}\,.\,\sqrt[mn]{a}\,.\,\sqrt[mn]{a}\ldots\ldots \text{ to } n \text{ factors},$$
$$\sqrt[mn]{a}\,.\,\sqrt[mn]{a}\,.\,\sqrt[mn]{a}\ldots\ldots$$
$$\begin{array}{c}\cdot\\\cdot\\\cdot\\\cdot\end{array}$$
$$\text{to } m \text{ factors}\ \ldots\ldots\ldots,$$

there being m horizontal lines and n factors in each line. Now each horizontal line gives a product equal to $(\sqrt[mn]{a})^n$, and m such products are equal to a; therefore one of the products must be the mth root of a, that is,

$$(\sqrt[mn]{a})^n=\sqrt[m]{a}.$$

b. If the exponent of the power, without being a divisor of the index, has one factor in common with it, as $(\sqrt[mn]{a})^{np}$, the quantity $\sqrt[mn]{a}$ may be raised first to the power n, and the result to the power p.

$$\text{Now } (\sqrt[mn]{a})^n=\sqrt[m]{a}, \text{ and } (\sqrt[m]{a})^p=\sqrt[m]{a^p};$$
$$\text{therefore } (\sqrt[mn]{a})^{np}=\sqrt[m]{a^p}.$$

Examples:

1st. $(\sqrt[4]{4a^3})^2$?..*Ans.* $2a\sqrt[4]{a^2}$.

2d. $(3\sqrt[3]{2a})^5$?..*Ans.* $486a\sqrt[3]{4a^2}$.

3d. $(\sqrt[4]{2a})^2$?..*Ans.* $\sqrt{2a}$.

4th. $(\sqrt[6]{3b})^2$?..*Ans.* $\sqrt[3]{3b}$.

5th. $(\sqrt[6]{4a^3})^2$?..*Ans.* $a\sqrt[3]{4}$.

6th. $(3\sqrt[3]{2a})^5$?..*Ans.* $486a\sqrt[3]{4a^2}$.

156. To extract any root of a radical it is only necessary to reverse the rules for the formation of powers.

First. If the quantity under the radical sign is affected with an exponent which is equal to or a multiple of the index of the root which is to be extracted; for example, if the radical is $\sqrt[m]{a^n}$, and the nth root is required.

$$\text{The } n\text{th root of } \sqrt[m]{a^n}=\sqrt[n]{\sqrt[m]{a^n}}=\sqrt[m]{a}.$$

For if $\sqrt[m]{a}$ is raised to the power n, the quantity $\sqrt[m]{a^n}$ is reproduced.

Second. Suppose that the exponent of the quantity under the radical sign is not divisible by the index of the root which is to be extracted, in this case the extraction can only be indicated.

$$\text{Thus the } n\text{th root of } \sqrt[m]{a} \text{ is } \sqrt[n]{\sqrt[m]{a}};$$
$$\text{but since } \sqrt[n]{\sqrt[m]{a^n}}=\sqrt[m]{a},$$
$$\text{and } (\sqrt[mn]{a})^n=\sqrt[m]{a},$$
$$\text{it follows that } \sqrt[n]{\sqrt[m]{a}}=\sqrt[mn]{a}.$$

Whence the root of a radical is extracted either, first, by extracting the root of the quantity which is under the radical sign, or, second, by multiplying the index of the radical by the index of the root which is to be extracted.

The second is the method more generally employed. Each of the rules shows that the order in which two successive roots are extracted is wholly indifferent, and it is evident that this conclusion extends to any number of successive roots.

If the exponent of the quantity under the radical sign and the index of the root which is to be extracted have a common factor, this factor may be cancelled in the index of the root and the exponent of the quantity;

for $\sqrt[mn]{a^{np}}=\sqrt[m]{\sqrt[n]{a^p})^n}=\sqrt[m]{a^p}$.

Examples:

1st. $\sqrt[2]{\sqrt[3]{3c}}$?..*Ans.* $\sqrt[6]{3c}$.

2d. $\sqrt[3]{\sqrt[4]{8a^3}}$? ....*Ans.* $\sqrt[4]{2a}$.

3d. $\sqrt[2]{\sqrt[5]{9a^2}}=\sqrt[5]{\sqrt[2]{9a^2}}$?..*Ans.* $\sqrt[5]{3a}$.

4th. $\sqrt[m]{\sqrt[n]{a^p}}$?..*Ans.* $\sqrt[mn]{a^p}$.

157. If any number of radicals which have the same index are multiplied together the product is a single radical; and if one radical is divided by another, both having the same index, the quotient is expressed by a single radical; but if radicals which have different indices are multiplied or divided by each other, the expressions are more complicated. As a means, therefore, of simplifying the products and quotients of radicals it is useful to reduce the radicals to a common index.

Since a radical is raised to a power by raising the quantity which is under the radical sign to that power, and the root of a radical is extracted by multiplying the index of the radical by the index of the root which is to be extracted, it follows that the value of a radical is not changed when its index is multiplied by a number, provided at the same time the quantity under the radical sign is raised to the power expressed by this number.

Whence any proposed radicals may be reduced to a common index by the rules for the reduction of fractions to the same denominator. In this case the indices of the radicals correspond to the denominators of the fractions.

Taking the example $P=\sqrt[3]{a^2b^2}\times\sqrt[4]{a^2b^3}\times\sqrt[5]{a^3b^4}$.

Since the indices 3, 4, 5 are prime to each other, their least common multiple is equal to their continued product, or to $3\times4\times5=60$; it is therefore to the index 60 that it is necessary to reduce these radical factors.

$$\text{Now } 60+3=20, \text{ and } \sqrt[3]{a^2b^2}=\sqrt[3\times20]{(a^2b^2)^{20}}=\sqrt[60]{a^{40}b^{40}}.$$

$$60+4=15, \text{ and } \sqrt[4]{a^2b^3}=\sqrt[4\times15]{(a^2b^3)^{15}}=\sqrt[60]{a^{30}b^{45}}.$$

$$60+5=12, \text{ and } \sqrt[5]{a^3b^4}=\sqrt[5\times12]{(a^3b^4)^{12}}=\sqrt[60]{a^{36}b^{48}}.$$

$$\text{Therefore } P=\sqrt[60]{a^{40}b^{40}}\times\sqrt[60]{a^{30}b^{45}}\times\sqrt[60]{a^{36}b^{48}}.$$

Effecting the multiplication (Art. 155),

$$P=\sqrt[60]{a^{40+30+36}\times b^{40+45+48}}=\sqrt[60]{a^{106}b^{133}},$$

$$\text{or } P=\sqrt[60]{a^{60}b^{120}}\times\sqrt[60]{a^{46}b^{13}}=ab^2\sqrt[60]{a^{46}b^{13}}.$$

Example 2. Let $P=\sqrt[4]{32a^2b^5}\times\sqrt[6]{128a^7b}\times\sqrt[8]{64a^3b^7}$.

In this example the least common multiple of the indices is 24.

Also the numerical factors under the radical sign are $32=2^5$, $128=2^7$, $64=2^6$. Making these substitutions, and reducing the radicals to the common index 24,

$$P=\sqrt[24]{2^{30}a^{12}b^{30}}\times\sqrt[24]{2^{28}a^{28}b^4}\times\sqrt[24]{2^{18}a^9b^{21}}=\sqrt[24]{2^{76}a^{49}b^{55}}.$$

$$\therefore P=2^3a^2b^2\sqrt[24]{2^4ab^7}=8a^2b^2\sqrt[24]{16ab^7}.$$

CALCULUS OF FRACTIONAL AND NEGATIVE EXPONENTS.

158. A fractional exponent is the symbol of the extraction of a root which cannot be executed. The expressions $a^{\frac{1}{n}}$, $a^{\frac{m}{n}}$ indicate, respectively, that the *n*th root of *a* and the *n*th root of the *m*th power of *a* is to be extracted.

$$a^{\frac{1}{n}} \text{ is equivalent to } \sqrt[n]{a}, \text{ and } a^{\frac{m}{n}} \text{ to } \sqrt[n]{a^m}.$$

a. A negative exponent is the symbol of a division which cannot be executed. Thus the expression a^{-p} indicates that a certain power of a, (a^n, for example) is divided by a^{n+p}, a higher power of the common root a. The true value of a^{-p} is $\frac{1}{a^p}$, which is the quotient of unity divided by a affected with the exponent p, taken positively.

b. From the combination of the extraction of a root and a division, neither of which can be executed, is derived another symbol, the negative fractional exponent.

Thus, if it is required to extract the *n*th root of $\frac{1}{a^m}$, the root is indicated by the expression $\sqrt[n]{\frac{1}{a^m}}$.

But since $\frac{1}{a^m}=a^{-m}$ and the *n*th root of a^{-m} is $\sqrt[n]{a^{-m}}=a^{-\frac{m}{n}}$, the expressions $\sqrt[n]{\frac{1}{a^m}}$ and $a^{-\frac{m}{n}}$ are considered equivalent.

It is consequently to be considered that the expressions

$$a^{\frac{m}{n}},\ a^{-p},\ a^{-\frac{m}{n}},$$

are the respective equivalents of

$$\sqrt[n]{a^m},\ \frac{1}{a^p},\ \sqrt[n]{\frac{1}{a^m}},$$

and that each may be used for its correspondent accordingly.

159. The origin and meaning of fractional and negative exponents being explained, it remains to investigate rules for the calculus of quantities affected with such exponents.

a. Multiplication. 1st. Let it be required to multiply $a^{\frac{m}{n}}$ by a^p.

$$a^{\frac{m}{n}}=\sqrt[n]{a^m}; \text{ and by Articles 154 and 157, } \sqrt[n]{a^m}\times a^p=\sqrt[n]{a^m}\times\sqrt[n]{a^{np}};$$

$$\sqrt[n]{a^m}\times\sqrt[n]{a^{np}}=\sqrt[n]{a^m\times a^{np}}=\sqrt[n]{a^{m+np}}=a^{\frac{m+np}{n}};$$

$$\therefore a^{\frac{m}{n}}\times a^p=a^{\frac{m+np}{n}}.$$

In like manner, in the following examples,

2d. $a^{\frac{m}{n}}\times a^{\frac{p}{q}}=\sqrt[n]{a^m}\times\sqrt[q]{a^p}=\sqrt[nq]{a^{mq}}\times\sqrt[nq]{a^{np}}=\sqrt[nq]{a^{mq}\times a^{np}}=\sqrt[nq]{a^{mq+np}}=a^{\frac{mq+np}{nq}}$.

3d. $a^m\times a^{-n}=a^m\times\frac{1}{a^n}=\frac{a^m}{a^n}=a^{m-n}$.

4th. $a^{-m}\times a^{-n}=\frac{1}{a^m}\times\frac{1}{a^n}=\frac{1}{a^m\times a^n}=\frac{1}{a^{m+n}}=a^{-m-n}$.

5th. $a^{\frac{m}{n}}\times a^{-p}=\sqrt[n]{a^m}\times\frac{1}{a^p}=\sqrt[n]{a^m}\times\sqrt[n]{\frac{1}{a^{np}}}=\sqrt[n]{\frac{a^m}{a^{np}}}=\sqrt[n]{a^{m-np}}=a^{\frac{m-np}{n}}$.

6th. $a^{-\frac{m}{n}}\times a^p=\sqrt[n]{\frac{1}{a^m}}\times a^p=\sqrt[n]{\frac{1}{a^m}}\times\sqrt[n]{a^{np}}=\sqrt[n]{\frac{a^{np}}{a^m}}=\sqrt[n]{a^{np-m}}=a^{\frac{np-m}{n}}$.

7th. $a^{\frac{m}{n}}\times a^{-\frac{p}{q}}=\sqrt[n]{a^m}\times\sqrt[q]{\frac{1}{a^p}}=\sqrt[nq]{a^{mq}}\times\sqrt[nq]{\frac{1}{a^{np}}}=\sqrt[nq]{\frac{a^{mq}}{a^{np}}}=\sqrt[nq]{a^{mq-np}}=a^{\frac{mq-np}{nq}}$.

8th. $a^{-\frac{m}{n}}\times a^{-\frac{p}{q}}=\sqrt[n]{\frac{1}{a^m}}\times\sqrt[q]{\frac{1}{a^p}}=\sqrt[nq]{\frac{1}{a^{mq}}}\times\sqrt[nq]{\frac{1}{a^{np}}}=\sqrt[nq]{\frac{1}{a^{mq+np}}}=a^{-\frac{mq+np}{nq}}$.

The product of two integer positive powers of the same root is obtained by adding the exponents together, and affecting the root with an exponent equal to the sum.

If this rule is applied to the preceding examples, the results already found are obtained; for

1st. $\frac{m}{n}+p=\frac{m+np}{n}$. 2d. $\frac{m}{n}+\frac{p}{q}=\frac{mq+np}{nq}$.
3d. $m+(-n)=m-n$. 4th. $(-m)+(-n)=-m-n$.
5th. $\frac{m}{n}+(-p)=\frac{m-np}{n}$. 6th. $\left(-\frac{m}{n}\right)+p=\frac{np-m}{n}$.
7th. $\frac{m}{n}+\left(-\frac{p}{q}\right)=\frac{mq-np}{nq}$. 8th. $\left(-\frac{m}{n}\right)+\left(-\frac{p}{q}\right)=-\frac{mq+np}{nq}$.

This rule consequently holds good for negative and fractional as well as whole positive exponents. Whence, in multiplying two monomials affected with any exponents, integer or fractional, positive or negative, it is necessary to add together the two exponents of the same letter.

b. Division. If two monomial quantities, which have any exponents whatever, are divided one by the other, the exponent of each letter in the quotient ought to be such that, if added to the exponent of the same letter in the divisor, the sum shall be equal to the exponent of that letter in the dividend. Whence the exponent of the quotient is equal to the difference between the exponents of the dividend and divisor.

Example, $a^{\frac{m}{n}}\div a^{\frac{p}{q}}=a^{\frac{m}{n}-\frac{p}{q}}=a^{\frac{mq-np}{nq}}$.

c. Formation of powers. The mth power of a monomial, affected with any exponent whatever, is obtained by forming a product of which the monomial is m times a factor. The exponent of this product is composed of m repetitions of the exponents of the monomial factor.

Whence to form the mth power of a monomial, affected with any exponents whatever, it is necessary to multiply the exponent of each letter by the exponent m of the power.

Example, $\left(a^{\frac{m}{n}}\right)^p=a^{\frac{mp}{n}}$.

d. Extraction of roots. In the nth root of a monomial, the exponent of each letter ought to be such that, if multiplied by n, the index of the root extracted, the exponent with which the letter is affected in the monomial shall be reproduced. Whence the exponent of the letter in the root ought to be equal to the quotient obtained by dividing the exponent of that letter in the proposed monomial by n, the index of the root.

Whence, to extract the nth root of any monomial, it is necessary to divide the exponent of each letter by n, the index of the root.

Example, $\left(a^{\frac{p}{q}}\right)^{\frac{1}{n}}$ or $\sqrt[n]{a^{\frac{p}{q}}}=a^{\frac{p}{nq}}$.

160. The preceding rules for the division, the formation of powers, and the extraction of roots of quantities affected with fractional and negative exponents are derived from the rule for the multiplication of such quantities; it is equally easy to deduce them from the rules for the calculus of radicals.

a. If it is required to divide $a^{\frac{m}{n}}$ by $a^{-\frac{p}{q}}$.

$$a^{\frac{m}{n}}=\sqrt[n]{a^m}, \text{ and } a^{-\frac{p}{q}}=\sqrt[q]{\frac{1}{a^p}}.$$

$$\therefore\ a^{\frac{m}{n}}\div a^{-\frac{p}{q}}=\sqrt[n]{a^m}\div\sqrt[q]{\frac{1}{a^p}}=\sqrt[nq]{a^{mq}}\div\sqrt[nq]{\frac{1}{a^{np}}}.$$

$$\text{But } \sqrt[nq]{a^{mq}}\div\sqrt[nq]{\frac{1}{a^{np}}}=\sqrt[nq]{a^{mq}\div\frac{1}{a^{np}}}=\sqrt[nq]{a^{mq}\times a^{np}}=\sqrt[nq]{a^{mq+np}}=a^{\frac{mq+np}{nq}}.$$

Now, if the exponent of the divisor is subtracted from the exponent of the dividend, the same result is obtained; for $\frac{m}{n}-\left(-\frac{p}{q}\right)=\frac{mq+np}{nq}$. Whence it is to be inferred, that the rule for the division of monomials having whole positive exponents is likewise applicable to the division of monomials which have fractional and negative exponents.

b. Next, let it be required to raise $a^{-\frac{m}{n}}$ to a power, p.

$$a^{-\frac{m}{n}}=\sqrt[n]{\frac{1}{a^m}} \quad \therefore\ \left(a^{-\frac{m}{n}}\right)^p=\left(\sqrt[n]{\frac{1}{a^m}}\right)^p.$$

$$\text{But } \left(\sqrt[n]{\frac{1}{a^m}}\right)^p=\sqrt[n]{\frac{1}{a^{mp}}}=\sqrt[n]{a^{-mp}}=a^{-\frac{mp}{n}}.$$

$$\therefore\ \left(a^{-\frac{m}{n}}\right)^p=a^{-\frac{mp}{n}}.$$

If $-\frac{m}{n}$, the exponent of the proposed monomial, is multiplied by p, the exponent of the power, the product $-\frac{m}{n}\times p=-\frac{mp}{n}$; the result obtained by the rules for the calculus of radicals.

c. Lastly, let it be required to extract the rth root of the quantity $a^{\frac{m}{n}}$.

$$a^{\frac{m}{n}}=\sqrt[n]{a^m} \quad \therefore\ \sqrt[r]{a^{\frac{m}{n}}}=\sqrt[r]{\sqrt[n]{a^m}}$$

$$\text{But } \sqrt[r]{\sqrt[n]{a^m}}=\sqrt[rn]{a^m}=a^{\frac{m}{rn}}.$$

$$\therefore\ \sqrt[r]{a^{\frac{m}{n}}}=a^{\frac{m}{rn}}.$$

The exponent $\frac{m}{rn}$ of the rth root of $a^{\frac{m}{n}}$ is the rth part of the exponent $\frac{m}{n}$ of the proposed monomial. This is the result given by the rule for the extraction of the rth root of a monomial whose exponent is integer and positive. The rule "Divide the exponent of the proposed quantity by the index of the root which is to be extracted," may therefore be considered perfectly general.

The investigations b, c of this Article may be combined; for the expression $\left(a^{\frac{m}{n}}\right)^{\frac{p}{r}}$ signifies that the power p of the quantity $a^{\frac{m}{n}}$ is to be formed, and the root, r, of the result extracted.

$$\text{Now } a^{\frac{m}{n}}=\sqrt[n]{a^m},$$

$$\text{and } \left(\sqrt[n]{a^m}\right)^{\frac{p}{r}}=\sqrt[r]{\left(\sqrt[n]{a^m}\right)^p}=\sqrt[rn]{a^{mp}}=a^{\frac{mp}{rn}}=a^{\frac{m}{n}\times\frac{p}{r}}.$$

$$\therefore\ \left(a^{\frac{m}{n}}\right)^{\frac{p}{r}}=a^{\frac{mp}{nr}}=a^{\frac{m}{n}\times\frac{p}{r}}.$$

$$\text{If } r=1,\ \frac{p}{r}=\frac{p}{1}=p;$$

$$\text{and } \left(a^{\frac{m}{n}}\right)^p=a^{\frac{m}{n}\times p}=a^{\frac{mp}{n}}.$$

$$\text{If } p=1,\ \frac{p}{r}=\frac{1}{r};$$

$$\text{and } \left(a^{\frac{m}{n}}\right)^{\frac{1}{r}}=a^{\frac{m}{n}\times\frac{1}{r}}=a^{\frac{m}{nr}}.$$

Whence, regarding a root as a fractional power whose exponent is expressed by the reciprocal of the index of the root, the rule for the formation of powers and the rule for the extraction of roots of monomials may be included in one; viz. "Multiply the exponent of the monomial by the integral or fractional exponent of the power."

d. An expression which involves fractional and negative exponents can always be transformed into another which involves only positive exponents and radicals.

Assuming as an example, $x=\left(\sqrt[-\frac{m}{n}]{a^{-\frac{p}{q}}}\right)^{-r}$.

$$\text{First, } a^{-\frac{p}{q}}=\sqrt[q]{\frac{1}{a^p}}.$$

$$\text{Second, } \sqrt[-\frac{m}{n}]{a^{-\frac{p}{q}}}=\sqrt[-\frac{m}{n}]{\sqrt[q]{\frac{1}{a^p}}}=\left.\sqrt[q]{\frac{1}{a^p}}\right)^{-\frac{n}{m}}=\left.\sqrt[mq]{\frac{1}{a^p}}\right)^{-n}=\sqrt[mq]{a^{np}}.$$

$$\text{Third, } \left(\sqrt[mq]{a^{np}}\right)^{-r}=\sqrt[mq]{\frac{1}{a^{npr}}}=\frac{1}{\sqrt[mq]{a^{npr}}}.$$

$$\therefore\ x=\frac{1}{\sqrt[mq]{a^{npr}}}.$$

161. It follows from Articles 159 and 160 that the only difference between the calculus of monomials having integral positive exponents and monomials having exponents which are fractional or negative consists in this, that, in the first instance, multiplication requires the addition, division the subtraction, the formation of powers the multiplication, and the extraction of roots the division of integer expressions.

While, in the second instance, these operations require, respectively, the addition, the subtraction, the multiplication, the division of fractional expressions.

162. In applying the rules of Articles 159 and 160 to incommensurable and imaginary exponents, it is to be observed,

First, that to render an incommensurable exponent intelligible, it is necessary to suppose it replaced by a commensurable exponent which differs from it by less than any assigned quantity.

Hence a formula involving incommensurable exponents is considered as the limit of the values which are obtained by substituting for the incommensurable exponents commensurable numbers which differ from these exponents by less than any assigned number; and in this sense it is understood that the proposed formula will represent exactly the same limit, after the required operations have been executed on the incommensurable exponents which it contains, in the same manner as if these exponents were commensurable.

For example, m and n being incommensurable, $a^m\times a^n$ ought always to be equal to a^{m+n}.

In effect, if, instead of m, n, commensurable numbers, m', n', m'', n'', which approach indefinitely near to m, n, are substituted,

$$a^{m'}\times a^{n'}=a^{m'+n'};\ a^{m''}\times a^{n''}=a^{m''+n''}.$$

The first members of these equalities tend to the same limit as the second.

Now $a^m\times a^n$ represents the limit of the quantities $a^{m'}\times a^{n'}$, $a^{m''}\times a^{n''}$; and a^{m+n} represents the limit of the quantities $a^{m'+n'}$, $a^{m''+n''}$. Therefore $a^m\times a^n=a^{m+n}$.

Second. With respect to imaginary exponents: that when imaginary quantities are introduced into a calculation, a tacit convention must always subsist; and this is, to regard as equivalent the expressions in which certain letters, as a, b, c, . . ., are replaced by imaginary quantities, whenever it can be demonstrated that these expressions must be equal if real values are substituted for the symbols of imaginary quantity.

Taking, for example, the expression $a^m \times a^n$; this expression having no meaning when m, n are imaginary, it is evident that without an express or implied convention $a^m \times a^n$ cannot be regarded as equivalent to a^{m+n}.

Examples: Articles 158-162.

1st. $a^{\frac{3}{4}} \times a^{\frac{5}{3}}$? .... *Ans.* $a^{\frac{29}{12}} = a^2 \times a^{\frac{5}{12}} = a^2 \sqrt[12]{a^5}$.

2d. $\dfrac{1}{a^{\frac{1}{2}}} \times a^{\frac{7}{4}} \times a^{-\frac{1}{5}}$? .... *Ans.* $a^{\frac{21}{20}} = a \sqrt[20]{a}$.

3d. $a^{-\frac{3}{4}} \times a^{-\frac{7}{8}}$? .... *Ans.* $a^{-\frac{13}{8}} = \dfrac{1}{a^{\frac{13}{8}}}$.

4th. $\sqrt[5]{a^{12}} \times \sqrt[7]{a^3} \times \sqrt[6]{a^4}$? .... *Ans.* $a^{\frac{367}{105}} = a^3 \sqrt[105]{a^{52}}$.

5th. $\dfrac{b}{\sqrt{a}} \times \sqrt[3]{ac} \times \dfrac{\sqrt[4]{c^3}}{\sqrt{b}}$? .... *Ans.* $\dfrac{b^{\frac{1}{2}} c^{\frac{13}{12}}}{a^{\frac{1}{6}}}$.

6th. $a^{\frac{m}{n}} \div a^{\frac{p}{q}}$? .... *Ans.* $a^{\frac{mq-np}{nq}}$.

7th. $a^{\frac{m}{n}} \div a^{-\frac{p}{q}}$? .... *Ans.* $a^{\frac{mq+np}{nq}}$.

8th. $a^{-\frac{m}{n}} \div a^{\frac{p}{q}}$? .... *Ans.* $a^{-\frac{mq+np}{nq}}$.

9th. $a^{-\frac{m}{n}} \div a^{-\frac{p}{q}}$? .... *Ans.* $a^{-\frac{mq-np}{nq}}$.

10th. $ca^{\frac{3}{4}} \div da^{\frac{5}{6}}$? .... *Ans.* $\dfrac{c}{d} a^{-\frac{1}{12}}$.

11th. $a^{\frac{2}{3}} b^{\frac{1}{2}} \div a^{-\frac{4}{3}} b^{-\frac{1}{4}} c$? .... *Ans.* $\dfrac{a^2 b^{\frac{3}{4}}}{c}$.

12th. $h \div \dfrac{a^{\frac{1}{3}} b^{\frac{1}{4}}}{cd^{\frac{1}{2}}}$? .... *Ans.* $\dfrac{chd^{\frac{1}{2}}}{a^{\frac{1}{3}} b^{\frac{1}{4}}}$.

13th. $\left(a^{\frac{3}{4}} - b^{\frac{3}{4}}\right) \div \left(a^{\frac{1}{4}} - b^{\frac{1}{4}}\right)$? .... *Ans.* $a^{\frac{1}{2}} + b^{\frac{1}{2}} + (ab)^{\frac{1}{4}}$.

14th. $\left(a^{\frac{m}{n}}\right)^{\frac{p}{q}}$? .... *Ans.* $a^{\frac{mp}{nq}}$.

15th. $\left(a^{-\frac{m}{n}}\right)^{\frac{p}{q}}$? .... *Ans.* $\dfrac{1}{a^{\frac{mp}{nq}}}$.

16th. $\left(a^{\frac{m}{n}}\right)^{-\frac{p}{q}}$? .... *Ans.* $\dfrac{1}{a^{\frac{mp}{nq}}}$.

17th. $\left(a^{-\frac{m}{n}}\right)^{-\frac{p}{q}}$? .... *Ans.* $a^{\frac{mp}{nq}}$.

18th. $\left(a^{\frac{3}{4}} b^{\frac{2}{3}}\right)^{\frac{1}{3}}$? .... *Ans.* $a^{\frac{1}{4}} b^{\frac{2}{9}} = \left(a^9 b^8\right)^{\frac{1}{36}}$.

19th. $\left(a^2 b^{-\frac{1}{3}} c^{\frac{2}{5}}\right)^{-\frac{1}{4}}$? .... *Ans.* $\dfrac{b^{\frac{1}{12}}}{a^{\frac{1}{2}} c^{\frac{1}{10}}}$.

20th. $\left[\left(a^{\frac{1}{2}}\right)^{\frac{2}{3}}\right]^{-\frac{1}{6}}$? .... *Ans.* $\dfrac{1}{a^{\frac{1}{18}}}$.

21st. $\left(\dfrac{ab^{\frac{1}{2}}}{a^{\frac{1}{3}}b^{\frac{1}{3}}}\right)^{\frac{3}{4}}$?..*Ans.* $a^{\frac{1}{2}}b^{\frac{1}{8}}$.

22d. $a+b\sqrt{-1}+c\sqrt{-1}-d\sqrt{-1}$?............*Ans.* $a+(b+c-d)\sqrt{-1}$.

23d. $a\times\sqrt{-a}$?..*Ans.* $a\sqrt{a}\sqrt{-1}$.

24th. $a+\sqrt{a}\sqrt{-1}$?..*Ans.* $-\sqrt{a}\sqrt{-1}$.

Of the Multiple Values of Algebraic Radicals.

163. The radicals hitherto considered are arithmetical, or those whose values are real and positive; it remains to consider algebraic radicals, under which description are included all the values which, being raised to the power marked by the index of the radical, reproduce the quantity placed under the radical sign.

Let any radical, $\sqrt[m]{A}$, be assumed, and let A be supposed successively positive, negative, and imaginary.

When A is positive, a positive quantity, a, can be found, either exactly or to any required degree of approximation, whose mth power shall be equal to A (Art. 146-148); and since every other quantity raised to this power must evidently give a result either greater than A or less than A, the value a is an arithmetical determination of the radical, and the only one of which it admits.

When A is negative, since there exists no positive magnitude whose powers are negative, the radical can have algebraic determinations only.

When A is imaginary, as it is evident that the powers of a real magnitude, whether positive or negative, are themselves real magnitudes, it follows that all the determinations of the radical are algebraic, and also imaginary.

164. The index m of a radical may be an even or an odd number. If m is even all the values of the radical, taken two by two, are equal, and of contrary signs; for, if a is one value such that $a^m=A$, since m is an even number, $(-a)^m=a^m=A$, whence $-a$ is also a value of the radical. All the values of the radical may therefore be represented by a series of quantities affected with the sign $\pm$, such, for example, as $\pm a$, $\pm b$, $\pm c$, &c.

When m is even and A positive, the radical has one real positive value. Supposing this to be a, it is evident that $-a$ is also a real, but negative, value of the radical.

With respect to the other values, $\pm b$, $\pm c$, &c., they can be only imaginary.

In the case of the index m odd; if the sign of A, the quantity under the radical sign, is changed, the values of the radical are also changed in respect of their signs; for if a is a value of $\sqrt[m]{A}$, it is evident that, m being odd, $(-a)^m=-a^m=-A$. Therefore, supposing that the values of $\sqrt{A}$ are $a, b, c, \ldots$, the values of $\sqrt[m]{-A}$ are $-a, -b, -c, \ldots$

If A is a real quantity, one of the values of the radical is real, and of the same sign as A; that is to say, it is positive when A is positive, and negative when A is negative.

The other values are imaginary.

165. It has been proved (Art. 103) that a radical of the second degree has two different values or roots, and it admits of proof that any radical, $\sqrt[m]{A}$, has m values or roots.

The only manner in which the general proposition can be established in this place is, by proving it true in many particular cases, and inferring the general proposition by analogy.

Taking, first, a radical of the third degree, $\sqrt[3]{A}$; every quantity which has A for cube is for this reason a value of $\sqrt[3]{A}$. Therefore the values of this radical are precisely the same as the values which satisfy the equation $x^3=A$, or $x^3-A=0$.

For the sake of precision suppose A positive; then there exists a positive quantity of which the cube is A, and which can be calculated either exactly or with any required degree of approximation.

Denoting this quantity by a, and replacing A by a^3, the equation $x^3-A=0$ becomes $x^3-a^3=0$.

The binomial x^3-a^3 is divisible by $x-a$ (Art. 22). Performing the division $(x^3-a^3)\div(x-a)=x^2+ax+a^2$.

$$\text{Therefore } x^3-a^3=(x-a)(x^2+ax+a^2)=0.$$

Now, in order that a product may be equal to zero, it is necessary and sufficient that one of its factors be equal to zero; therefore all the solutions of the equation x^3-a^3 must be obtained by making successively $x-a=0$;

$$x^2+ax+a^2=0;$$

and deducing the values of x from these equations.

$$\text{The first gives } x=a,$$

$$\text{the second gives } x=a\left(\frac{-1\pm\sqrt{-3}}{2}\right).$$

Consequently a, $a\left(\frac{-1+\sqrt{-3}}{2}\right)$, $a\left(\frac{-1-\sqrt{-3}}{2}\right)$ are the three cube roots of a^3 or A.

Amongst these roots the value a is re-obtained, and this was to be expected, since a is by hypothesis a cube root of A. But, besides a, two other values are obtained by the preceding analysis; and this analysis, let it be observed, must subsist, whatever be the quantity A, provided that a expresses a value of $\sqrt[3]{A}$, whether real or imaginary.

$$\text{Similarly, if } x^4=A \text{ and } \sqrt[4]{A}=a,\ x^4=a^4, \text{ or } x^4-a^4=0.$$

$$\text{But } x^4-a^4=(x^2-a^2)(x^2+a^2);$$

$$\therefore x^2-a^2=0,\ x^2+a^2=0.$$

Now $x^2-a^2=0$ gives $x=\pm\sqrt{a}$, and $x^2+a^2=0$ gives $x=\pm a\sqrt{-1}$.

Whence four different algebraic expressions are obtained for the fourth root of the quantity A.

The proposition is easily extended to every radical whose index is a power of 2 or of 3, or a number composed of the factors 2, 3.

Suppose that $x^{12}=A$ or $x=\sqrt[12]{A}$, the index $12=2\times2\times3$.

The quantity A has therefore two square roots, each of which, again, has two square roots; this makes four different quantities. Then each of these quantities has three cube roots, which makes twelve different quantities; each of the twelve is a value of $\sqrt[12]{A}$.

Let a be one of the two square roots of A,
a', one of the two square roots of a,
a'', one of the three cube roots of a',

it is evident that a'' is one of the twelve quantities under consideration.

$$\text{For, since } (a'')^3=a', \text{ and } (a')^2=a,$$

$$\therefore (a'')^6=a \text{ and } (a'')^{12}=a^2.$$

$$\text{But } a^2=A \quad \therefore (a'')^{12}=A \quad \therefore a'' \text{ is a value of } \sqrt[12]{A}.$$

Whatever the radical $\sqrt[m]{A}$ may be, the determination of its different values is always reducible to the resolution of an equation. In effect $\sqrt[m]{A}$ represents, indifferently, all the quantities, real and imaginary, which, raised to the power m, reproduce A. Consequently these quantities are the values of x, which satisfy the equation $x^m=A$.

It appears, therefore, that the general proposition which forms the subject of this Article requires it to be proved that in the equation $x^m=A$ the unknown quantity x has m different values.

166. Denoting by a one value of $\sqrt[3]{A}$, it has been found (Art. 165) that $\sqrt[3]{A}$ has the three values,

$$a,\ a\left(\frac{-1+\sqrt{-3}}{2}\right),\ a\left(\frac{-1-\sqrt{-3}}{2}\right).$$

Making $A=1$, $a=\sqrt[3]{1}=1$, consequently the three values of $\sqrt[3]{1}$ are

$$1,\ \frac{-1+\sqrt{-3}}{2},\ \frac{-1-\sqrt{-3}}{2}.$$

Comparing the values of $\sqrt[3]{1}$ with the values of $\sqrt[3]{A}$, this consequence is obtained: the three cube roots of any quantity are found by multiplying any one of the three by the three cube roots of unity.

This property belongs to radicals of every order.

For, representing by a a value of any radical, $\sqrt[m]{A}$, and by $1, \alpha, \beta, \gamma \ldots$ the m values of $\sqrt[m]{1}$, the products of the root a by $1, \alpha, \beta, \gamma \ldots.$ are

$$a\times 1,\ a\times\alpha,\ a\times\beta,\ a\times\gamma \ldots.$$

and if these products are raised to the power m, the results are

$$a^m,\ a^m\alpha^m,\ a^m\beta^m,\ a^m\gamma^m \ldots.$$

Now a^m, $a^m\alpha^m$, $a^m\beta^m$, &c. are all equal to A; for it is evident that $a^m=A$, and that, α being the mth root of 1, $\alpha^m=1$; that β being the mth root of 1, $\beta^m=1$, &c. Therefore the products a, $a\alpha$, $a\beta$, $a\gamma$, ... are the m values of $\sqrt[m]{A}$.

Whence the roots of every order, of any quantity whatever, are formed by multiplying one of the roots by the corresponding roots of 1.

What has been proved of the roots of any quantity may be applied to the roots of unity. Thus (taking the cube roots of 1),

$$\frac{-1+\sqrt{-3}}{2}\times 1=\frac{-1+\sqrt{-3}}{2},$$

$$\frac{-1+\sqrt{-3}}{2}\times\frac{-1+\sqrt{-3}}{2}=\frac{1+2\sqrt{-3}+(\sqrt{-3})^2}{4}=\frac{1-2\sqrt{-3}-3}{4}$$

$$=\frac{-2-2\sqrt{-3}}{4}=\frac{-1-\sqrt{-3}}{2}.$$

$$\frac{-1+\sqrt{-3}}{2}\times\frac{-1-\sqrt{-3}}{2}=\frac{1-(\sqrt{-3})^2}{4}=\frac{1-(-3)}{4}=\frac{1+3}{4}=\frac{4}{4}=1.$$

Whence this property: the different roots of unity reproduce each other in a different order when each of them is multiplied in succession by all the other roots.

If this theorem is verified (as above) by means of the cube roots of 1, it will be found that the two imaginary roots are each the square of the other, and that the real root is the product of the two imaginary roots.

CALCULUS OF ALGEBRAIC RADICALS.

167. Expressions which contain algebraic radicals may have many values, which ought to be preserved in any calculation performed with the expressions in which they are contained.

The simplification of arithmetical radicals is founded on the equality,

$$\sqrt[m]{a^m b}=a\sqrt[m]{b}.$$

Since the second member, considered as an algebraic radical, has m values the mth power of each of which is $a^m b$, the expression $a\sqrt[m]{b}$ represents all the m values of $\sqrt[m]{a^m b}$. Therefore algebraic can be simplified in the same manner as arithmetical radicals.

168. $\sqrt[m]{a}$, $\sqrt[m]{b}$, being arithmetical radicals which have the same index, $\sqrt[m]{a}\times\sqrt[m]{b}=\sqrt[m]{ab}$ (Art. 165).

$\sqrt[m]{a}$, $\sqrt[m]{b}$, considered as algebraic radicals, have each m values; and if each of the m values of $\sqrt[m]{a}$ is multiplied by each of the m values of $\sqrt[m]{b}$, the number of products is $m(m-1)$.

Now $\sqrt[m]{ab}$ has only m values. Therefore before it can be admitted that $\sqrt[m]{a}\times\sqrt[m]{b}=\sqrt[m]{ab}$, it must be proved that the number of different products of the m values of $\sqrt[m]{a}$ by the m values of $\sqrt[m]{b}$ is m, and that these are equal to the corresponding values of $\sqrt[m]{ab}$.

Let a' be the arithmetical value of $\sqrt[m]{a}$, b' the arithmetical value of $\sqrt[m]{b}$, and $1, \alpha, \beta, \gamma, \ldots$ the m values of $\sqrt[m]{1}$.

If a', b' are each multiplied by the m roots of $\sqrt[m]{1}$, the results are the m values of $\sqrt[m]{a}$, $\sqrt[m]{b}$. Therefore the different products of the m values of $\sqrt[m]{a}$ by the m values of $\sqrt[m]{b}$ are expressed by the product of the quantities,

$a', a'\alpha, a'\beta, a'\gamma, \ldots\ldots$ to m terms,
by $b', b'\alpha, b'\beta, b'\gamma, \ldots\ldots$ to m terms;

or of $a'(1, \alpha, \beta, \gamma, \ldots\ldots)\times b'(1, \alpha, \beta, \gamma, \ldots\ldots)$.

Now the product of a' $(1, \alpha, \beta, \gamma, \ldots\ldots$ to m terms$)$ by $b'\times 1$ is,
$a'b'\times 1(1, \alpha, \beta, \gamma, \ldots\ldots$ to m terms$)$;
and the product of $a'(1, \alpha, \beta, \gamma, \ldots\ldots$ to m terms$)$ by $b'\times\alpha$ is
$a'b'\times\alpha(1, \alpha, \beta, \gamma, \ldots\ldots$ to m terms$)$.

But since the different roots of 1 reproduce each other in different orders when each of them is multiplied by all the other roots,

the m products $a'b'$ by $1\times(1, \alpha, \beta, \gamma, \ldots\ldots)$,
and the m products $a'b'$ by $\alpha(1, \alpha, \beta, \gamma, \ldots\ldots)$,

must be the same m quantities differently arranged.

In the same manner the m products,

$a'b'\times\beta(1, \alpha, \beta, \gamma, \ldots\ldots$ to m terms$)$,
$a'b'\times\gamma(1, \alpha, \beta, \gamma, \ldots\ldots$ to m terms$)$,

contain only the same m distinct products. Therefore the whole number of different values obtained by multiplying each of the m values of $\sqrt[m]{a}$ by each of the m values of $\sqrt[m]{b}$ is m.

The values of $\sqrt[m]{ab}$ being also in number m, and equal to the different products of one value of $\sqrt[m]{ab}$ (the arithmetical value, for example, which may be denoted by $a'b'$) by the m values of $\sqrt[m]{1}$, it follows that the values of $\sqrt[m]{a}\times\sqrt[m]{b}$ are the same as the values of $\sqrt[m]{ab}$, and therefore that $\sqrt[m]{a}\times\sqrt[m]{b}=\sqrt[m]{ab}$.

It is proved in this investigation that if the m values of a radical are multiplied by one of the m values of another radical, the m results comprehend all the distinct products which can arise from the multiplication of each of the m values of the one by each of the m values of the other.

169. When $\sqrt[m]{a}$, $\sqrt[m]{b}$, express arithmetical radicals, it is proved that $\dfrac{\sqrt[m]{a}}{\sqrt[m]{b}}=\sqrt[m]{\dfrac{a}{b}}$.

Let $\sqrt[m]{a}$, $\sqrt[m]{b}$ be algebraic radicals. Then the values obtained by dividing the m values of $\sqrt[m]{a}$ by the m values of $\sqrt[m]{b}$ ought to be such that, multiplied by the m values of $\sqrt[m]{b}$, the product shall be the m values of $\sqrt[m]{a}$; whence, by the last Article, the quotient ought to have m values. But $\sqrt[m]{\dfrac{a}{b}}$ has also m values; whence $\dfrac{\sqrt[m]{a}}{\sqrt[m]{b}}$ has the same number of values as $\sqrt[m]{\dfrac{a}{b}}$.

170. $\sqrt[m]{a}$ being an arithmetical radical, and n the exponent of a power to which $\sqrt[m]{a}$ is raised, it is proved (Art. 155), that $(\sqrt[m]{a})^n=\sqrt[m]{a^n}$. This equality is to be demonstrated for algebraic radicals.

Three cases may be distinguished:
1st. Let m, n be prime to each other.

Raising $(\sqrt[m]{a})^n$ to the power m,

$$[(\sqrt[m]{a})^n]^m=(\sqrt[m]{a})^{mn}=[(\sqrt[m]{a})^m]^n=a^n.$$

From this it is to be inferred that all the values of the expression $(\sqrt[m]{a})^n$ are found among the values of $\sqrt[m]{a^n}$. It remains to be proved that the number of the values of $(\sqrt[m]{a})^n$ is m.

Let a', a'', a''' . . . . . . be the m values of $\sqrt[m]{a}$. Then a'^n, a''^n, a'''^n . . . . are the m values of $(\sqrt[m]{a})^n$; it is to be shown that a'^n, a''^n, a'''^n are all different.

Suppose that all the values are not different, and that $a'^n=a''^n$ - 1
Then, since both a', a'' are mth roots of a, $a'^m=a''^m$ - - 2
Let m be greater than n, and let $m=nq+r$; therefore
since $a'^m=a''^m$, $a'^{nq+r}=a''^{nq+r}$ - - - - 3
But $a'^n=a''^n$; $\therefore (a'^n)^q=(a''^n)^q$, or $a'^{nq}=a''^{nq}$ - - - - 4
Whence, if equation 3 is divided by equation 4, $a'^r=a''^r$.

Dividing, next, n by r, let the quotient be q', and the remainder r', so that $n=rq'+r'$. Substituting $rq'+r'$ for n in equation 1,

$$a'^{rq'+r'}=a''^{rq'+r'};$$

$$\text{and since } a'^r=a''^r;\ a'^{rq'}=a''^{rq'};$$

$$\text{therefore by division, } a'^{r'}=a''^{r'}.$$

By continuing this process a result is always obtained of the form $a'^s=a''^s$, in which the exponent s is one of the remainders obtained by applying to m and n the process for the discovery of their greatest common measure.

Now, as m, n are prime to each other, the last remainder must be 1. Therefore $a'=a''$.

That is, a', a'' are not two different values of $\sqrt[m]{a}$, which is contrary to the hypothesis. Therefore the values of a'^n, a''^n are different, and $(\sqrt[m]{a})^n$ has m different values.

2d. Let mn, the index of the radical, be a multiple of n, the exponent of the power, it is to be proved that

$$(\sqrt[mn]{a})^n=\sqrt[m]{a}.$$

Let a' be a value of $\sqrt[mn]{a}$, then $a'^{mn}=a$, or $(a'^n)^m=a$.

Therefore the values of a'^n are those of $\sqrt[m]{a}$.

$$\text{But } a'^n=(\sqrt[mn]{a})^n.\quad \therefore (\sqrt[mn]{a})^n=\sqrt[m]{a}.$$

3d. Let mp, the index of the radical, and np, the exponent of the power, have a common factor, p, the other factors, m, n, being prime to each other,

$$\sqrt[mp]{a})^{np}=\sqrt[m]{a^n}.$$

For, from the preceding cases 1 and 2,

$$(\sqrt[mp]{a})^{np}=[(\sqrt[mp]{a})^p]^n=(\sqrt[m]{a})^n=\sqrt[m]{a^n}.$$

171. The rule which relates to the extraction of the roots of algebraic radicals is contained in the formula,

$$\sqrt[n]{\sqrt[m]{a}}=\sqrt[mn]{a}.$$

For, making $x=\sqrt[n]{\sqrt[m]{a}}$, $x^n=\sqrt[m]{a}$, and $x^{mn}=a$,
Therefore x has the same values as $\sqrt[mn]{a}$.

OF IMAGINARY EXPRESSIONS OF THE SECOND DEGREE.

171′. If two or more imaginary expressions of the form $a+b\sqrt{-1}$ (in which a, b are real quantities) are combined together, the result is always reducible to the form $a+b\sqrt{-1}$.

Let $a+b\sqrt{-1}$, $a'+b'\sqrt{-1}$ be expressions of this form. Then,

$$a+b\sqrt{-1}+a'+b'\sqrt{-1}=a+a'+(b+b')\sqrt{-1}.$$

$$(a+b\sqrt{-1})-(a'+b'\sqrt{-1})=a-a'+(b-b')\sqrt{-1}.$$

$$(a+b\sqrt{-1})\times(a'+b'\sqrt{-1})=aa'+ab'\sqrt{-1}+a'b\sqrt{-1}-bb'=aa'-bb'$$
$$+(ab'+a'b)\sqrt{-1}.$$
$$\frac{a+b\sqrt{-1}}{a'+b'\sqrt{-1}}=\frac{(a+b\sqrt{-1})\ (a'-b'\sqrt{-1})}{(a'+b'\sqrt{-1})(a'-b'\sqrt{-1})}=\frac{aa'+bb'}{a'^2+b'^2}+\frac{a'b-ab'}{a'^2+b'^2}\sqrt{-1}.$$

Consequently the results obtained by the four fundamental operations are of the form $a+b\sqrt{-1}$.

172. The powers of $\sqrt{-1}$ are expressions which occur frequently.

Of these, $(\sqrt{-1})^1=+\sqrt{-1}$.
$$(\sqrt{-1})^2=\sqrt{-1}^2=-1.$$
$$(\sqrt{-1})^3=(\sqrt{-1})^2\times\sqrt{-1}=-1\times\sqrt{-1}=-\sqrt{-1}.$$
$$(\sqrt{-1})^4=(\sqrt{-1})^2\times(\sqrt{1}-(^2=-1\times-1=+1.$$

Denoting by e any whole positive number whatever,
$$(\sqrt{-1})^{4e}=[(\sqrt{-1})^4]^e=+1^e=+1.$$
$$(\sqrt{-1})^{4e+1}=(\sqrt{-1})^{4e}\times\sqrt{-1}=+1\times\sqrt{-1}=+\sqrt{-1}.$$
$$(\sqrt{-1})^{4e+2}=(\sqrt{-1})^{4e}\times(\sqrt{-1})^2=+1\times-1=-1.$$
$$(\sqrt{-1})^{4e+3}=(\sqrt{-1})^{4e}\times(\sqrt{-1})^3=+1\times-\sqrt{-1}=-\sqrt{-1}.$$

Whence it follows that for any power whatever of $\sqrt{-1}$ one or other of the results $+\sqrt{-1}$, -1, $-\sqrt{-1}$, $+1$, is obtained.

These formulæ ought to be employed, respectively, according as, on dividing the exponent of $\sqrt{-1}$ by 4, the remainder is 0, 1, 2, 3; and they evidently comprehend all cases.

Considering, next, the expression $(a+b\sqrt{-1})^n$, in which n is any whole positive number; by the binomial formula
$$(a+b\sqrt{-1})^n=a^n(1+\frac{b}{a}\sqrt{-1})^n=a^n\left\{1+\frac{n}{1}\cdot\frac{b}{a}\sqrt{-1}-\frac{n(n-1)b^2}{1.\ 2.\ a^2}\right.$$
$$\left.-\frac{n(n-1)\ (n-2)b^3}{1.\ 2.\ \ 3.\ a^3}\sqrt{-1}+\frac{n(n-1)\ (n-2)\ (n-3)b^4}{1.\ 2.\ \ 3.\ \ 4.\ a^4}+,\text{ \&c.}\right\};$$

Combining the terms in $\sqrt{-1}$,
$$(a+b\sqrt{-1})^n=a^n\left\{1-\frac{n(n-1)b^2}{1.\ 2.\ a^2}+\frac{n(n-1)\ (n-2)\ (n-3)b^4}{1.\ 2.\ \ 3.\ \ 4.\ a^4}-,\text{ \&c.}\right\}$$
$$+a^n\left\{\frac{n\cdot b}{1\cdot a}-\frac{n(n-1)\ (n-2)b^3}{1.\ 2.\ \ 3.\ a^3}+,\text{ \&c.}\right\}\sqrt{-1}.$$

Since the quantities a, b, are, by hypothesis, real, this result is of the form $A+B\sqrt{-1}$, A and B being also real quantities.

To develope $(a-b\sqrt{-1})^n$ it is sufficient to change b into $-b$ in the preceding result; by this substitution no change is produced except that of +B into −B, for b enters by even powers (which are +) into all the terms of A, and by odd powers (which are −) into all the terms of B.

Making, therefore, for the sake of abbreviation,
$$A=a^n\left(1-\frac{n(n-1)b^2}{1.\ 2.\ a^2}+\frac{n(n-1)\ (n-2)\ (n-3)b^2}{1.\ 2.\ \ 3.\ \ 4.\ a^4}-,\text{ \&c.}\right)$$
$$B=a^n\left(\frac{n\cdot b}{1\cdot a}-\frac{n(n-1)\ (n-2)\cdot b^3}{1.\ 2.\ \ 3.\ a^3}+,\text{ \&c.}\right);$$
$$(a+b\sqrt{-1})^n=A+B\sqrt{-1}\quad\text{- - - - - 1}$$
$$(a-b\sqrt{-1})^n=A-B\sqrt{-1}\quad\text{- - - - 2}$$

To pass to negative powers, it is to be observed that
$$(a+b\sqrt{-1})\ (a-b\sqrt{-1})=a^2+b^2,$$
$$\text{whence }(a+b\sqrt{-1})^{-1}=\frac{1}{a+b\sqrt{-1}}=\frac{a-b\sqrt{-1}}{a^2+b^2},$$
$$(a-b\sqrt{-1})^{-1}=\frac{1}{a-b\sqrt{-1}}=\frac{a+b\sqrt{-1}}{a^2+b^2}.$$
$$\text{Wherefore, also, }(a+b\sqrt{-1})^{-n}=\frac{1}{(a+b\sqrt{-1})^n}=\frac{(a-b\sqrt{-1})^n}{(a^2+b^2)^n}.$$

Consequently by formulæ 1 and 2,

$$(a+b\sqrt{-1})^{-n}=\frac{A-B\sqrt{-1}}{(a^2+b^2)^n} \quad - \quad - \quad - \quad 3$$

$$(a-b\sqrt{-1})^{-n}=\frac{A+B\sqrt{-1}}{(a^2+b^2)^n} \quad - \quad - \quad - \quad 4$$

In Article 175 it is proved that the binomial theorem is true for any exponent whatever; the transformations 1, 2, 3, 4, are therefore true for every value of n; but when n is negative or fractional, the values of A, B, are composed of an infinite number of terms.

173. When it is required to reduce the radical expression $\sqrt[n]{a\pm b\sqrt{-1}}$ to the form $A\pm B\sqrt{-1}$ the radical is replaced by the fractional power $(a\pm b\sqrt{-1})^{\frac{1}{n}}$, which is developed in the manner already mentioned.

Algebra supplies no other general method for this transformation; but when n is a power of 2 the transformation can be effected without the help of series.

Taking, first, the two radicals $\sqrt{a+b\sqrt{-1}}$, $\sqrt{a-b\sqrt{-1}}$.

$$\text{Making } \sqrt{a+b\sqrt{-1}}+\sqrt{a-b\sqrt{-1}}=x \quad - \quad - \quad - \quad 5$$

$$\sqrt{a+b\sqrt{-1}}-\sqrt{a-b\sqrt{-1}}=y \quad - \quad - \quad - \quad 6$$

and forming the squares, the results are

$$2a+2\sqrt{a^2+b^2}=x^2,$$

$$2a-2\sqrt{a^2+b^2}=y^2,$$

in which, whatever be the sign of a, the value of x^2 is positive, and that of y^2 negative.

$$\left.\begin{array}{l}\text{Now, since } x^2=2a+2\sqrt{a^2+b^2},\ x=\pm\sqrt{2a+2\sqrt{a^2+b^2}} \\ \text{and since } y^2=2a-2\sqrt{a^2+b^2},\ y=\pm\sqrt{-2a+2\sqrt{a^2+b^2}}\sqrt{-1}\end{array}\right\} \ - \ 7$$

Again, by adding together equations 5 and 6,

$$2\sqrt{a+b\sqrt{-1}}=x+y \ \therefore\ \sqrt{a+b\sqrt{-1}}=\frac{x+y}{2};$$

And if equation 6 is subtracted from equation 5,

$$2\sqrt{a-b\sqrt{-1}}=x-y \ \therefore\ \sqrt{a-b\sqrt{-1}}=\frac{x-y}{2}.$$

Substituting for x, y in these expressions the values of x, y, from equation 7,

$$\sqrt{a+b\sqrt{-1}}=\tfrac{1}{2}\sqrt{2a+2\sqrt{a^2+b^2}}+\tfrac{1}{2}\sqrt{-2a+2\sqrt{a^2+b^2}}\sqrt{-1} \quad - \quad - \quad 8$$

$$\sqrt{a-b\sqrt{-1}}=\tfrac{1}{2}\sqrt{2a+2\sqrt{a^2+b^2}}-\tfrac{1}{2}\sqrt{-2a+2\sqrt{a^2+b^2}}\sqrt{-1} \quad - \quad - \quad 9$$

If the radical expressions, $\sqrt[4]{a\pm b\sqrt{-1}}$, $\sqrt[8]{a\pm b\sqrt{-1}}$, $\sqrt[16]{a\pm b\sqrt{-1}}$, &c. are considered, it is evident that the extraction of a root whose index is a power of 2, may be replaced by successive extractions of the square root; and that, by consequence, the repeated employment of formulæ 8 and 9 must always reduce expressions such as the preceding to the form $A\pm B\sqrt{-1}$.

BINOMIAL THEOREM, THE EXPONENT BEING FRACTIONAL OR NEGATIVE.

174. Let the expressions $A+Bx+Cx^2+Dx^3+\ \ldots\ldots$
$A'+B'x+C'x^2+D'x^3+\ \ldots\ldots$ (in which A, A′, B, B′, &c. do not contain x) be equal to each other, whatever the value of x, then, if $x=0$, the terms which contain x are reduced to zero, and $A=A'$.

Again, since $A=A'$,

$$Bx+Cx^2+Dx^3+\ \ldots\ldots=B'x+C'x^2+D'x^3+\ \ldots\ldots$$

Dividing all the terms of this equality by x,

$$B+Cx+Dx^2+\ .\ .\ .\ =B'+C'x+D'x^2+\ .\ .\ .\ .$$

Therefore, if $x=0$, $B=B'$;

and, similarly, $C=C'$, $D=D'$.

Wherefore, if two polynomial expressions, which are arranged according to the powers of x, are equal, whatever be the value of x, the coefficients of the same powers of x in the two expressions are equal.

175. $a+x$ being a binomial, and $\frac{m}{n}$ a positive fractional exponent,

$$(a+x)^{\frac{m}{n}}=\sqrt[n]{(a+x)^m}=\sqrt[n]{a^m+ma^{m-1}x+\frac{m(m-1)}{1.2}a^{m-2}x^2+, \&c.}$$

Then, since the first and second terms of the nth root of the quantity under the radical sign are $a^{\frac{m}{n}}$, $\frac{m}{n}a^{\frac{m}{n}-1}x$, and the succeeding terms are of the form $Aa^{\frac{m}{n}-2}x^2$, $Ba^{\frac{m}{n}-3}x^3$, &c. (A, B, . . denoting coefficients which contain neither a nor x), the development of $(a+x)^{\frac{m}{n}}$ is of the form

$$a^{\frac{m}{n}}+\frac{m}{n}a^{\frac{m}{n}-1}x+Aa^{\frac{m}{n}-2}x^2+Ba^{\frac{m}{n}-3}x^3+\ .\ .\ .\ .\ .$$

When the exponent is any negative number, integral or fractional,

$$(a+x)^{-m}=\frac{1}{(a+x)^m}=\frac{1}{a^m+ma^{m-1}x+\underline{m(m-1)}a^{m-2}x^2,\ \&c.}$$
$$1.2$$

Performing the division indicated in this expression, the first and second terms of the quotient are a^{-m}, $-ma^{-m-1}x$; and the quotient, which is composed of an indefinite number of terms, is of the form

$$a^{-m}-ma^{-m-1}x+Aa^{-m-2}x^2+Ba^{-m-3}x^3+\ .\ .\ .\ .$$

The first and second terms of the development $(a+x)^m$ being a^m and $ma^{m-1}x$, and the third, fourth, &c. being of the form $Aa^{m-2}x^2$, $Ba^{m-3}x^3$, &c., it appears that if m denote any exponent, positive or negative, integral or fractional, the development of $(a+x)^m$ may be put under the form

$$a^m+ma^{m-1}x+Aa^{m-2}x^2+Ba^{m-3}x^3+\ .\ .\ .\ .\ .$$

in which the first and second terms are determined; and it is necessary to find the coefficients A, B, . . .

To render the investigation more general, let

$$(a+x)^m=a^m+ma^{m-1}x+\ .\ .\ .\ .\ .\ .\ +Ma^{m-n}x^n+Na^{m-n-1}x^{n+1}+\ .\ .\ ,$$

$Ma^{m-n}x^n$, $Na^{m-n-1}x^{n+1}$, being two consecutive terms of any rank.

If x is changed into $x+y$ (as the indeterminate coefficients contain neither a nor x) the result is

$$\left.\begin{array}{l}(a+x+y)^m=a^m+ma^{m-1}(x+y)+\ .\ .\ .\ .\ .\ .\ +Ma^{m-n}(x+y)^n+\\ \qquad Na^{m-n-1}(x+y)^{n+1}+\quad .\quad .\quad .\quad .\quad .\end{array}\right\}\ -\ 1$$

And if a is changed into $a+y$, the result is

$$\left.\begin{array}{l}(a+x+y)^m=(a+y)^m+m(a+y)^{m-1}x+\ .\ .\ .\ .\ .\ .\ +M(a+y)^{m-n}x^n+\\ \qquad N(a+y)^{m-n-1}x^{n+1}+\quad .\quad .\quad .\quad .\quad .\end{array}\right\}2$$

The first members of equalities 1, 2 are identical, therefore the second members must be equal, whatever the values of x, y, and also identical, if arranged according to the powers of y. The second members involve certain powers of the binomials $x+y$, $a+y$; but the first and second terms of each of these binomials are known, so that the polynomial coefficient of the first power of y can be formed in both the second members, and this is sufficient.

The parts of the binomials which contain the first power of y are,

Equation 1. $ma^{m-1}y$ - - from the term $ma^{m-1}(x+y)$.

. . . .

$Mna^{m-n}x^{n-1}y$ - - - - $Ma^{m-n}(x+y)^n$.

$N(n+1)a^{m-n-1}x^ny$ - - $Na^{m-n-1}(x+y)^{n+1}$.

. . . .

Equation 2. $ma^{m-1}y$ - - - from the term $(a+y)^m$.

. . . .

$M(m-n)a^{m-n-1}x^ny$ - - - $M(a+y)^{m-n}x^n$.

$N(m-n-1)a^{m-n-2}x^{n+1}y$ - - $N(a+y)^{m-n-1}x^{n+1}$.

. . . .

Denoting the assemblage of these parts by Yy, $Y'y$ respectively,

$$Yy=(ma^{m-1}+\ldots\ldots+Mna^{m-n}x^{n-1}+N(n+1)a^{m-n-1}x^n+\ldots.)y,$$

$$Y'y=(ma^{m-1}+\ldots.+M(m-n)a^{m-n-1}x^n+N(m-n-1)a^{m-n-2}x^{n+1}+\ldots.)y.$$

These quantities, being the coefficients of the same power of y, are equal; and since they must be equal, whatever be the value of x, the coefficients of the same powers of x in each of the expressions Yy, $Y'y$ are also equal (Art. 174); therefore $N(n+1)$ the coefficient of $a^{m-n-1}x^n$ in the first expression is equal to $M(m-n)$ the coefficient of $a^{m-n-1}x^n$ in the second.

And since $N(n+1)=M(m-n)$,

$$N=\frac{M(m-n)}{n+1}.$$

Now M, N are the coefficients of two consecutive terms; $m-n$ is the exponent of a, and n the exponent of x, in the term whose coefficient is M; it therefore appears that the coefficient of any term is formed from the coefficient of the preceding term by the rule (Art. 141) for the case of an integer positive exponent. Whence, since the first and second terms of the development are composed in the same manner, whatever be the exponent, it follows that the coefficients of all the terms of a binomial are formed by the rule of Article 141, whether the exponent be positive or negative, integer or fractional. That is,

$$(a+x)^m=a^m+\frac{m}{1}a^{m-1}x+\frac{m(m-1)}{1.2}a^{m-2}x^2+\frac{m(m-1)(m-2)}{1.2.3}a^{m-3}x^3+\text{, \&c.}$$

When m is integer and positive, the last term of the formula is x^m; in all other cases the number of terms is infinite.

176. Writing the development of $(x+a)^m$ as in Article 144,

$$(x+a)^m=x^m(1+\frac{m}{1}\frac{a}{x}+\frac{m(m-1)}{1.2}\frac{a^2}{x^2}+\frac{m(m-1)(m-2)}{1.2.3}\frac{a^3}{x^3}+\text{, \&c.,}$$

and replacing m by $\frac{1}{n}$, $(x+a)^m$ becomes $(x+a)^{\frac{1}{n}}$ or $\sqrt[n]{x+a}$.

$$\therefore\ \sqrt[n]{x+a}=x^{\frac{1}{n}}(1+\frac{1}{n}\cdot\frac{a}{x}+\frac{1}{n}\,\frac{\frac{1}{n}-1}{2}\cdot\frac{a^2}{x^2}+\frac{1}{n}\cdot\frac{\frac{1}{n}-1}{2}\cdot\frac{\frac{1}{n}-2}{3}\cdot\frac{a^2}{x^2}+\text{, \&c.}$$

Reducing the coefficients, $\frac{1}{n}\ \frac{\frac{1}{n}-1}{2}$, &c.

$$\sqrt[n]{x+a}=x^{\frac{1}{n}}(1+\frac{1}{n}\frac{a}{x}-\frac{1}{n}\frac{n-1}{2n}\cdot\frac{a^2}{x^2}+\frac{1}{n}\cdot\frac{n-1}{2n}\cdot\frac{2n-1}{3n}\cdot\frac{a^3}{x^3}+\text{, \&c.)}$$

The fifth term of this expression is evidently obtained by multiplying the fourth by $\frac{3n-1}{4n}$ and by $\frac{a}{x}$, the sixth by multiplying the fifth by $\frac{4n-1}{5n}\cdot\frac{a}{x}$, &c.

As an application of the binomial formula thus reduced, let it be required to find the cube root of 31:

$$31=27+4=3^3+4.$$

Making in the formula $n=3$, $x=27$, $a=4$;

$$\sqrt[3]{31}=\sqrt[3]{27+4}=27^{\frac{1}{3}}(1+\frac{4}{27})^{\frac{1}{3}};$$

$$\sqrt[3]{31}=3(1+\frac{1}{3}\frac{4}{27}-\frac{1}{3}\frac{1}{3}\frac{16}{729}+\frac{1}{3}\frac{1}{3}\frac{5}{9}\frac{64}{19683}-\text{, \&c.,}$$

$$\text{or }\sqrt[3]{31}=3+\frac{4}{27}-\frac{16}{2187}+\frac{320}{531441}-\text{, \&c.}$$

In this particular case the fifth term

$$=\frac{320}{531441}\times\frac{3n-1}{4n}\cdot\frac{a}{x}=\frac{320}{531441}\times\frac{2}{3}\times\frac{4}{27}=\frac{2560}{43046721};$$

$$\text{and the sixth term}=\frac{2560}{43046721}\times\frac{11}{15}\times\frac{4}{27}=\frac{112640}{17433922005}.$$

Reducing the terms to decimals, and forming their algebraic sum,

$$\left.\begin{array}{lcl} +3 & = & +3\cdot00000 \\ +\frac{4}{27} & = & +0\cdot14815 \\ -\frac{16}{2187} & = & -0\cdot00731 \\ +\frac{320}{531441} & = & +0\cdot00060 \\ -\frac{2560}{43046721} & = & -0\cdot00006 \end{array}\right\}=\left\{\begin{array}{l} +3\cdot14875 \\ -0\cdot00737 \end{array}\right.$$

$$\sqrt[3]{31}\ \ =\ \ \underline{3\cdot14138}$$

To determine the degree of approximation obtained by taking any number of terms of a series like the preceding, in which the terms are alternately positive and negative, and the absolute value of each term is less than that of the preceding term,

Let the series be expressed by $a-b+c-d+e-f+g-h\ldots\ldots$ and let x denote the value of the series.

Taking the two consecutive sums,

$$a-b+c-d,$$
$$a-b+c-d+e,$$

the terms which come after $a-b+c-d$ are $+(e-f)+(g-h)+(k-l)+$, &c. and, since the series is decreasing, the differences $e-f$, $g-h$, $k-l$, &c. are positive numbers; whence, to obtain the complete value of x, it is necessary to add an absolute number to the sum $a-b+c-d$;

$$\text{therefore } a-b+c-d<x.$$

Again, the terms which follow $+e$ are $(-f+g)$, $(-l+m)$, &c.; and the differences $g-f$, $m-l$ are negative. Consequently, to obtain the true value of x it is necessary to add a negative quantity to the sum,

$$a-b+c-d+e,$$

or to diminish this sum by an absolute number;

$$\text{therefore } a-b+c-d+e>x.$$

Consequently the numerical value of x is comprehended between any two consecutive sums of the terms of the series.

Since the numerical value of the difference of the sums $a-b+c-d$ and $a-b+c-d+e$ is e, it follows that the error committed by taking a certain number of terms, $a-b+c-d$, for the value of x is numerically less than e, the term which follows the last term taken.

Hence the error of the value $\sqrt[3]{31}=3\cdot14138$, to form which five terms have been taken, is less than the sixth term $\frac{112640}{17433922005}=0\cdot0000064+$.

Other examples:

1st. $\sqrt[5]{39}$?..*Ans.* $\sqrt[5]{32+7}=2\cdot0807$.

2d. $\sqrt[3]{65}$?..*Ans.* $\sqrt[3]{64+1}=4\cdot02073$.

3d. $\sqrt[4]{260}$?..*Ans.* $\sqrt[4]{256+4}=4\cdot01553$.

4th. $\sqrt[7]{108}$?..*Ans.* $\sqrt[7]{128-20}=1\cdot95204$.

SECTION VIII.

OF ARITHMETICAL PROPORTION, RATIO, GEOMETRICAL PROPORTION, VARIATION, ARITHMETICAL AND GEOMETRICAL PROGRESSION, AND PILES OF BALLS.

Of Arithmetical Proportion.

177. If the difference between two quantities, a, b, is equal to the difference between two other quantities, c, d, the four quantities a, b, c, d, form an arithmetical proportion or equi-difference.

This relation of the quantities a, b, c, d, is expressed thus, $+a.b.c.d$; or by the equation $a-b=c-d$.

By transposing the terms b, d, this equation becomes

$$a+d=b+c.$$

Consequently the sum of the extreme terms of an arithmetical proportion is equal to the sum of the mean terms.

a. Again, from $a+d=b+c$ or $b+c=a+d$ is obtained, by transposing the terms a, c,

$$b-a=d-c.$$

Therefore, when four quantities are such that the sum of any two is equal to the sum of the other two, the first two are the extremes, and the second the means; or the first two are the means, and the second the extremes of an equi-difference or arithmetical proportion.

Also, from $b-a=d-c$, is obtained, by transposing b, c,

$$c-a=d-b.$$

Whence the extreme and mean terms may be interchanged without destroying the equi-difference.

b. When $b=c$, the proportion is termed *continued.*

In this case the equi-difference $a-b=c-d$ becomes

$$a-b=b-d. \quad \therefore a+d=b+b=2b.$$

Whence, in a continued arithmetical proportion, the sum of the extreme terms is equal to twice the mean term.

Of Ratios.

178. The term Ratio is employed to express the relation which exists between two quantities of the same kind with respect to magnitude. The first quantity is termed the antecedent, and the second the consequent, of the ratio. As in arithmetic (Part I. Art. 329) a ratio, that of a to b, for example, is expressed by the symbols $a : b$, or by writing the antecedent and consequent as the numerator and denominator of a fraction. The theory of ratios becomes thus reduced to the theory of fractions.

179. Since all fractions, whose terms are like multiples or like parts of the terms of a proposed fraction, are equal to that fraction, it follows that if one antecedent is the same multiple part or parts of its consequent that another antecedent is of its consequent, the ratios are equal. Consequently, if the terms of a ratio are both multiplied or both divided by the same quantity, the ratio is not altered.

Ratios are compared with each other by comparing the fractions by which they are denoted. The ratios of 3 to 5 and of 7 to 8 are expressed by the fractions $\frac{3}{5}$ and $\frac{7}{8}$. These fractions are compared, in respect of their magnitude, by reducing them to the common denominator 40, $\frac{3}{5}=\frac{24}{40}$ and $\frac{7}{8}=\frac{35}{40}$. The second being the greater, the ratio of 7 to 8 is greater than the ratio of 3 to 5.

According as the antecedent of a ratio is greater than the consequent, less than the consequent, or equal to the consequent, the ratio is termed a ratio of greater inequality, less inequality, or equality. The first of these ratios is expressed by an improper fraction, the second by a proper fraction, and the third by unity.

Since the value of an improper fraction is diminished, and the value of a proper fraction increased, by the addition of the same quantity to both its terms (Art. 39), it follows that a ratio of greater inequality is diminished, and a ratio of less inequality increased, by adding the same quantity to both its terms.

Conversely, a ratio of greater inequality is increased, and a ratio of less inequality diminished, by subtracting from the terms a quantity less than either of them.

A ratio of equality is not altered by adding the same quantity to both its terms, or by subtracting from both its terms a quantity which is less than either of them.

180. If the antecedents of any ratios are multiplied together for a new antecedent, and the consequents for a new consequent, the resulting ratio is said to be compounded of the ratios whose corresponding terms are multiplied together. It is also called the sum of the component ratios.

The sum of two or more ratios is consequently the product of the fractions which express these ratios. Thus the sum of the ratios $\frac{a}{b}$, $\frac{c}{d}$, (or the ratio which is compounded of these ratios,) is $\frac{ac}{bd}$.

The sum of the ratios $\frac{a}{b}$, $\frac{a}{b}$ is $\frac{a^2}{b^2}$. The ratio $\frac{a^2}{b^2}$ is called the double or duplicate of the ratio $\frac{a}{b}$. Hence the sum of two equal ratios, or the duplicate of any ratio, is the ratio of the square of the antecedent to the square of the consequent.

In like manner the triple or triplicate of any ratio is the ratio of the cube of its antecedent to the cube of its consequent; for the product $\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}$ is $\frac{a^3}{b^3}$; and generally the sum of m times any ratio $\frac{a}{b}$ is the ratio $\frac{a^m}{b^m}$, for the product $\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}$ to m factors is $\frac{a^m}{b^m}$.

The ratios $\frac{a^2}{1}$, $\frac{a^3}{1}$, ... $\frac{a^n}{1}$ are twice, three times, ... n times the ratio $\frac{a}{1}$ or $a:1$. In these ratios the exponents 2, 3, ... n show what multiple or part of the ratios $\frac{a^2}{1}$, $\frac{a^3}{1}$, ... $\frac{a^n}{1}$ the ratio $\frac{a}{1}$ is; on this account the exponents 1, 2, 3, .. n are called measures of the ratios $\frac{a^1}{1}$, $\frac{a^2}{1}$, $\frac{a^3}{1}$ ... $\frac{a^n}{1}$.

If the consequent of each ratio is the antecedent of the following ratio, the sum of any number of such ratios, or the ratio which results from their composition, is that of the first antecedent to the last consequent. Let $\frac{a}{b}$, $\frac{b}{c}$, $\frac{c}{d}$, $\frac{d}{e}$ be the ratios; then $\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}\times\frac{d}{e}=\frac{abcd}{bcde}=\frac{a}{e}$.

A ratio of greater inequality, compounded with another ratio, increases that other ratio; and a ratio of less inequality diminishes it. For a ratio of greater inequality being expressed by an improper fraction, and a ratio of less inequality by a proper fraction, it follows (Part I. Art. 209) that in the first instance the product is greater, and in the second less, than the ratio or fraction which forms the multiplicand.

OF GEOMETRICAL PROPORTION.

181. Proportion consists in the equality of two ratios.

A proportion is consequently composed of four terms, of which the first and third are the antecedents, and the second and fourth the consequents, of the two ratios which are required to be equal to each other. If $\frac{a}{b}$ and $\frac{c}{d}$ denote the ratios which constitute a proportion, the proportion is written (Part I. Art. 330)

$$a : b :: c : d.$$

But since the fractions which express these ratios must be equal, the proportion can be also expressed by the equation

$$\frac{a}{b}=\frac{c}{d}.$$

The equality of the ratios, and by consequence the proportionality of the quantities a, b, c, d, depends on the equality of the fractions $\frac{a}{b}$, $\frac{c}{d}$; therefore when the equality of the fractions can be established, the quantities must compose a proportion.

A proportion of this kind is termed a geometrical proportion or equi-quotient, to distinguish it from the arithmetical proportion or equi-difference.

182. If the equal fractions $\frac{a}{b}$, $\frac{c}{d}$ are reduced to the common denominator bd, $\frac{ad}{bd}=\frac{bc}{bd}$. Suppressing the common denominator, $ad=bc$.

Whence, if four quantities, a, b, c, d, compose a proportion, the product of the extreme terms, a, d, is equal to the product of the mean terms, b, c.

a. Again, if ad, bc are two equal products, and each is divided by bd, $\frac{ad}{bd}=\frac{bc}{bd}$, or $\frac{a}{b}=\frac{c}{d}$, or $\frac{c}{d}=\frac{a}{b}$;

that is, $a : b :: c : d$,
or $c : d :: a : b$.

Wherefore, if the product of two quantities is equal to the product of two others, the four quantities are convertible into a proportion of which the multiplicand and multiplier of either product form the extreme terms, and the multiplicand and multiplier of the other product the mean terms.

b. If the consequent of the one ratio is the antecedent of the other ratio, the product of the extremes is equal to the square of the mean; for if $\frac{a}{b}=\frac{b}{c}$, $a : b :: b : c$ and $ac=b^2$.

c. If $\frac{a}{b}=\frac{c}{d}$, and 1 is divided by each of these equal fractions, the quotients are equal. Now $1 \div \frac{a}{b}=\frac{b}{a}$ and $1 \div \frac{c}{d}=\frac{d}{c}$, therefore $\frac{b}{a}=\frac{d}{c}$, or $b : a :: d : c$.

Consequently, if $a : b :: c : d$, $b : a :: d : c$; that is, if the first of four quantities is to the second as the third is to the fourth, then also the second is to the first as the fourth is to the third; or the quantities are proportional when taken inversely.

d. If $\frac{a}{b}=\frac{c}{d}$, $ad=bc$ and $\frac{ad}{dc}=\frac{bc}{dc}$ or $\frac{a}{c}=\frac{b}{d}$; but if $\frac{a}{c}=\frac{b}{d}$, $a:c::b:d$.

Whence, if $a:b::c:d$, then also $a:c::b:d$; which is expressed by saying, that if four quantities are proportional, they are also proportional when taken alternately, or that the first is to the third as the second is to the fourth.

e. Since $\frac{m}{m}=\frac{n}{n}=1$, and $\frac{m}{n}=\frac{m}{n}$, it follows that, if

$$\frac{a}{b}=\frac{c}{d},$$

then also $\frac{a}{b}=\frac{c}{d}\times\frac{m}{m}=\frac{mc}{md}$, or $a:b::mc:md$,

$\frac{a}{b}\times\frac{m}{m}=\frac{c}{d}\times\frac{m}{m}$, or $ma:mb::mc:md$,

$\frac{a}{b}\times\frac{m}{n}=\frac{c}{d}\times\frac{m}{n}$, or $ma:nb::mc:nd$.

Whence, if four quantities are proportional, and like multiples or parts are taken of corresponding terms or of all the terms, the results are proportional. (In this and the following expression m, n may denote whole numbers or fractions.)

f. If the ratios $\frac{a}{b}$, $\frac{c}{d}$ are equal, and both are raised to the power m, $\left(\frac{a}{b}\right)^m=\left(\frac{c}{d}\right)^m$ or $\frac{a^m}{b^m}=\frac{c^m}{d^m}$;

therefore $a^m:b^m::c^m:d^m$.

Whence, if four quantities are proportional, the same powers or roots of these quantities are proportional.

g. If $\frac{a}{b}=\frac{c}{d}$, or $a:b::c:d$,

$\frac{e}{f}=\frac{g}{h}$, or $e:f::g:h$,

$\frac{k}{l}=\frac{m}{n}$, or $k:l::m:n$,

$\frac{a}{b}\times\frac{e}{f}\times\frac{k}{l}=\frac{c}{d}\times\frac{g}{h}\times\frac{m}{n}$, or $\frac{aek}{bfl}=\frac{cgm}{dhn}$,

or $aek:bfl::cgm:dhn$.

Whence, if any number of proportions is taken, and the corresponding terms are multiplied together, the products are proportional.

h. If $\frac{a}{b}=\frac{c}{d}$, then also $\frac{b}{a}=\frac{d}{c}$.

Let 1 be added to and subtracted from each member of these equalities; the sums are equal, and also the differences.

$\therefore\ \frac{a}{b}\pm1=\frac{c}{d}\pm1$ or $\frac{a\pm b}{b}=\frac{c\pm d}{d}$, $\frac{b}{a}\pm1=\frac{d}{c}\pm1$ or $\frac{b\pm a}{a}=\frac{d\pm c}{c}$.

Since $\frac{a\pm b}{b}=\frac{c\pm d}{d}$, $\frac{a\pm b}{c\pm d}=\frac{b}{d}$. Taking the upper signs together and the lower signs together, $\frac{a+b}{c+d}=\frac{b}{d}$ and $\frac{a-b}{c-d}=\frac{b}{d}$.

$\therefore\ \frac{a+b}{c+d}=\frac{a-b}{c-d}$ and $a+b:c+d::a-b:c-d$, or $a+b:a-b::c+d:c-d$.

Again, since $\frac{b\pm a}{a}=\frac{d\pm c}{c}$, $\frac{b\pm a}{d\pm c}$ or $\frac{a\pm b}{c\pm d}=\frac{a}{c}$;

but $\frac{a\pm b}{c\pm d}=\frac{b}{d}$; consequently $a\pm b : a$ or $b :: c\pm d : c$ or d.

Whence, if $a:b::c:d$, the sum of the two terms of the first ratio is to their difference as the sum of the two terms of the second ratio is to their difference, also the sum or the difference of the two terms of the first ratio is to the antecedent or to the consequent of the first ratio as the sum or the difference of the two terms of the second ratio is to the antecedent or to the consequent of the second ratio.

k. Let $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\frac{g}{h}$, &c., or $a:b::c:d::e:f::g:h$, &c.;

Also let $\frac{a}{b}=q$, and therefore $\frac{c}{d}=q, \frac{e}{f}=q, \frac{g}{h}=q \ldots$

$$\therefore a=bq,\ c=dq,\ e=fg,\ g=hq \ldots.$$

$$\text{and } a+c+e+g \ldots. =(b+d+f+h \ldots)q.$$

$$\therefore \frac{a+c+e+g \ldots}{b+d+f+h \ldots}=q=\frac{a}{b}=\frac{a+c}{b+d}=\frac{a+c+e}{b+d+f}, \text{\&c.}$$

Whence in every series of equal ratios the sum of all the antecedents is to the sum of all the consequents as one antecedent, a, is to its consequent b, or as a sum of antecedents, $a+c$, $a+c+e$, &c., is to a like sum of corresponding consequents, $b+d$, $b+d+f$, &c.

l. If three terms of a proportion are given to find the fourth term; that is, if the antecedent a and consequent b of one ratio $\frac{a}{b}$, and the antecedent c of a second ratio $\frac{c}{x}$, are given to determine the consequent x of the second ratio, since $\frac{a}{b}=\frac{c}{x}$, the required value is found to be $x=\frac{bc}{a}$.

Let it be supposed that the fraction which expresses the ratio of a to b is in the lowest terms, or that a, b are prime to each other; then,

1st. If c is a multiple of a, the value of x is expressed by a whole number, which (since $\frac{c}{a}$ is a whole number) is a multiple of b.

2d. If c is prime to a, since b is by hypothesis prime to a, the product bc is prime to a (Part I. Art. 114), and there is no whole number which exactly expresses the value of x; but if the divison of bc by a is made decimally, the value of x can be determined in such a manner that it shall differ from the exact value of the fourth proportional to a, b, c, by less than any assigned quantity, how small soever; consequently three numbers, a, b, c, being given, it is always possible to find a fourth number, x, such that the fraction $\frac{c}{x}$ shall differ from the fraction $\frac{a}{b}$ by less than any assigned quantity, however small. But two quantities which differ by less than any quantity which can be assigned are considered equal; whence $\frac{a}{b}=\frac{c}{x}$, or $a:b::c:x\left(=\frac{cb}{a}\right)$, and consequently the properties which have been proved to belong to proportionals belong also to quantities such as the preceding, a, b, c and x.

OF VARIATION.

182. Let a denote a constant quantity, and x a variable, or quantity which is increased or diminished.

The product of the quantities a, x being denoted by X, it is evident that if x is increased or diminished X is increased or diminished in the same proportion; so that if x, x' are two values of x, and X, X′ the corresponding values of ax, ax', $ax:ax'::\mathrm{X}:\mathrm{X}'$,

or $x:x'::\mathrm{X}:\mathrm{X}'$ (Art. 181. e).

Under these circumstances X is said to vary directly as x.

The symbol of variation is $\propto$; and the expression, X varies directly as x, is indicated by the combination of symbols, $X \propto x$.

a. If the product of x and y is constant, and x, y are both variable, since $xy=x'y'=C$,

$$x:x'::y':y \text{ or } x:x'::\frac{1}{y}:\frac{1}{y'} \text{ (Art. 181 e).}$$

In this case, since x varies as the reciprocal of y, x is said to vary inversely as y, and the symbolical expression is $x \propto \frac{1}{y}$.

b. If $xy=X$, and $x'y'=X'$,

$$X:X'::xy:x'y'.$$

The variation of X depends in this case on the variation of the two quantities x, y. Whence X is said to vary as x, y jointly, which is expressed thus:

$$X \propto xy.$$

c. When $x=y$, $X \propto xx$, that is, $X \propto x^2$.

d. If $xy=X$, and $x'y'=X'$,

$$x=\frac{X}{y}, \text{ and } x'=\frac{X'}{y'};$$

$$\therefore x:x'::\frac{X}{y}:\frac{X'}{y'}.$$

In this case x varies as X directly, and as y inversely; the symbolical expression of which is $x \propto \frac{X}{y}$.

e. Let $x \propto y$, that is, let $x:x'::y:y'$, or $\frac{x}{x'}=\frac{y}{y'}$,

and $y \propto z$, that is, let $y:y'::z:z'$, or $\frac{y}{y'}=\frac{z}{z'}$;

$$\therefore \frac{x}{x'}=\frac{y}{y'}=\frac{z}{z'}, \quad \therefore \frac{x}{x'}=\frac{z}{z'},$$

$$\text{or } x:x'::z:z'.$$

$$\therefore x \propto z.$$

Whence, if one quantity vary as a second, and the second as a third, the first quantity varies as the third.

In like manner, if $x \propto y$ and $y \propto \frac{1}{z}$, $x \propto \frac{1}{z}$.

f. Let $x \propto y$ and $z \propto y$, $x \propto z$ or $x:x'::z:z'$.

$$\therefore x:z::x':z' \text{ and } x\pm z:z::x'\pm z':z',$$

$$\text{or } x\pm z:x'\pm z'::z:z'.$$

$$\text{But } z:z'::y:y'. \quad \therefore x\pm z:x'\pm z'::y:y'.$$

$$\text{That is, } y \propto x\pm z.$$

$$\text{Again, since } x \propto y, \quad x:x'::y:y',$$

$$\text{and since } z \propto y, \quad z:z'::y:y';$$

$$\therefore xz:x'z'::y^2:y'^2.$$

$$\text{And } \sqrt{xz}:\sqrt{x'z'}::y:y',$$

$$\text{or } y \propto \sqrt{xz}.$$

Whence, if two quantities vary respectively as a third, their sum or difference, or the square root of their product, varies as the third quantity.

g. If $x \propto y$ and m is a constant quantity, integer or fractional, since $x \propto y$, $x:x'::y:y'$;

$$\therefore x:x'::my:my' \text{ (Art. 181. e).}$$

$$\text{That is, } x \propto my.$$

Whence, if one quantity vary as another, it will also vary as any multiple or part of that other quantity.

h. When $x \propto y$, then also $x \propto my$, or $x : x' :: my : my'$;

$$\therefore\ x : my :: x' : my'.$$

Consequently if m is so assumed that $x = my$, then in all cases $x' = my'$.

Whence, if x vary as y, x is equal to y multiplied by some constant quantity.

In general, if X, Y are two corresponding values of x, y,

$$\mathrm{X} = m\mathrm{Y}.$$

Whence, since from $\mathrm{X} = m\mathrm{Y}$ is obtained $m = \frac{\mathrm{X}}{\mathrm{Y}}$, it follows that when two corresponding values of x, y are known the constant m can be determined.

k. Let $x \propto y$, and let m denote any exponent, integer or fractional.

$$\text{Since } x \propto y,\ x : x' :: y : y';$$

$$\therefore\ x^m : x'^m :: y^m : y'^m;\ \text{that is, } x^m \propto y^m.$$

Whence, if one quantity vary as another, any power or root of the first quantity varies as the same power or root of the second quantity.

l. Let $x \propto y$, and let t be another quantity, either variable or constant, and of which t, t' are either different or equal values;

$$\text{Then since } x \propto y,\ x : x' :: y : y',$$

$$\text{and } t : t' :: t : t' \text{ (whether } t \text{ is variable or constant),}$$

$$xt : x't' :: yt : y't',\ \text{or } xt \propto yt,$$

$$\frac{x}{t} : \frac{x'}{t'} :: \frac{y}{t} : \frac{y'}{t'},\ \text{or } \frac{x}{t} \propto \frac{y}{t}.$$

Whence, if one quantity vary as another, and if each of them be multiplied or divided by any quantity, variable or constant, the products or quotients will vary as each other.

$$\text{Consequently if } x \propto y,\ \frac{x}{y} \propto \frac{y}{y},\ \text{or } \frac{x}{y} \propto 1.$$

$$\text{Whence, if } x \propto y,\ \frac{x}{y} \text{ is constant.}$$

m. Let $xy \propto \mathrm{X}$, that is, let $xy : x'y' :: \mathrm{X} : \mathrm{X}'$;

$$\therefore\ \text{by alternation } xy : \mathrm{X} :: x'y' : \mathrm{X}';$$

$$\therefore\ y : \frac{\mathrm{X}}{x} :: y' : \frac{\mathrm{X}'}{x'}.$$

$$\text{Consequently } y \propto \frac{\mathrm{X}}{x};\ \text{and, similarly, } x \propto \frac{\mathrm{X}}{y}.$$

Whence, if two quantities jointly vary as a third quantity, each of the two quantities varies as the third directly and the other of the two inversely.

n. If $\mathrm{X} = \mathrm{X}' = \text{constant}$. $xy : 1 :: x'y' : 1$.

$$\therefore\ x : \frac{1}{y} :: x' : \frac{1}{y'},\ \text{or } x \propto \frac{1}{y}.$$

Whence, if the product of two variable quantities is constant, these quantities vary inversely as each other.

o. Let a be a constant and x, y, z variables, and let $a : x :: y : z$, $a : x' :: y' : z'$, &c.

$$\therefore\ az = xy,\ az' = x'y',\ \text{\&c.}$$

$$\therefore\ az : az' :: xy : x'y',\ \text{or } z : z' :: xy : x'y'.$$

$$\therefore\ z \propto xy.$$

Consequently if four quantities are always proportional, and one or two of them are constant, the others being variable, it can be found how the latter vary.

p. Let x, y, z be three quantities, of which $x \propto y$ when z is constant and $x \propto z$ when y is constant; it is required to determine the variation of x when y, z are both variable.

Suppose, first, that x is made to vary as y, and that when y becomes y', x becomes x'.

Next, that x' (varied from x by the variation of y) is made further to vary as z, and that when z becomes z', x' becomes x''.

$$\begin{gathered}\text{Then since } x : x' :: y : y', \\ \text{and } x' : x'' :: z : z'; \\ \therefore xx' : x'x'' :: yz : y'z', \\ \text{or } x : x'' :: yz : y'z'; \\ \therefore x \propto yz.\end{gathered}$$

Therefore if x vary as y when z is constant, and as z when y is constant, when yz are both variable x varies as the product yz.

Similarly, it can be proved that if t varies as v, x, y, z singly, the others being constant, when v, x, y, z are all variable t varies as the product $vxyz$.

OF ARITHMETICAL PROGRESSION.

183. If the continued arithmetical proportion $a-b=b-c$ is prolonged indefinitely, the series of equidifferences composes an arithmetical progression, or progression by difference.

An arithmetical progression is, consequently, a series composed of an indefinite number of terms, such that the difference of any two consecutive terms is a constant quantity. The constant difference, when the series is increasing, is additive; and when the series is decreasing, subtractive. This difference is termed the ratio of the progression.

The notation $a-b=b-c=c-d=d-e$ . . . . is replaced by

$$\div\, a \,.\, b \,.\, c \,.\, d \,.\, e \,.\,.\,.\,.\,.$$

which has the same meaning.

$$\div\, 3 \,.\, 7 \,.\, 11 \,.\, 15 \,.\, 19 \,.\, 23 \,.\, 27 \,.\, 31, \text{ \&c.}$$

$$\div\, 45 \,.\, 41 \,.\, 37 \,.\, 33 \,.\, 29 \,.\, 25 \,.\, 21 \,.\, 17, \text{ \&c.}$$

are examples of arithmetical progressions. In the first the ratio is 4, and in the second -4. The first is an increasing, the second a decreasing series.

184. Let $\div\, a \,.\, b \,.\, c \,.\, d \,.\, e$ . . . . be an arithmetical progression whose ratio is δ. Then, since by the definition each term is equal to the preceding term plus the ratio,

$$b=a+\delta,\ c=b+\delta=a+2\delta,\ d=c+\delta=a+3\delta, \text{ \&c.}$$

Therefore the corresponding terms of the series

$$\div\, a \,.\, b \,.\, c \,.\, d \,.\, e \,.\,.\,.\,.$$

$$\div\, a \,.\, a+\delta \,.\, a+2\delta \,.\, a+3\delta \,.\, a+4\delta \text{ . . . are equal.}$$

$$\begin{aligned}\text{Since the second term} &= a+1\times\delta \ .\ (1=2-1), \\ \text{the third term} &= a+2\times\delta \ .\ (2=3-1), \\ \text{the fourth term} &= a+3\times\delta \ .\ (3=4-1);\end{aligned}$$

it follows that the nth term (which may be denoted by l, and which has $n-1$ terms before it,) is expressed by $l=a+(n-1)\delta$; and generally, that any term whatever is equal to the first term plus as many times the ratio as there are terms before it.

Since the expression $l=a+(n-1)\delta$ is changed into the first term by making $n=1$, into the second by making $n=2$, into the third by making $n=3$, &c., it is called the general term.

By substituting $-\delta$ for δ in l, the general term of a decreasing series is obtained; viz. $l=a-(n-1)\delta$.

The formula $l=a+(n-1)\delta$, expressing a relation between a, δ, n, l, is sufficient for the resolution of questions in which any three of these quantities are given to find the fourth.

For example, if a, l, n are given to find δ.

Since $l=a+(n-1)\delta$,

$$\delta=\frac{l-a}{n-1} \quad - \quad - \quad - \quad - \ 1.$$

Whence, to find the ratio, it is necessary to subtract the first term from the last, and to divide the remainder by the number of mean terms augmented by 1.

As an illustration of the use of this formula, let it be required to insert m arithmetical means between the two given numbers a, l; in other words, to find m quantities comprehended between a, l, and, forming with a, l, an arithmetical progression beginning with a and ending with l.

In this question, the given quantities are, the first term a, the last term l; the number of terms which, including a and l, is $m+2$, and the quantity which is required is the ratio δ; for when δ is known, the other terms of the progression can be formed by successive additions.

In formula 1, n denotes the number of terms, making $n=m+2$, and by consequence $n-1=m+1$.

$$\delta=\frac{l-a}{m+1}.$$

If $a=2$, $l=17$, $m=4$, $\delta=\frac{17-2}{4+1}=\frac{15}{5}=3$, and the four arithmetical means between 2 and 17 are,

$$\begin{array}{lllll} & 2+3, & 2+2\times3, & 2+3\times3, & 2+4\times3, \\ \text{or} & 5, & 8, & 11, & 14. \end{array}$$

Whence the $m+2$ terms of the series beginning with 2 and ending with 17 are $\div 2 \,.\, 5 \,.\, 8 \,.\, 11 \,.\, 14 \,.\, 17$.

This particular case is a solution of the problem; to insert four arithmetical means between 2 and 17.

It is to be inferred, from the preceding example and conclusion, that, an arithmetical progression being given, if the same number of arithmetical means is inserted between each term and the following, the new series will still be an arithmetical progression. For partial progressions will then be formed, of which all will have the same ratio; and as the last term of each successive partial progression is also the first term of the following progression, the whole of these partial progressions, taken in order, must form an arithmetical progression.

185. Let it be next required to find the sum of the terms of an arithmetical progression $\div a \,.\, b \,.\, c \,.\, d \,.\,.\,.\,.\, i \,.\, k \,.\, l$, the number of terms being n, and the ratio δ.

Denoting the required sum by S, and arranging the terms consecutively, first, from a to l, second from l to a,

$$S=a+b+c+d+\;.\;.\;.\;.\;.+i+k+l,$$
$$S=l+k+i+h+\;.\;.\;.\;.\;.+c+b+a.$$

Adding, $2\,S=(a+l)+(b+k)+(c+i)+(d+h)+$ &c.

Now, $a+\delta=b$, $l-\delta=k$ $\therefore (a+\delta)+(l-\delta)=b+k$ or $a+l=b+k$,
$b+\delta=c$, $k-\delta=i$ $\therefore (b+\delta)+(k-\delta)=c+i$ or $b+k=c+i$,
$c+\delta=d$, $i-\delta=h$ $\therefore (c+\delta)+(i-\delta)=d+h$ or $c+i=d+h$,

. . . .

Wherefore $a+l=b+k=c+i=d+h=\;.\;.\;.\;.\;.\;.$; that is, the sum of any two terms, which are equally distant from the extremes, is equal to the sum of the extremes. Consequently, the sum of all the terms, in both series, is equal to the sum of the extremes as often repeated as there are terms in one series. The sum of the extremes being $a+l$, and the number of terms n, the sum of all the terms in the two equal series is $(a+l)n$. The sum of all the terms of the progression $\div a \,.\, b \,.\, c \,.\, d \,.\,.\,.\,.\,.\, i \,.\, k \,.\, l$ is, therefore, $\frac{(a+l)n}{2}$.

$$\text{Therefore } S=\frac{(a+l)n}{2} \quad - \quad - \quad - \quad 2.$$

In words, the sum of the terms of an arithmetical progression is equal to half the sum of the extreme terms multiplied by the number of terms.

186. Five quantities, a, δ, l, n, S, enter into the formulæ. Whence, as two equations are sufficient for the discovery of two unknown quantities, it is possible to find any two of the quantities a, δ, l, n, S, when the three

others are given. Formulæ 1, 2 give, therefore, the solutions of as many distinct problems as there are combinations of five quantities, taken two by two. The number of such combinations is $\frac{5\times4}{2}=10$.

In order that these problems may be possible, it is necessary that the value of n be not only real, but also integer and positive, for the number of terms can neither be fractional nor negative.

The ten problems, with their solutions, are,

1st. Given a, δ, n, to find l, S? *Ans.* $l=a+(n-1)\delta$, $S=\frac{1}{2}n[2a+(n-1)\delta]$.

2d. Given l, δ, n to find a, S? *Ans.* $a=l-(n-1)\delta$, $S=\frac{1}{2}n[2l-(n-1)\delta]$.

3d. Given a, n, l to find δ, S?.................*Ans.* $\delta=\frac{l-a}{n-1}$, $S=\frac{1}{2}n(a+l)$.

4th. Given δ, n, S to find a, l? *Ans.* $a=\frac{2S-n(n-1)\delta}{2n}$, $l=\frac{2S+n(n-1)\delta}{2n}$.

5th. Given a, n, S to find δ, l?...............*Ans.* $\delta=\frac{2(nl-S)}{n(n-1)}$, $l=\frac{2S}{n}-a$.

6th. Given l, n, S to find a, δ?................*Ans.* $a=\frac{2S}{n}-l$, $\delta=\frac{2(nl-S)}{n(n-1)}$.

7th. Given a, δ, l to find n, S?....*Ans.* $n=\frac{l-a}{\delta}+1$, $S=\frac{(l+a)(l-a+\delta)}{2\delta}$.

8th. Given a, l, S to find n, δ?.............*Ans.* $\delta=\frac{(l+a)(l-a)}{2S-(l+a)}$, $n=\frac{2S}{a+l}$.

9th. Given a, δ, S to find l, n?

Ans. $l=a+(n-1)\delta$, $n=\frac{\delta-2a\pm\sqrt{(\delta-2a)^2+8\delta S}}{2\delta}$.

10th. Given δ, l, S to find a, n?

Ans. $a=l-(n-1)\delta$, $n=\frac{\delta+2l\pm\sqrt{(\delta+2l)^2-8\delta S}}{2\delta}$.

Examples:

1st. Find the 13th term of the series 3, 9, 15, &c.?.................*Ans.* 75.
2d. Find the 8th term of the series −5, −3, −1, &c.?............*Ans.* 9.
3d. Find the 7th term of the series $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$, &c.?....................*Ans.* $-\frac{1}{2}$.
4th. Find the sum of 3+9+15+, &c. to 11 terms?.............*Ans.* 363.
5th. Find the sum of −5−3−1−, &c. to 8 terms?...............*Ans.* 16.
6th. Find the sum of $\frac{1}{2}-\frac{2}{3}-\frac{11}{6}-$, &c. to 13 terms?.........*Ans.* $-84\frac{1}{2}$.
7th. Insert 9 arithmetical means between 3 and 9?
Ans. $3\frac{3}{5}$, $4\frac{1}{5}$, $4\frac{4}{5}$, $5\frac{2}{5}$, 6, $6\frac{3}{5}$, $7\frac{1}{5}$, $7\frac{4}{5}$, $8\frac{2}{5}$.
8th. Insert 8 arithmetical means between −3 and $-\frac{3}{4}$?
Ans. $-2\frac{3}{4}$, $-2\frac{1}{2}$, $-2\frac{1}{4}$, −2, $-1\frac{3}{4}$, $-1\frac{1}{2}$, $-1\frac{1}{4}$, −1.
9th. Insert 7 arithmetical means between −13 and 3?
Ans. −11, −9, −7, −5, −3, −1, +1.

OF GEOMETRICAL PROGRESSION.

187. If the continued geometrical proportion $a:b::b:c$ is extended indefinitely, the series of equal ratios forms a geometrical progression, or progression by quotient.

In writing the geometrical progression the terms which are consequent and antecedent, in succession, are not repeated as in the continued geometrical proportion, the notation of which $a:b::b:c::c:d::d:e$, &c. is replaced by the following $\div\!\!:a:b:c:d:e$, &c. Both forms of expression have the same meaning.

The geometrical progression is hence composed of a series of terms such that, if each term is divided by the term which precedes it, the quotient is a constant quantity; this quotient is the ratio of the progression. Denoting the ratio by q, when $q>1$ the series is increasing, and when $q<1$ the series is decreasing.

$$\div\!\div\, 2 : 6 : 18 : 54 : 162 : 486 :, \text{\&c.}$$
$$\div\!\div\, 180 : 60 : 20 : \tfrac{20}{3} : \tfrac{20}{9} : \tfrac{20}{27} :, \text{\&c.}$$

are examples of geometrical progressions in numbers. The ratio of the first, which is an increasing series, is 3, and the ratio of the second, which is a decreasing series, is $\frac{1}{3}$.

188. Let $\div\!\div\, a : b : c : d : \ldots\ldots : k : l$, be the general expression of a geometrical progression whose ratio is q.

$$\text{Since } b=aq \text{ and } c=bq \therefore c=aq^2.$$
$$d=cq=bq^2=aq^3 : e=dq=aq^4 \ldots\ldots$$

The exponent of q in the first term a, which has 0 term before it, is $0=1-1$.

The exponent of q in the second term aq or b, which has 1 term before it, is $1=2-1$.

The exponent of q in the third term aq^2 or c, which has 2 terms before it, is $2=3-1$.

The exponent of q in the fourth term aq^3 or d, which has 3 terms before it, is $3=4-1$.

. . . .

The exponent of q in the nth term aq^{n-1} or l, which has $n-1$ terms before it, is $n-1$.

Whence the exponent of q in any term is indicated by the number of preceding terms, and any term whatever is equal to the first term multiplied by the ratio raised to a power marked by the number of the terms which precede the term proposed.

The formula $l=aq^{n-1}$ - - - - 1

is called the general term of a geometrical progression.

If a and l are the extremes, and n the number of terms of a progression, $\div\!\div\, a : b : c : d : e : f : \ldots k : l$, whose ratio is q.

Since $b=aq$, $c=aq^2 \ldots k=aq^{n-2}$, $l=aq^{n-1}$, the same progression may be expressed also thus, $\div\!\div\, a : aq : aq^2 : aq^3 \ldots\ldots aq^{n-2} : aq^{n-1}$.

If it is required to insert m geometrical means between a and l, since the progression is to consist of two extremes, a, l, and m mean terms, the whole number of terms is $m+2$. Substituting $m+2$ for n, in formula 1,

$$l=aq^{m+1} \therefore q^{m+1}=\frac{l}{a},$$
$$\text{and } q=\sqrt[m+1]{\frac{l}{a}} \quad - \quad - \quad - \quad 2.$$

Hence the determination of the constant ratio by means of which the second and following terms are formed from the first term and from each other (or from the general term, formula 1) depends on the extraction of the $m+1$th root; an operation which, in particular cases, is much facilitated by the use of logarithms.

From formula 2 it follows that if the same number of geometrical means is inserted between the 1st and 2d, the 2d and 3d, ... the $(n-1)$th and nth terms of a geometrical progression, the series composed of these partial progressions is a geometrical progression.

For if $\frac{l}{a}$ is replaced, in formula 2, successively by $\frac{\text{2d term}}{\text{1st term}}=q'$; $\frac{\text{3d term}}{\text{2d term}}=q' \ldots \frac{n\text{th term}}{(n-1)\text{th term}}=q'$; since q' is constant, $q=\sqrt[m+1]{q'}$, the ratio of all the partial progressions must be constant; and as that term which is the last of any one of the partial progressions is also the first term of the following progression, it is evident that the series composed of these partial progressions is a geometrical progression.

189. Let the sum of all the terms of the series $\div\!\div\, a : b : c : d : \ldots\ldots : k : l$, be denoted by S; that is, let $S=a+b+c+d+\ldots\ldots+k+l$.

Then, if all the terms are multiplied by q,

$$qS=qa+qb+qc+qd+\ldots\ldots\ldots+qk+ql.$$

But $qa=b$, $qb=c$, $qc=d$, $\ldots\ldots\, qk=l$; therefore, by substitution, $qS=b+c+d+\ldots\ldots\ldots+l+ql$.

From this equality subtract $S=a+b+c+d+\ldots\ldots\ldots+l$;

$$\therefore qS-S=ql-a, \text{ or } (q-1)S=ql-a,$$

$$\text{and } S=\frac{ql-a}{q-1} \quad - \quad - \quad 3.$$

Whence the sum of the terms of a geometrical progression is obtained by multiplying the last term by the ratio, subtracting the first term from the product, and dividing the remainder by the ratio, less 1.

190. If l is replaced by its value, aq^{n-1}, formula 3 becomes

$$S=\frac{a(q^n-1)}{q-1} \quad - \quad - \quad 4.$$

In formula 4, suppose 1st $q>1$, which is the case of increasing progressions.

In this case the quantity q^n is great in proportion as n is great. There is no limit even which it may not exceed if a sufficiently considerable value is given to n. For making $q=1+\alpha$, $q^n=(1+\alpha)^n=1+n\alpha+\frac{n(n-1)}{1\cdot 2}\alpha^2+$, &c. $\therefore$ $q^n>1+n\alpha$. Now, however small the quantity α, it is evident that, by taking n sufficiently great, $1+n\alpha$ can be made to exceed any assigned quantity however great; therefore, à fortiori, this must be true of q^n. Hence when $q>1$, by taking a sufficient number of terms of the progression a, aq, $aq^2\ldots\ldots$ S can be made to exceed any assigned quantity.

Suppose, 2d, $q<1$, which is the case of decreasing progressions.

If the signs of the numerator and denominator are changed, formula 4 may be written thus;

$$S=\frac{a}{1-q}-\frac{aq^n}{1-q}.$$

In this expression the quantity q^n is small in proportion as n is great; it can even be rendered less than any assigned quantity. To prove this, let $q=\frac{1}{\beta}$, β being greater than 1; then $q^n=\frac{1}{\beta^n}$; now if n is made greater and greater, β^n may be augmented to infinity, and by consequence $\frac{1}{\beta^n}$, or q^n, reduced to zero; by which means $\frac{aq^n}{1-q}$ is also reduced to zero. Hence it is evident that the sum S must increase as n is increased, and that it can never exceed the limit $S=\frac{a}{1-q}$, from which it may, however, be made to differ in defect by less than any assigned quantity.

Consequently making $n=\infty$, $\frac{aq^n}{1-q}$ is reduced to zero, and

$$S=\frac{a}{1-q} \quad - \quad - \quad 5.$$

That is to say, the sum of the terms of a decreasing geometrical progression, continued to infinity, is equal to the quotient of the first term divided by unity, less the constant ratio.

To obtain the sum of the terms of the series $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+$, &c. make $a=1$ and $q=\frac{1}{3}$; then,

$$S=\frac{1}{1-\frac{1}{3}}=\frac{1}{\frac{2}{3}}=\frac{3}{2}.$$

Suppose, 3d, $q=1$. In this case formula 4 gives $S=\frac{0}{0}$, an indeterminate result. But the indetermination is apparent only, for if q^n-1 is divided by $q-1$, the expression is reduced to $S=a\,(q^{n-1}+q^{n-2}+\ldots+q+1)$, which for the hypothesis $q=1$ gives $S=an$. That this must be the result is evident à priori, for the ratio being 1 each of the n terms of the progression is equal to the first term a.

The division of q^n-1 by $q-1$ reproduces, in a contrary order, that progression of which formula 4 represents the sum.

Writing the formula thus, $S=\frac{a(1-q^n)}{1-q}$, and effecting the division of $1-q^n$ by $1-q$, the terms of the progression are obtained in the original order.

In like manner formula 5, which expresses the sum of the terms of a decreasing progression prolonged to infinity, may be made to reproduce the progression $\div\!\!\div a : aq : aq^2 :$, &c.

Dividing a by $1-q$, as follows,

$$\begin{array}{l} 1-q\big)\,a\qquad\big(a+aq+aq^2+aq^3+, \&c. \\ \qquad\ \underline{a-aq} \\ \qquad\qquad aq \\ \qquad\qquad \underline{aq-aq^2} \\ \qquad\qquad\quad aq^2 \\ \qquad\qquad\quad \underline{aq^2-aq^3} \\ \qquad\qquad\qquad \underline{aq^3\ \ldots\ldots} \end{array}$$

it appears that according as the division is ended at the 1st, the 2d, the 3d.... remainder, it is necessary, in order to complete the corresponding quotients, to add to them respectively the fractions $\frac{aq}{1-q}, \frac{aq^2}{1-q}, \frac{aq^3}{1-q}\ldots$.

Now q being less than 1, these fractions have decreasing values; and since the division may be continued until the exponent of q in the remainder is greater than any assigned number, and the value of the numerator of the complementary fraction by consequence less than any assigned number, it follows that, supposing the terms of the quotient continued indefinitely, the complementary fraction or remainder ought to be regarded as zero, therefore the series of these terms continued to infinity is the exact value of the quotient.

This conclusion is legitimate only in the hypothesis $q<1$. If $q>1$, the complementary fractions or remainders, instead of approximating to zero, increase to $-\infty$.

191. In geometrical as in arithmetical progression all the questions which can be proposed are reducible to ten. The ten questions, with their solutions (which are deduced from the two equations $l=aq^{n-1}$, $S=\frac{ql-a}{q-1}$), are,

1st. Given a, q, n, to find l, S?......*Ans.* $l=aq^{n-1}$; $S=\frac{ql-a}{q-1}=\frac{a(q^{n-1})}{q-1}$.

2d. Given l, q, n, to find a, S?............*Ans.* $a=\frac{l}{q^{n-1}}$; $S=\frac{l(q^n-1)}{q^{n-1}(q-1)}$.

3d. Given a, l, n, to find q, S?...*Ans.* $q=\sqrt[n-1]{\frac{l}{a}}$; $S=\frac{\sqrt[n-1]{l^n}-\sqrt[n-1]{a^n}}{\sqrt[n-1]{l}-\sqrt[n-1]{a}}$.

4th. Given n, q, S, to find a, l?........*Ans.* $a=\frac{S(q-1)}{q^n-1}$; $l=\frac{Sq^{n-1}(q-{}^1)}{q^n-1}$.

5th. Given a, n, S, to find l, q?

Ans. $q^{n-1}+q^{n-2}+\ \ldots\ +1=\frac{S}{a}$; $l=aq^{n-1}$.

6th. Given l, n, S, to find a, q?

Ans. $\left(\frac{1}{q}\right)^{n-1}+\left(\frac{1}{q}\right)^{n\ 2}\ldots\ +1=\frac{S}{l}$; $a=l\left(\frac{1}{q}\right)^{n-1}$.

7th. Given a, l, q, to find S, n?...*Ans.* $S=\frac{ql-a}{q-1}$; $n=1+\frac{\log. l-\log. a}{\log. q}$.

8th. Given a, l, S, to find q, n?....*Ans.* $q=\frac{S-a}{S-l}$; $n=1+\frac{\log. l-\log. a}{\log. q}$.

9th. Given a, q, S, to find l, n?

Ans. $l=\frac{a+S(q-1)}{q}$; $n=1+\frac{\log. l-\log. a}{\log. q}$.

10th. Given l, q, S, to find a, n?

Ans. $a=lq-S(q-1)$; $n=1+\frac{\log. l-\log. a}{\log. q}$.

Remarks. In case 5, l is eliminated between the equations $l=aq^{n-1}$ and $S=\frac{ql-a}{q-1}$, the result is an equation in q of the degree $n-1$, which, unless in the particular case $n=3$, cannot be resolved by any of the preceding rules. The value of the ratio q being determined, l is found from the equation $l=aq^{n-1}$

In case 6, $\frac{1}{q}$ instead of q is made the unknown quantity. The preceding remark respecting case 5 applies to case 6 also.

In cases 7, 8, 9, 10, the equation $S=\frac{ql-a}{q-1}$ gives each of the quantities a, l, q, S, in terms of the three others. The only difficulty, therefore, in these cases lies in the determination of the unknown quantity n from the equation $l=aq^{n-1}$, in which n is an exponent. The solution given depends on certain properties of logarithms, for which see Section X.

Examples:

1st. Find the 7th term of the series 1, 4, 16, &c.?............*Ans.* 4096.

2d. Find the 8th term of the series 81, -27, 9, &c.?,........*Ans.* $-\frac{1}{27}$.

3d. Find the 5th term of the series 4, -12, 36, &c.?.........*Ans.* 324.

4th. Find the 6th term of the series 3, $\frac{1}{2}$, $\frac{1}{12}$, &c.?............*Ans.* $\frac{1}{2592}$.

5th. Find the sum of $3+6+12+$, &c. to 6 terms?.............*Ans.* 189.

6th. Find the sum of $1-4+16-$, &c. to 7 terms?............*Ans.* 3277.

7th. Find the sum of $\frac{1}{3}+\frac{1}{6}+\frac{1}{12}+$, &c. to 8 terms?.............*Ans.* $\frac{85}{128}$.

8th. Find the sum of $3-\frac{1}{2}+\frac{1}{12}-$, &c. to 5 terms?............*Ans.* $2\frac{247}{432}$.

9th. Find the sum of the series $1+\frac{1}{2}+\frac{1}{4}+$, &c., continued to infinity?

Ans. 2.

10th. Find the sum of the infinite series $4+2+1+$, &c.?..........*Ans.* 8.

11th. Find the sum of the infinite series $1-\frac{1}{2}+\frac{1}{4}-$, &c.?.........*Ans.* $\frac{2}{3}$.

12th. Find the sum of the infinite series $-3\frac{1}{5}+1\frac{3}{5}-\frac{4}{5}+$, &c.?

Ans. $-2\frac{2}{15}$.

13th. Insert 3 geometrical means between 2 and 32?.*Ans.* ±4, 8, ±16.

14th. Insert 4 geometrical means between $-\frac{1}{10}$ and $3\frac{1}{5}$?

Ans. $\frac{1}{5}$, $-\frac{2}{5}$, $\frac{4}{5}$, $-1\frac{3}{5}$.

15th. Insert 3 geometrical means between $\frac{1}{2}$ and 128?*Ans.* ±2, 8, ±32.

16th. What is the sum of the infinite geometrical progression

$a-b+\frac{b^2}{a}-\frac{b^3}{a^2}+\frac{b^4}{a^3}-$, &c.?..................................*Ans.* $\frac{a^2}{a+b}$.

OF THE NUMBER OF BALLS IN A PILE.

192. If all the terms of a geometrical progression

$$∺a : aq : aq^2 : aq^3 : \dots . aq^{n-1},$$

are raised to the same power, m, the result is the series,

$$a^m, a^mq^m, a^mq^{2m}, a^mq^{3m}, \dots . a^mq^{(n-1)m},$$

which is a geometrical progression, of which the first term is a^m, the ratio q^m, and the number of terms n.

Whence, if all the terms of a geometrical progression are raised to the same power, the result is another geometrical progression.

a. Next, if $a, a+\delta, a+2\delta, a+3\delta$, are consecutive terms of an arithmetical progression whose ratio is δ, and if each of these terms is raised to the power m, and the differences of the mth powers of the consecutive terms are taken; since

$$a^m=a^m;$$

$$(a+\delta)^m=a^m+ma^{m-1}\delta+\frac{m(m-1)}{1.2}a^{m-2}\delta^2+, \text{&c.}$$

$$(a+2\delta)^m=a^m+ma^{m-1}2\delta+\frac{m(m-1)}{1.2}a^{m-2}4\delta^2+, \text{&c.}$$

$$(a+3\delta)^m=a^m+ma^{m-1}3\delta+\frac{m(m-1)}{1.2}a^{m-2}9\delta^2+, \text{&c.}$$

$$(a+\delta)^m-a^m=ma^{m-1}\delta+\frac{m(m-1)}{1.2}a^{m-2}\delta^2+, \text{&c.}$$

$$(a+2\delta)^m-(a+\delta)^m=ma^{m-1}\delta+\frac{m(m-1)}{1.2}a^{m-2}3\delta^2+, \text{&c.}$$

$$(a+3\delta)^m-(a+2\delta)^m=ma^{m-1}\delta+\frac{m(m-1)}{1.2}a^{m-2}5\delta^2+, \text{&c.},$$

the differences are not constant; therefore the same powers of the terms of an arithmetical progression are not in arithmetical progression.

If, however, all the terms of an arithmetical progression are raised to the same power, m (m being integer and positive), the sum of the new series can be found.

Let $\div a.\ b.\ c.\ d \ldots k : l$, be any arithmetical progression, and let the common ratio be denoted by δ.

Then $b=a+\delta;\ c=b+\delta;\ \ldots l=k+\delta$.

Raising these equalities to the power $m+1$,

$$b^{m+1}=a^{m+1}+(m+1)a^m\delta+\frac{(m+1)m}{1.2}a^{m-1}\delta^2+, \text{&c.}$$

$$c^{m+1}=b^{m+1}+(m+1)b^m\delta+\frac{(m+1)m}{1.2}b^{m-1}\delta^2+, \text{&c.}$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$l^{m+1}=k^{m+1}+(m+1)k^m\delta+\frac{(m+1)m}{1.2}k^{m-1}\delta^2+, \text{&c.}$$

Adding these equalities member by member, suppressing the common terms b^{m+1}, c^{m+1}, &c., and transposing a^{m+1},

$$l^{m+1}-a^{m+1}=(m+1)\delta(a^m+b^m+\ldots\ldots+k^m)$$

$$+\frac{(m+1)m}{1.2}\delta^2(a^{m-1}+b^{m-1}+\ldots\ldots+k^{m-1})+, \text{&c.}$$

To abridge this expression, let $a+b+c+\ldots\ldots+k+l=S_1$;

- - - - $a^2+b^2+c^2+\ldots\ldots+k^2+l^2=S_2$;

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

- - - - $a^m+b^m+c^m+\ldots\ldots+k^m+l^m=S_m$.

$$\therefore\ l^{m+1}-a^{m+1}=\frac{m+1}{1}\delta(S_m-l^m)+\frac{(m+1)m}{1.2}\delta^2(S_{m-1}-l^{m-1})+, \text{&c.}$$

The value of S_m deduced from this series is

$$S_m=l^m+\frac{l^{m+1}-a^{m+1}}{(m+1)\delta}-\frac{m}{2}\delta(S_{m-1}-l^{m-1})-\frac{m(m-1)}{2\ 3}\delta^2(S_{m-2}-l^{m-2})-, \text{&c.} \quad 1$$

The law of the unwritten terms is sufficiently apparent, and the series must evidently end with the term preceding that which contains the factor $m-m$, or 0.

By formula 1 the sum S_m can be found when the inferior sums are known; for, making $m=0$, the formula gives S_0; making $m=1$, it gives S_1; making $m=2$, it gives S_2, and so on to the sum which is required.

b. If the progression $\div a \,.\, a+\delta \,.\, a+2\delta \,.\, a+3\delta \ldots\ldots$ is replaced by $\div 1 \,.\, 2 \,.\, 3 \,.\, 4 \ldots\ldots \mathrm{N}$ (or the series of numbers from 1 to N), $a=1$, $\delta=1$, $l=\mathrm{N}$, and formula 1 becomes

$$S_m=\mathrm{N}^m+\frac{\mathrm{N}^{m+1}-1}{m+1}-\frac{m}{2}(S_{m-1}-\mathrm{N}^{m-1})-\frac{m(m-1)}{2.3}(S_{m-2}-\mathrm{N}^{m-2})-, \text{ \&c.} \quad - \; 2$$

If $m=0$, formula 2 becomes

$$S_0=\mathrm{N}^0+\frac{\mathrm{N}^{0+1}-1}{0+1}=1+\frac{\mathrm{N}-1}{1};$$

$$\text{or } S_0=\mathrm{N}. \quad - \quad - \quad - \quad - \; 3$$

If $m=1$, formula 2 becomes

$$S_1=\mathrm{N}+\frac{\mathrm{N}^{1+1}-1}{1+1}-\frac{1}{2}(S_0-\mathrm{N}^{1-1})=\mathrm{N}+\frac{\mathrm{N}^2-1}{2}-\frac{1}{2}(\mathrm{N}-1);$$

$$\text{or } S_1=\frac{\mathrm{N}^2+\mathrm{N}}{2}=\frac{\mathrm{N}(\mathrm{N}+1)}{2}. \quad - \quad - \quad - \quad - \; 4$$

If $m=2$, formula 2 becomes

$$S_2=\mathrm{N}^2+\frac{\mathrm{N}^3-1}{3}-(S_1-\mathrm{N})-\frac{1}{3}(S_0-\mathrm{N}^0); \text{ or}$$

$$S_2=\mathrm{N}^2+\frac{\mathrm{N}^3-1}{3}-\left(\frac{\mathrm{N}^2+\mathrm{N}}{2}-\mathrm{N}\right)-\frac{1}{3}(\mathrm{N}-1); \text{ or}$$

$$S_2=\mathrm{N}^2+\frac{\mathrm{N}^3}{3}-\frac{1}{3}-\frac{\mathrm{N}^2}{2}-\frac{\mathrm{N}}{2}+\mathrm{N}-\frac{\mathrm{N}}{3}+\frac{1}{3}; \text{ or}$$

$$S_2=\frac{\mathrm{N}^3}{3}+\frac{\mathrm{N}^2}{2}+\frac{\mathrm{N}}{6}=\frac{2\mathrm{N}^3+3\mathrm{N}^2+\mathrm{N}}{6}; \text{ or}$$

$$S_2=\frac{\mathrm{N}(\mathrm{N}+1)(2\mathrm{N}+1)}{6}. \quad - \quad - \quad - \quad - \quad - \quad - \; 5$$

If $m=3$, it is found in the same manner that

$$S_3=\frac{\mathrm{N}^4+2\mathrm{N}^3+\mathrm{N}^2}{4}=\frac{\mathrm{N}^2(\mathrm{N}+1)^2}{4}. \quad - \quad - \quad - \quad - \quad - \; 6$$

. . . .

Formula 3 expresses the sum of $1^0+2^0+3^0+\ldots$ to N terms, or $\frac{1}{1}+\frac{2}{2}+\frac{3}{3}+\ldots\ldots+\frac{\mathrm{N}}{\mathrm{N}}$; or $1+1+1+1+\ldots\ldots$ to N terms; or N.

Example: If $m=0$ and $\mathrm{N}=10$, $S_0=\mathrm{N}=10$.

Formula 4 expresses the sum of $1+2+3+4+\ldots\ldots+\mathrm{N}$.

Example: If $m=1$ and $\mathrm{N}=10$, $S_1=\frac{10(10+1)}{2}=\frac{110}{2}=55$.

Formula 5 expresses the sum of $1^2+2^2+3^2+4^2+\ldots\ldots+\mathrm{N}^2$, or the sum of the squares of the series of natural numbers from 1 to N inclusive.

Example:
If $m=2$ and $\mathrm{N}=10$, $S_2=\frac{\mathrm{N}(\mathrm{N}+1)(2\mathrm{N}+1)}{6}=\frac{10\times11\times21}{6}=385$.

Formula 6 expresses the sum of $1^3+2^3+3^3+4^3+\ldots\ldots+\mathrm{N}^3$, or the sum of the cubes of the series of natural numbers from 1 to N inclusive.

Example: If $m=3$ and $\mathrm{N}=10$, $S_3=\frac{10^2(10+1)^2}{4}=\frac{100\times121}{4}=3025$.

193. In arsenals balls of the same size are arranged in piles; and it is by means of these formulæ (Art. 192) that the number of balls in a pile is computed.

These piles are either, 1st, triangular; or, 2d, quadrangular; or, 3d, rectangular.

1st. The triangular pile is formed thus: A layer of balls is placed horizontally on the ground in such a manner that the centres of the balls which are at the angles form an equilateral triangle. On this layer is arranged a second, each ball of which is placed in the hollow between three balls of the first layer. This second layer has also the form of an equilateral triangle, but its side contains one ball fewer than the side of the first layer.

Layer is placed upon layer in this manner, the side of each containing one ball fewer than the side of the layer on which it rests, until, by successive diminutions of one, the side of the last or uppermost layer (and by consequence that layer itself) contains only one ball.

The pile thus formed is a pyramid, having an equilateral triangle for its base; and it is evident that since the uppermost layer of a pyramid of this form contains one ball, the side of the second layer must contain two, the side of the third, three, ... and the side of the nth, n balls.

Denoting by N the number of balls contained in one side of the equilateral triangle which forms the base of a triangular pile, and by Y the number of balls in the base, it is evident that $Y=1+2+3+\ldots\ldots+N$.

But this is the sum S_1 of formula 4 (Art. 192);

$$\text{Therefore } Y=\frac{N^2+N}{2}.$$

If, in this expression, N is successively replaced by the numbers 1, 2, 3, ... the number of balls in the successive layers, beginning from the top of the pile, will be obtained.

$$\text{These are, in the first, } \frac{1^2+1}{2}=\frac{2}{2}=1.$$

$$\text{- - in the second, } \frac{2^2+2}{2}=\frac{6}{2}=3.$$

$$\text{- - in the third, } \frac{3^2+3}{2}=\frac{12}{2}=6.$$

$$\text{- - in the fourth, } \frac{4^2+4}{2}=\frac{20}{2}=10.$$

Whence the sum X of the whole number of balls contained in the pile is

$$X=\frac{1^2+1}{2}+\frac{2^2+2}{2}+\frac{3^2+3}{2}+\ldots\ldots+\frac{N^2+N}{2}.$$

Which expression may be otherwise arranged thus,

$$X=\frac{1^2+2^2+3^2+\ldots\ldots+N^2}{2}+\frac{1+2+3+\ldots\ldots+N}{2}.$$

The numerator of the first term of the second member of this equation is the sum S_2 (formula 5, Art. 192), and the numerator of the second term the sum S_1 (formula 4). Therefore

$$X=\tfrac{1}{2}\frac{2N^3+3N^2+N}{6}+\tfrac{1}{2}\frac{N^2+N}{2}=\frac{N^3+3N^2+2N}{6}.$$

Which by reduction becomes

$$X=\frac{N(N+1)(N+2)}{6} \quad - \quad - \quad - \quad 7$$

The number of balls contained in a triangular pile is found by this formula (7).

Example: How many balls are contained in a triangular pile, the side of whose base contains 35 balls?

Substituting 35 for N in formula 7,

$$X=\frac{35(35+1)(35+2)}{6}=\frac{35\times36\times37}{6}=7770.$$

2d. Quadrangular pile. In the pile of this form the layer of balls which composes the base is arranged in a square; the other layers are also squares. Each layer has one ball fewer in its side than the layer which is immediately below it, and the layers are continued until the last is composed of one ball only, which forms the top of the pile.

The quadrangular pile is, therefore, a pyramid with a square base.

Denoting by N the number of balls in a side of the base, and by X the number of balls in the pile, it follows from the construction of the pile that the uppermost layer contains $1^2=1$ ball; the second, $2^2=4$, the third, $3^2=9$; ...... the Nth, N^2.

$$\text{So that X}=1^2+2^2+3^2+ \ldots\ldots +\text{N}^2.$$

This sum is that of S_2 (formula 6, Art. 192).

$$\text{Whence X}=\frac{\text{N}(\text{N}+1)\ (2\text{N}+1)}{6} \quad - \quad - \quad - \quad - \quad 8$$

Example: If a side of the base of a quadrangular pile contains 35 balls the whole number of balls in the pile is

$$\text{X}=\frac{35\times 36\times 71}{6}=14910.$$

3d. Rectangular pile. In this pile the balls in all the layers except the uppermost are arranged in rectangles.

Each side of the layer resting on the base contains 1 ball fewer than the corresponding side of the base; and, similarly, the sides of every layer contain each one ball fewer than the corresponding sides of the immediately inferior layer.

When the number of layers is equal to the number of balls in the smaller side of the base, a layer is encountered which consists of only a single row of balls.

This row forms the top or ridge of the pile.

Representing by $p+1$ the number of balls in the single row, the layer below it (or the second layer) must contain 2 rows of $p+2$ balls each; the third, 3 rows of $p+3$ balls each; ... the nth, n rows of $p+n$ balls each.

If, therefore, n be the number of balls in the smaller side of the base, and X the number in the pile,

$$\text{X}=1(p+1)+2(p+2)+3(p+3)+ \ldots\ldots +n(p+n)\text{; or}$$

$$\text{X}=p(1+2+3+ \ldots\ldots +n)+(1^2+2^2+3^2+ \ldots\ldots n^2)=pS_1+S_2\text{, or}$$

$$\text{X}=\frac{p\times n(n+1)}{2}+\frac{n(n+1)\ (2n+1)}{6};$$

or, reducing to the same denominator, and observing that $3p$ and $2n+1$ have the common factor $n(n+1)$,

$$\text{X}=\frac{n(n+1)\ (3p+2n+1)}{6}.$$

In this formula p is the number of balls less one in the single row which forms the ridge of the pile, n is the number in the smaller side, and $p+n$ the number in the greater side of the base.

To express X in terms of the number of balls contained in the sides of the base, let $p+n=\text{N}$, and therefore $p=\text{N}-n$, $3p+2n=3\text{N}-n$; then, by substitution,

$$\text{X}=\frac{n(n+1)\ (3\text{N}-n+1)}{6} \quad - \quad - \quad - \quad - \quad 9$$

Example: If the greater side of the base of a rectangular pile contain 50 balls and the less side 35, how many balls are contained in the pile?

$$\text{X}=\frac{35\times 36(150-35+1)}{6}=\frac{35\times 36\times 116}{6}=24360.$$

4th. Each kind of pile may be truncated, that is, terminated by an upper layer composed of several rows of balls; in this case the pile is the difference of two complete piles.

Examples:

1st. How many balls are contained in a triangular pile, the side of whose base contains 10 balls? *Ans.* 220.

2d. How many balls are contained in two quadrangular piles, the sides of whose bases contain 9 and 12 balls respectively? *Ans.* First 285, second 650.

3d. If the greater side of the base of a rectangular pile contain 19 balls and the less side 10, how many balls are contained in the pile? *Ans.* 880.

4th. The side of the base of a truncated triangular pile contains 15 balls and the side of the top 7, required the number of balls in the pile? *Ans.* 625.

SECTION IX.

OF CONTINUED FRACTIONS.

194. The name Continued Fraction is given to an expression of the form

$$x=a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+,\ \&c.,}}}$$

in which the letters a, b, c, d, ... express any whole positive numbers, the series of which may be limited or not.

An expression whose numerators and denominators are any quantities whatever may have the form of a continued fraction; but continued fractions of which the numerators are 1, and the denominators whole positive numbers, are those whose properties are most deserving of attention.

The numbers a, b, c ... are termed incomplete quotients, or, simply, quotients; the complete quotients are obtained by taking each incomplete quotient together with the whole of the quantity which is joined to it by the sign +; such is $b+\cfrac{1}{c+,\ \&c.}$

The fractions $\frac{1}{b}$, $\frac{1}{c}$ .... are called partial or integrant fractions.

195. If in a continued fraction the first partial fraction, the first and second partial fractions, the first, second, and third partial fractions, &c. are successively taken, different expressions are obtained which are reducible to the form of vulgar fractions; these fractions are distinguished by the name of Converging Fractions or Convergents.

In the preceding expression of a continued fraction the first term or partial fraction is a or $\frac{a}{1}$, and the first convergent is $\frac{a}{1}$; the expression composed of the first and second terms or partial fractions is $a+\frac{1}{b}$, which reduced to the form of a vulgar fraction is $\frac{ab+1}{b}$; $\frac{ab+1}{b}$ is the second convergent.

The expression composed of the first, second, and third terms, or partial fractions, is $a+\cfrac{1}{b+\cfrac{1}{c}}$

Now $a+\cfrac{1}{b+\cfrac{1}{c}}=a+\cfrac{1}{\cfrac{bc+1}{c}}=a+\frac{c}{bc+1}=\frac{abc+a+c}{bc+1}=\frac{(ab+1)c+a}{bc+1}$,

$\therefore\ a+\cfrac{1}{b+\cfrac{1}{c}}=\frac{(ab+1)c+a}{bc+1}$; $\frac{(ab+1)c+a}{bc+1}$ is the third convergent.

In like manner $\left. a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}}\right\}=\left\{a+\cfrac{1}{b+\cfrac{d}{cd+1}}\right\}=a+\frac{cd+1}{b(cd+1)+d}$

$=\frac{abcd+ab+ad+cd+1}{bcd+b+d}$, an expression which, being arranged thus,

$\frac{abcd+cd+ad+ab+1}{bcd+d+b}$, is reducible to the form $\frac{\left\{(ab+1)c+a\right\}d+ab+1}{(bc+1)d+b}$;

$\frac{\left\{(ab+1)c+a\right\}d+ab+1}{(bc+1)d+b}$ is the fourth convergent.

The 5th, 6th, ... mth convergents may be formed in the same manner; but the process becomes very laborious when the number of partial fractions is considerable. Any convergent can be found by this method of reduction, without the help of the preceding convergents.

If the convergents are formed consecutively in the order second, third, fourth, &c., the second being found by the rule for the reduction of a mixed quantity to an improper fraction, the third can be obtained by changing b into $b+\frac{1}{c}$ in both terms of the second convergent, and making the necessary reductions; thus the second convergent is $\frac{ab+1}{b}$; changing b into $b+\frac{1}{c}$ $\frac{ab+1}{b}$ becomes $\frac{a\left(b+\frac{1}{c}\right)+1}{b+\frac{1}{c}}=\frac{(ab+1)c+a}{bc+1}$.

In the same manner the fourth convergent can be deduced from the third by changing c into $c+\frac{1}{d}$, the fifth from the fourth by changing d into $d+\frac{1}{e}$, &c. To obtain any convergent by this process it is necessary to form all the preceding convergents.

If the first, second, and third convergents are compared it is found that the third can be formed from the second and first by multiplying $ab+1$ and b, the terms of the second by c, the third quotient, and adding a and 1, the terms of the first, to the products; and if the second, third, and fourth convergents are compared it will appear that the fourth can be formed from the third and second convergents and the fourth quotient in the same manner.

To determine whether every convergent of a series can be formed in this manner from the two preceding convergents, and the quotient of the same rank with itself, let $\frac{N}{D}, \frac{N'}{D'}, \frac{N''}{D''}$ be three consecutive convergents, m the quotient of the same rank as the convergent $\frac{N''}{D''}$, and $\frac{1}{n}$ the partial fraction which follows $\frac{1}{m}$, and let $N''=N'm+N$, $D''=D'm+D$.

The convergent which follows $\frac{N''}{D''}$ is formed by substituting $m+\frac{1}{n}$ for m, in the terms of the convergent, $\frac{N''}{D''}=\frac{N'm+N}{D'm+D}$.

Making this substitution, and denoting the new convergent by $\frac{N'''}{D'''}$;

$$\frac{N'''}{D'''}=\frac{N'(m+\frac{1}{n})+N}{D'(m+\frac{1}{n})+D}=\frac{(N'm+N)n+N'}{(D'm+D)n+D'}=\frac{N''n+D'}{D''n+D'}.$$

Whence, if the convergent $\frac{N''}{D''}$ is formed by this law from the convergents $\frac{N}{D}$, $\frac{N'}{D'}$, and the quotient m, it follows that the convergent $\frac{N'''}{D'''}$ is formed from the convergents $\frac{N'}{D'}$, $\frac{N''}{D''}$, and the quotient n, by the same law.

Now it has been shown that the third convergent is formed from the first and second, and the fourth from the second and third, by it; therefore the law is general. Whence the $(n-1)$th and nth convergents of a series and the $n+1$ quotient being given to find the $(n+1)$th convergent, multiply the terms of the nth convergent by the $(n+1)$th quotient, and to the products add the corresponding terms of the $(n-1)$th convergent.

In applying this rule, it is convenient to arrange all the quotients a, b, c, ... in a horizontal line, and to place under them the corresponding convergents as they are formed. If, for example, the continued fraction is

$$x=3+\cfrac{1}{5+\cfrac{1}{2+\cfrac{1}{7}}};$$

the arrangement of the quotients and convergents is made as follows:

$$\begin{array}{lcccc} \text{quotients,} \quad - & 3, & 5, & 2, & 7, \\ \text{convergents,} \quad - & \frac{3}{1}, & \frac{16}{5}, & \frac{35}{11}, & \frac{261}{82}. \end{array}$$

When the continued fraction is not terminated (that is, when it is prolonged indefinitely) it follows, from the law of formation of the convergents, that the numerators and denominators form two series which increase to infinity.

196. The convergents are alternately less and greater than x, the value of the continued fraction.

For the first convergent is equal to a; and as the fractional part, which is added to a, is neglected, $a<x$.

The second convergent is equal to $a+\frac{1}{b}$; and since the divisor of 1 is b, a quantity less than $b+\frac{1}{c}$, the divisor in the continued fraction, it follows that the second convergent is greater than x.

The third convergent is equal to $a+\frac{1}{b+\frac{1}{c}}$, in which the divisor $b+\frac{1}{c}$ is too great; therefore the fraction $\frac{1}{b+\frac{1}{c}}$ is too small, therefore $a+\frac{1}{b+\frac{1}{c}}<x$.

The same reasoning may be continued indefinitely.

197. The difference between any two consecutive convergents is equal to 1, divided by the product of the denominators of the two convergents.

Let $\frac{N}{D}$, $\frac{N'}{D'}$, $\frac{N''}{D''}$ be three successive convergents, and m the quotient of the same rank as the convergent $\frac{N''}{D''}$; then $N''=N'm+N$; $D''=D'm+D$.

$$\text{But } \frac{N'}{D'}-\frac{N}{D}=\frac{DN'-D'N}{DD'} \quad \text{- - - } 1$$

$$\therefore \frac{N''}{D''}-\frac{N'}{D'}=\frac{N'm+N}{D'm+D}-\frac{N'}{D'}=\frac{N'D'm+D'N-N'D'm-DN'}{D'(D'm+D)}$$

$$=\frac{D'N-DN'}{D'(D'm+D)}=\frac{D'N-DN'}{D'D''} \quad \text{- - - - } 2$$

The differences 1, 2 have numerators which are equal, but of contrary signs, and denominators which are the products of the denominators of the two convergents whose difference is taken; hence the numerator (making abstraction of its sign) is a constant number. To determine this constant from the second convergent $\frac{ab+1}{b}$, subtract the first $\frac{a}{1}$,

$$\frac{ab+1}{b}-\frac{a}{1}=\frac{ab+1-ab}{1\times b}=\frac{1}{1\times b}.$$

The numerator of the difference of any two consecutive convergents being constant, and that of the difference of the first and second convergents being 1, it follows that the constant difference, $DN'-D'N$, $D'N-DN'=\pm 1$, and that $\frac{N'}{D'}-\frac{N}{D}=\frac{\pm 1}{DD'}$, $\frac{N''}{D''}-\frac{N'}{D'}=\frac{\pm 1}{D'D''}$......

The sign of the numerator of the difference is + or −, according as a convergent of an odd rank is subtracted from a convergent of an even rank or the contrary, for the convergents of odd rank are less, and the convergents of even rank greater, than the value of the continued fraction.

198. Since the denominators increase to infinity it is evident that it is possible to assign two consecutive convergents whose difference shall be less than any assigned number. Wherefore, as two consecutive convergents comprehend between them the value of the continued fraction, it is to be concluded that a convergent can be found whose value shall differ from that of the fraction by less than any assigned number.

For example, if it is required to find the value of a continued fraction to within the $\frac{1}{1000}$th part of 1; the calculation of the convergents must be continued till the product of the denominators of the last but one and the last is at least equal to 1000; then the last convergent but one has the degree of approximation required.

199. The convergents are fractions in the lowest terms; for if a convergent $\frac{N}{D}$ admits of reduction to lower terms, some quantity, q, must be a common measure of N and D. Whence q must also be a measure of $DN'-ND'$ (Part I. Art. 104-106). Now this is impossible, since $DN'-ND'=\pm 1$ (Art. 197).

200. Each convergent is a nearer approximation to the true value of the continued fraction than the preceding convergent.

Let $\frac{N''}{D''}=\frac{N'm+N}{D'm+D}$ be a convergent in which m is the last quotient employed; then, if the complete quotient $m+\frac{1}{n}+$, &c. is denoted by y, and y is substituted for m in the expression of $\frac{N''}{D''}$, it is evident that (employing x to denote the value of the fraction)

$$x=\frac{N'y+N}{D'y+D}.$$

Subtracting each of the convergents $\frac{N}{D}, \frac{N'}{D'}$ from this value of x,

$$x-\frac{N}{D}=\frac{N'y+N}{D'y+D}-\frac{N}{D}=\frac{(DN'-ND')y}{D(D'y+D)}.$$

$$x-\frac{N'}{D'}=\frac{N'y+N}{D'y+D}-\frac{N'}{D'}=\frac{ND'-DN'}{D'(D'y+D)}.$$

The numerators $(DN'-ND')y$, $ND'-DN'$ of these differences have contrary signs (Art. 4), and the absolute value of $DN'-ND'$ or $ND'-DN'$ is 1 (Art. 197); therefore

$$x-\frac{N}{D}=\frac{\pm y}{D(D'y+D)};\quad x-\frac{N'}{D'}=\frac{\mp 1}{D'(D'y+D)}.$$

But since y is a complete quotient, and D' the denominator of the convergent which comes after D, $y>1$ and $D'>D$.

Therefore $D'(D'y+D)>D(D'y+D)$, and (making abstraction of the sign) $\frac{\pm y}{D(D'y+D)}>\frac{\mp 1}{D'(D'y+D)}$.

Whence $\frac{N'}{D'}$ is a nearer approximation to the value of x than $\frac{N}{D}$.

From this property, the results $\frac{N}{D}, \frac{N'}{D'}$ receive their name, convergents or converging fractions.

201. The complete quotient y is comprehended between m and $m+1$; therefore, if, in the denominator of the difference, $\frac{\mp 1}{D'(D'y+D)}$, y is replaced by m the fraction is too great, and if y is replaced by $m+1$ the fraction is too small. Consequently (abstraction being made of the sign $\mp$)

$$x-\frac{N'}{D'}<\frac{1}{D'(D'm+D)}\text{ and }x-\frac{N'}{D'}>\frac{1}{D'(D'm+D'+D)};$$

or since $D'm+D=D''$,

$$x-\frac{N'}{D'}<\frac{1}{D'D''},\text{ and }x-\frac{N'}{D'}>\frac{1}{D'(D'+D'')}.$$

$\frac{1}{D'D''}, \frac{1}{D'(D'+D'')}$ are limits of the difference $x-\frac{N'}{D'}$.

The first (which is that of Art. 197) shows that the error committed by taking any convergent, as the value of x, is less than 1 divided by the product of the denominators of that and the following convergent. The second shows that the error is greater than 1, divided by the product of the denominator of that convergent into the sum of the denominators of that and the following convergent.

202. Every convergent approaches x more nearly than any fraction whose denominator is less than that of the convergent.

Let $\frac{N}{D}, \frac{N'}{D'}$ be two consecutive convergents; and let it be supposed that the irreducible fraction $\frac{n}{d}$ is nearer x than $\frac{N'}{D'}$ is.

Since x is between the convergents $\frac{N}{D}, \frac{N'}{D'}$, and differs less from $\frac{N'}{D'}$ than from $\frac{N}{D}$, therefore by the supposition $\frac{n}{d}$ must also be between $\frac{N}{D}$ and $\frac{N'}{D'}$. Whence (considering only the absolute values of the differences)

$$\frac{n}{d}-\frac{N}{D}<\frac{N'}{D'}-\frac{N}{D},$$

$$\text{or }\frac{Dn-Nd}{Dd}<\frac{1}{DD'}.$$

Now $Dn-Nd$ must be equal at least to 1, therefore, in order that $\frac{Dn-Nd}{Dd}$ may be less than $\frac{1}{DD'}$, the denominator Dd must be greater than DD', that is d must be greater than D'. Consequently that a fraction $\frac{n}{d}$ may be nearer in value to x than the convergent $\frac{N'}{D'}$, it is necessary that the denominator of the fraction $\frac{n}{d}$ be greater than the denominator of the convergent $\frac{N'}{D'}$.

203. Every periodic continued fraction is equal to one of the roots of an equation of the second degree with rational coefficients.

Let a, b, . . . be the quotients which form the non-periodic part of the continued fraction; and p, q, . . . the following quotients which are periodic. Then making

$$x=a+\frac{1}{b+}\;\vdots\;+\frac{1}{p+}\ \&c. \qquad\qquad y=p+\frac{1}{q+}\;\vdots\;+\frac{1}{p+}\ \&c.$$

it is evident that in these expressions the part $p+\frac{1}{q}+$, &c. may be replaced by y; or that

$$x=a+\frac{1}{b+}\;\vdots\;+\frac{1}{y} \qquad\qquad y=p+\frac{1}{q+}\;\vdots\;+\frac{1}{y}.$$

Considering these continued fractions to terminate with the partial fraction $\frac{1}{y}$, and deducing the convergents (Art. 195), two equations of the following form are obtained.

$$x=\frac{P'y+P}{Q'y+Q},\ y=\frac{R'y+R}{S'y+S}.$$

The value of y given by the first equation is

$$y=\frac{P-Qx}{Q'x-P'};$$

substituting this value of y in the second equation, and reducing

$$\frac{P-Qx}{Q'x-P'}=\frac{R'(P-Qx)+R(Q'x-P')}{S'(P-Qx)+S(Q'x-P')}.$$

which is an equation of the second degree in x.

204. The process of developing any quantity x, in a continued fraction, consists in making successively $x=a+\frac{1}{x'}$, $x'=b+\frac{1}{x''}$, $x''=c+\frac{1}{x'''}$, &c., a being the greatest whole number contained in x; b, the greatest whole number ontained in x'; c, the greatest whole number contained in x'', &c.

The numbers a, b, c, &c. being found, it is evident that if x', x'' are replaced by their values, the required development is obtained, viz.

$$x=a+\frac{1}{b+\frac{1}{c+}}\ \&c.$$

When x is an incommensurable quantity the continued fraction is prolonged indefinitely; for otherwise the convergents are limited in number, and the last is an exact value of x. Therefore x is not incommensurable. On the other hand, when x is commensurable the continued fraction always terminates.

Let it be required to convert the vulgar fraction $x=\frac{N}{D}$ into a continued fraction, and to form the convergents.

Dividing first N by D, denoting the quotient by a, and the remainder by R; $\frac{N}{D}=a+\frac{R}{D}$.

Dividing next D by R, denoting the quotient by b and the remainder by R′; $\frac{D}{R}=b+\frac{R'}{R}$, and therefore $\frac{R}{D}=\cfrac{1}{b+\cfrac{R'}{R}}$.

Dividing again R by R′, denoting the quotient by c and the remainder by R″; $\frac{R}{R'}=c+\frac{R''}{R'}$, and therefore $\frac{R'}{R}=\cfrac{1}{c+\cfrac{R''}{R'}}$.

Continuing this series of operations (which is the same as that for finding the greatest common measure of N and D) a remainder equal to zero is necessarily obtained; since the successive remainders are decreasing whole numbers.

Supposing therefore that $\frac{R'}{R''}=d$, and therefore $\frac{R''}{R'}=\frac{1}{d}$;

$$\text{then } \frac{N}{D}=a+\frac{R}{D}=a+\cfrac{1}{b+\cfrac{R'}{R}}=a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{R''}{R'}}}=a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}}.$$

By means of the quotients a, b, c, &c. the convergents are calculated as in Article 195.

Example. Let $x=\frac{86400}{20929}$.

The quotients are found in the following manner:

```
20929)86400(4
      83716
      ---------
       2684)20929(7
            18788
            ---------
             2141)2684(1
                  2141
                  ---------
                   543)2141(3
                       1629
                       ---------
                        512)543(1
                            512
                            ---------
                             31)512(16
                                31
                                --
                                202
                                186
                                ---------
                                 16)31(1
                                    16
                                    ---------
                                    15)16(1
                                       15
                                       ---------
                                        1)15(15
                                          15
                                          --
                                           0
```

Consequently the continued fraction is

$$\tfrac{86400}{20929}=4+\cfrac{1}{7+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{16+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{15}}}}}}}}$$

The quotients being 4, 7, 1, 3, 1, 16, 1, 1, 15.

The convergents are $\frac{4}{1}, \frac{29}{7}, \frac{33}{8}, \frac{128}{31}, \frac{161}{39}, \frac{2704}{655}, \frac{2865}{694}, \frac{5569}{1349}, \frac{86400}{20929}$.

The given fraction, which is expressed by large numbers, may be replaced by one of these convergents; then it is certain that no fraction whose terms are expressed by smaller numbers can approach so nearly in value to $\frac{86400}{20929}$ as the assumed convergent.

If the convergent $\frac{161}{39}$ is taken, the error committed is less than $\frac{1}{39\times 655}$ or $\frac{1}{25545}$.

Let it be required to reduce the irrational quantity $\sqrt{a^2+1}$ (in which a is any whole number) to a continued fraction.

The greatest whole number contained in $\sqrt{a^2+1}$ being a, let

$$\sqrt{a^2+1}=a+\frac{1}{x'};$$

$$\text{then } \frac{1}{x'}=\sqrt{a^2+1}-a, \text{ and } x'=\frac{1}{\sqrt{a^2+1}-a}.$$

$$\text{Now } \frac{1}{\sqrt{a^2+1}-a}=\frac{1}{\sqrt{a^2+1}-a}\times\frac{a+\sqrt{a^2+1}}{a+\sqrt{a^2+1}}=a+\sqrt{a^2+1}.$$

$$\text{Therefore } x'=a+\sqrt{a^2+1}.$$

The greatest whole number contained in $\sqrt{a^2+1}$ being a, the integer part of the value of x' is $2a$; let the irrational part be denoted by $\frac{1}{x''}$, then $x'=2a+\frac{1}{x''}$,

$$\text{or } a+\sqrt{a^2+1}=2a+\frac{1}{x''};$$

$$\therefore \frac{1}{x''}=\sqrt{a^2+1}-a, \text{ and } x''=\frac{1}{\sqrt{a^2+1}-a}.$$

The value of x'' being equal to the value of x', it follows that $\sqrt{a^2+1}$ is expressed by the continued periodic fraction

$$\sqrt{a^2+1}=a+\cfrac{1}{2a+\cfrac{1}{2a+}} \ \&c.$$

The square root of any whole number which is not an exact square can be reduced in this manner to a continued periodic fraction.

205. A property of convergents may be employed in the resolution of indeterminate equations of the first degree.

To explain the manner, let it be required to resolve in whole numbers the indeterminate equation

$$ax+by=c, \quad - \quad - \quad - \quad - \quad 1,$$

in which a, b, c are whole numbers; and a, b prime to each other.

Suppose that the ratio $\frac{a}{b}$ is reduced to a continued fraction, and that all the convergents are calculated, the last must be the ratio $\frac{a}{b}$ itself. Subtracting from $\frac{a}{b}$ the last convergent but one (which may be denoted by $\frac{a'}{b'}$), the numerator of the difference is $ab'-ba'$; and by Article 197,

$$ab'-ba'=\pm 1 \quad - \quad - \quad - \quad - \quad 2.$$

Multiplying the terms of the equality 2 by $\pm c$,

$$a\times \pm b'c+b\times \mp a'c=c.$$

Whence equation 1 is satisfied by taking

$$x=\pm b'c, \; y=\mp a'c.$$

This solution being known, all the others are given by the formula (Art. 96),

$$x=\pm b'c-bt, \; y=\mp a'c+at,$$

t being any whole number.

The upper or lower sign is to be taken according as the convergent $\frac{a}{b}$ is of the even or the odd rank.

Example. Let the equation $261x-82y=117$ be given to find the integer values of x, y.

Reducing $\frac{261}{82}$ to a continued fraction,

the quotients are 3, 5, 2, 7,

and the convergents, $\frac{3}{1}$, $\frac{16}{5}$, $\frac{35}{11}$, $\frac{261}{82}$.

Taking the numerator of the difference $\frac{261}{82}-\frac{35}{11}$, and observing that $\frac{261}{82}$ is of the even rank,

$$ab'-ba'=261\times 11-82\times 35=+1.$$

Multiplying by $c=117$,

$$261\times 11\times 117-82\times 35\times 117=117.$$

Consequently the proposed equation is satisfied by making

$$x=11\times 117=1287, \; y=35\times 117=4095;$$

and the general values of x and y are

$$x=1287+82t, \; y=4095+261t.$$

If 1287 is divided by 82, and 4095 by 261,

$$1287=82\times 15+57, \; 4095=261\times 15+180.$$

Whence, as t is any whole number whatever, the general values can be more simply expressed by

$$x=57+82t, \; y=180+261t.$$

Examples. It is required to reduce the following expressions to continued fractions, and to form the convergents.

1st. $\frac{251}{764}$?.................. *Ans.* The quotients are $\frac{1}{3}$, 22, 1, 4, 2.

And the convergents, $\frac{1}{3}$, $\frac{22}{67}$, $\frac{23}{70}$, $\frac{114}{347}$, $\frac{251}{764}$.

2d. $\frac{1769}{5537}$?

Ans. The quotients are $\frac{1}{3}$, 7, 1, 2, 4, 5, 1, 2.

And the convergents, $\frac{1}{3}$, $\frac{7}{22}$, $\frac{8}{25}$, $\frac{23}{72}$, $\frac{100}{313}$, $\frac{523}{1637}$, $\frac{623}{1950}$, $\frac{1769}{5537}$.

3d. $\frac{31415926535}{10000000000}$?

Ans. The quotients are 3, 7, 15, 1, 292, 1, 1, &c.

And the convergents, $\frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}, \frac{104348}{33215}, \frac{208341}{66317}$, &c.

4th. $\sqrt{28}$?............*Ans.* The quotients are 5, 3, 2, 3, 10, &c.

And the convergents, $\frac{5}{1}, \frac{16}{3}, \frac{37}{7}, \frac{127}{24}, \frac{1307}{247}$, &c.

5th. $\sqrt{31}$?.........*Ans.* The quotients are 5, 1, 1, 3, 5, 3, &c.

And the convergents, $\frac{5}{1}, \frac{6}{1}, \frac{11}{2}, \frac{39}{7}, \frac{206}{37}, \frac{657}{118}$, &c.

6th. $\sqrt{45}$?

Ans. The quotients are 6, 1, 2, 2, 2, 1, 12, &c.

And the convergents, $\frac{6}{1}, \frac{7}{1}, \frac{20}{3}, \frac{47}{7}, \frac{114}{17}, \frac{161}{24}, \frac{2046}{305}$.

SECTION X.

OF LOGARITHMS.

206. Let ÷ 0 . 1 . 2 . 3 . 4 . 5 . 6 . 7 . 8 . 9 1
∺ 1 : 3 : 9 : 27 : 81 : 243 : 729 : 2187 : 6561 : 19683 : 2
be two progressions, the first arithmetical and beginning with 0, the second geometrical and beginning with 1. If any terms of the second progression, 27 and 243 for example, are multiplied together, and the corresponding terms, 3 and 5, of the first progression are added together, the product, 6561, of the terms of the second, and the sum, 8, of the terms of the first, are corresponding terms of the two progressions; and, generally, the number which expresses the product of any terms of the geometrical progression and the number which expresses the sum of the corresponding terms of the arithmetical progression are corresponding terms of the two series.

Since the product of any terms of the second series is obtained by forming the sum of the corresponding terms of the first series, and taking, in the second series, the number which is the term corresponding to that sum, it follows that the multiplication of numbers contained in the second series is reducible to the addition of numbers contained in the first series.

207. Numbers through every change of magnitude may be considered as the terms of a geometrical progression; for in the series

$$∺ 1 : 1+a : (1+a)^2 : (1+a)^3 : (1+a)^4 : \ldots\ldots$$

if a is supposed a very small quantity, the terms increase by very small increments; and since a may be diminished indefinitely, the terms may be considered in this case to vary in a continuous manner. A progression in which continuity exists cannot indeed be written; but the understanding is capable of conceiving such a progression, and this is sufficient. All numbers greater than 1 may therefore be considered as comprised in the geometrical progression.

If again an arithmetical progression,

$$÷ 0 . \beta . 2\beta . 3\beta . 4\beta \ldots\ldots$$

is taken, the terms of which begin from 0 and increase by very small differences, and the terms of this series are considered in connexion with the terms of the geometrical progression, the former are called the logarithms of the latter.

Whence, the logarithms of numbers are the terms of an arithmetical progression commencing with 0, which correspond to other numbers considered as forming the terms of a geometrical progression which commences with 1.

No logarithms appear to be assigned by this definition to numbers less than 1. In order, however, that the geometrical progression may include such numbers, it is sufficient to conceive the progression prolonged below 1. This is accomplished by dividing 1 by the successive powers of the ratio $1+a$.

If the geometrical progression is decreasing, as,

$$∺ \ldots\ldots (1+a)^2 : 1+a : 1 : \frac{1}{1+a} : \ldots\ldots$$

the corresponding terms of the arithmetical progression,

$$\div \ldots\ldots 2\beta \,.\, \beta \,.\, 0 \ldots\ldots$$

are obtained by subtracting the ratio β from each term to form the succeeding term. When from β (which is the term corresponding to $1+a$) β is subtracted, the remainder, which is the term corresponding to 1, is 0. The next term of the geometrical progression is $\frac{1}{1+a}$ or $(1+a)^{-1}$, and the corresponding term of the arithmetical progression is $0-\beta$ or $-\beta$.

The corresponding terms of the decreasing progressions are,

$$\div 0 \,.\, -\beta \,.\, -2\beta \,.\, -3\beta \ldots\ldots$$

$$∺ 1 : \frac{1}{1+a} : \frac{1}{(1+a)^2} : \frac{1}{(1+a)^3} : \ldots\ldots$$

Negative terms, therefore, being employed, the arithmetical progression is made to descend below zero; and these negative terms are the logarithms of numbers less than 1.

If the descending terms are written on the left, the two series may be exhibited thus,

$$\div \ldots -3\beta \;.\; -2\beta \;.\; -\beta \;.\; 0 \;.\; \beta \;.\; 2\beta \;.\; 3\beta \ldots\ldots \quad 3$$

$$∺ \ldots \frac{1}{(1+a)^3} : \frac{1}{(1+a)^2} : \frac{1}{(1+a)} : 1 : 1+a : (1+a)^2 : (1+a)^3 \ldots \quad 4$$

and the first gives the logarithms of the corresponding terms of the second. It is an essential condition in the definition of logarithms that the terms 1 and 0 must always correspond to each other; in other words that the logarithm of 1 must be 0.

The ascending part of both the series 3, 4, augments to infinity; but in 4 the descending part tends indefinitely towards zero, while in 3 it augments to negative infinity. Therefore $\log. \infty = \infty$; and $\log. 0 = -\infty$.

It appears from a comparison of the series 3, 4, that if any number, n, is situated at a certain distance from 1 in the ascending part of 4, the number $\frac{1}{n}$ ought to occupy the same rank in the descending part. Therefore this last number has the same logarithm, but taken negatively; that is,

$$\log. \frac{1}{n} = -\log. n.$$

208. The fundamental property of logarithms is this. The logarithm of a product is equal to the sum of the logarithms of the factors of that product.

To give a general demonstration of this theorem,

1st. Let a, b be two numbers in the ascending part of progression 4; for example, let

$$a=(1+a)^2;\ b=(1+a)^5.$$

$$\text{Then } ab=(1+a)^2\times(1+a)^5=(1+a)^{5+2}.$$

Now, in progression 3, the logarithms of a, b, are 2β, 5β;

$$\text{therefore, } \log. a+\log. b=2\beta+5\beta=(5+2)\beta.$$

But $(5+2)\beta$ or 7β is the logarithm of $(1+a)^{5+2}$ or $(1+a)^7$;

$$\text{therefore, } \log. ab=\log. a+\log. b.$$

2d. Let a be in the descending part, b in the ascending part of series 4, and b more remote than a from the term 1.

Suppose $a=\frac{1}{(1+\alpha)^2}$; $b=(1+\alpha)^5$;

$$\text{then } ab=\frac{1}{(1+\alpha)^2}\times(1+\alpha)^5=\frac{(1+\alpha)^5}{(1+\alpha)^2}=(1+\alpha)^{5-2}.$$

But log. $a=-2\beta$; log. $b=5\beta$; therefore log. $a+$log. $b=(5-2)\beta$.

Whence, as $(5-2)\beta$ is evidently the logarithm of $(1+\alpha)^{5-2}$, it is concluded again that log. $ab=$log. $a+$log. b.

3d. Let a be in the descending part, b in the ascending part of series 4, b being nearer than a to the term 1.

Suppose $a=\frac{1}{(1+\alpha)^5}$; $b=(1+\alpha)^2$;

$$\text{then } ab=\frac{1}{(1+\alpha)^5}\times(1+\alpha)^2=\frac{(1+\alpha)^2}{(1+\alpha)^5}=\frac{1}{(1+\alpha)^{5-2}}.$$

But log. $a=-5\beta$; log. $b=2\beta$; therefore log. $a+$log. $b=-(5-2)\beta$, which is the log. of $\frac{1}{(1+\alpha)^{5-2}}$.

Whence log. $ab=$log. $a+$log. b.

4th. Let a, b, be both in the descending part of series 4.

Suppose $a=\frac{1}{(1+\alpha)^2}$; $b=\frac{1}{(1+\alpha)^5}$;

$$\text{then } ab=\frac{1}{(1+\alpha)^2}\times\frac{1}{(1+\alpha)^5}=\frac{1}{(1+\alpha)^{5+2}}.$$

In this case log. $a=-2\beta$; log. $b=-5\beta$. Consequently log. $a+$log. $b=-(2+5)\beta$, which is the log. of the product ab; therefore log. $ab=$log. $a+$log. b.

The equality log. $ab=$log. $a+$log. b being true in all cases if b is changed into bc.

$$\text{log. }(a\times bc)=\text{log. }a+\text{log. }bc.$$

But log. $(a\times bc)=$log. abc, and log. $bc=$log. $b+$log. c;

therefore, log. $abc=$log. $a+$log. $b+$log c.

This process may be extended to any number of factors.

Wherefore log. $abcd\ldots.=$log. $a+$log. $b+$log. $c+$log. $d+\ldots.$

209. Let $q=\frac{a}{b}$, and therefore $bq=a$.

Then (Art. 208) log. $b+$log. $q=$log. a;

and transposing, log. $q=$log. $a-$log. b.

Whence the log. of a quotient is equal to the log. of the dividend less the log. of the divisor.

210. If a product is composed of n factors, each equal to a,

$a\times a\times a\times\ldots.$ to n factors $=a^n$;

$\therefore$ log. $(a\times a\times a\times\ldots.$ to n factors$)=$log. a^n.

But log. $(a\times a\times a\times\ldots.$ to n factors$)=$log. $a+$log. $a+$log. $a\ldots.$ to n repetitions $=n$ log. a.

Therefore log. $a^n=n$ log. a.

To extend this formula to fractional exponents, let $x=a^{\frac{m}{n}}$;

$\therefore x^n=a^m$; log. $x^n=$log. a^m; n log. $x=m$ log. a;

and log. $x=\frac{m}{n}$ log. a.

If a is affected with a negative exponent, as a^{-n},

Let $x=a^{-n}=\frac{1}{a^n}$. $\therefore xa^n=1$; and log. $(xa^n)=$log. $1=0$.

But $\log.(xa^n) = \log. x + \log. a^n = \log. x + n \log. a$.

Therefore $\log. x + n \log. a = 0$, and $\log. x = -n \log. a$.

Consequently, $\log. a^{-n} = -n \log. a$.

Whence the logarithm of any power whatever of a number is equal to the product of the logarithm of the number by the exponent of the power.

211. Let $r = \sqrt[n]{a}$, and therefore $r^n = a$.

Then, by Article 210, $n \log. r = \log. a$.

Dividing by n, $\log. r = \dfrac{\log. a}{n}$.

Whence $\log. r = \log. (\sqrt[n]{a}) = \dfrac{\log. a}{n}$.

Wherefore the logarithm of the root of a number is obtained by dividing the logarithm of the number by the index of the root.

212. In the geometrical series ∺ $1 : 1+a : (1+a)^2 : (1+a)^3$, &c., a may be assumed so small that the difference of any two consecutive terms shall be less than any assigned number. Then, as $1+a$ is greater than 1, and the series may be indefinitely prolonged, it is evident that each term is greater than the preceding, and that a term may be found which shall exceed any given number.

Since the series can be prolonged until a term greater than 2 is found, then either one of the terms must be equal to 2, or 2 must fall between two consecutive terms. The difference of these terms being less than any assigned number, and the difference between 2 and either term being less than the difference of the two terms, either term may be considered equal to 2.

In the same manner it can be made to appear that other terms of the series are either equal exactly to 3, 4, 5 . . . or differ from them by less than any assigned numbers. Consequently certain terms of the series are equal respectively to the numbers 2, 3, 4, 5, . . . 10, . . . 100, . . . 1000, &c.

An arithmetical series $\div 0 . \beta . 2\beta . 3\beta . 4\beta$ . . . . whose ratio β is a very small quantity, being connected with the preceding geometrical series by the conditions that the terms 0 and 1 shall correspond, and that the exponent of any terms of the geometrical and the coefficient of the corresponding term of the arithmetical series shall be the same number, the terms of the arithmetical series corresponding to those terms of the geometrical series which are considered equal to the numbers 2, 3, 4, . . . are the logarithms of the numbers 2, 3, 4, ...

213. Another condition may be imposed on the series, viz., that between the simultaneous increments of 1 and 0, the first terms of the geometrical and arithmetical series, an arbitrary ratio must subsist; as, for example, that the one increment shall be equal to M times the other.

Then, if the increment of the term 1 of the geometrical series is denoted by ω, that of the term 0 of the arithmetical series is $M\omega$, and the progressions become,

∺ $1 : 1+\omega : (1+\omega)^2 : (1+\omega)^3 :$ - - - 1,
$\div 0 . M\omega \quad 2M\omega . 3M\omega$. - - - 2.

According as different particular values are given to M, different systems of logarithms are formed.

The ratio M is termed the modulus of the system of logarithms.

Assuming $M = 1$, the second of the preceding series becomes,

$\div 0 . \omega . 2\omega . 3\omega$ - - - - - - 3.

By means of the hypothesis $M = 1$, the multiplications by M are avoided. In the system published by Napier, the inventor of logarithms, $M = 1$. To change the series 3 into 2, it is only necessary to multiply the terms of 3 by M. Wherefore, to change the logarithms of Napier into logarithms of any other system, it is sufficient to multiply Napier's logarithms by the modulus of that other system.

214. A system is rendered determinate by fixing, arbitrarily, the number to which a particular logarithm is to correspond; as, for example, by assigning the number whose logarithm shall be 1.

In every system the number whose logarithm is 1 is called the base of that system.

In Napier's system, the base or number whose logarithm is 1 is 2·718281828. The base of the system of the common tabular logarithms is 10.

Let Napier's logarithm of the number a be denoted by La;
and the logarithm of a in any other system by $\log. a$;
then $M \times La = \log. a$, and, therefore, $M = \frac{\log. a}{La}$.

Whence the modulus of any system of logarithms is obtained by dividing the logarithm of any number a in that system by Napier's logarithm of the number a.

Therefore the logarithms of Napier being given, if the logarithm of one number in any other system is known, the modulus, and by consequence the logarithms also, of that system can be found.

Since the logarithm of the base of every system is 1, if the base of any system is denoted by a, the formula,

$$M = \frac{\log. a}{La} \text{ becomes } M = \frac{1}{La}.$$

Consequently the modulus of any system is equal to the reciprocal of Napier's logarithm of the base of that system. Napier's log. of 10 being 2·302585092, $\frac{1}{2·302585092}$ or ·434294481 is the modulus of the common tabular logarithms. The tabular logarithm of any number can consequently be obtained by multiplying Napier's log. of that number by ·434294481.

215. In the system whose base is 10, the logarithms of 10, 100, 1000, &c. are 1, 2, 3, &c., and, in general, the log. of $10^m = m$.

Of numbers falling between 1 and 10, the logs. fall between 0 and 1,
- - - 10 and 100 - - - 1 and 2,
- - - 100 and 1000 - - - 2 and 3,
. . . .

The integer part of a logarithm is named the characteristic, and the decimal part the mantissa.

It may, therefore, be concluded that in this system the characteristic of the log. of a number contains as many repetitions of 1, less one, as the number contains figures.

If a number is multiplied by 10, 100, 1000, . . . its log. is augmented by 1, 2, 3, . . . and if a number is divided by 10, 100, 1000, . . . its logarithm is diminished by 1, 2, 3 . . . Therefore, if a number is multiplied or divided by a power of 10, so long as the result is greater than 1 the mantissa is constant, and the characteristic only varies.

Taking the number 658, the log of which is 2·818226,
$\text{Log.} (658 \times 10) = \log. 658 + \log. 10 = 2·818226 + 1 = 3·818226.$
$\text{Log.} (658 \times 100) = \log. 658 + \log. 100 = 2·818226 + 2 = 4·818226.$
$\text{Log.} (658 \times 1000) = \log. 658 + \log. 1000 = 2·818226 + 3 = 5·818226.$
. . . .
$\text{Log.} (658 \div 10 \text{ or } 65·8) = \log. 658 - \log. 10 = 2·818226 - 1 = 1·818226.$
$\text{Log.} (658 \div 100 \text{ or } 6·58) = \log. 658 - \log. 100 = 2·818226 - 2 = 0·818226.$
$\text{Log.} (658 \div 1000 \text{ or } ·658) = \log. 658 - \log. 1000 = 2·818226 - 3 = -0·181774.$
$\text{Log.} (658 \div 10000 \text{ or } ·0658) = \log. 658 - \log. 10000 = 2·818226 - 4 = -1·181774.$
. . . .

From these instances it appears that when the divisor is a power of 10, and the result of the division is less than 1, the log. becomes negative, and all the figures are changed. In this case, instead of subtracting the whole log. of the dividend from the log. of the divisor and affecting the remainder with the sign −, it is usual to subtract only the characteristic of the log. of the dividend from the log. of the divisor, and to consider the remainder as a negative characteristic. The negative sign is not prefixed to, but placed over, the characteristic, thus,

$2{\cdot}818226-3$ is written $\bar{1}{\cdot}818226$;
$2{\cdot}818226-4$ - - $\bar{2}{\cdot}818226$;

expressions which are to be considered as abbreviations, the first of $0{\cdot}818226-1$ or of $-1+0{\cdot}818226$, and the second of $0{\cdot}818226-2$ or of $-2+0{\cdot}818226$.

By admitting, in this manner, logarithms whose characteristics alone are negative, it becomes permissible to say that, a decimal number being given, the place of the decimal point may be changed at pleasure without change of the decimal part of the logarithm.

216. In the system whose base is 10, the logarithms of the series of whole numbers, from 1 to 10000, for example, may be computed in the following manner.

Since 1 and 10 are two terms of the geometrical progression, and 0, 1 the corresponding terms of the arithmetical progression, if a great number of geometrical means is inserted between 1 and 10, and the same number of arithmetical means between 0 and 1, and the two progressions are extended until the first reaches 10000; then certain terms of the first progression differ from the numbers 2, 3, 4, ... by less than any assigned number (Art. 212).

The corresponding terms of the second progression differ from the logarithms of these numbers by less than any assigned number. For, let N be a whole number which falls between two terms n, n' of the geometrical progression, and let the corresponding terms of the arithmetical progression or the logarithms of n, n' be l, l'; then if the number of geometrical and arithmetical means is more and more augmented, the terms of the two progressions are made nearly to increase continuously; and the number N falling between n, n' is found to have a corresponding term L between l and l'.

Now, since L falls between l and l', $L-l<l'-l$ and $l'-L<l'-l$; therefore the error committed by taking either l or l' for the logarithm of N is less than the difference $l'-l$, which can be rendered less than any assigned number.

If the ratio of the arithmetical progression is $\frac{1}{1000000}$, $l'-l=\frac{1}{1000000}$ $=\cdot000001$; and the progression gives logarithms with six correct decimal figures. That a geometrical progression may correspond with an arithmetical possessing this degree of accuracy, 10 must have 1000000 terms before it. To obtain the ratio of this progression it is necessary to extract the 1000000th root of 10. Since $1000000=10^6=2^6\times5^6$, the extraction of the millionth root may be affected by repeated extractions of the 2d and 5th roots.

217. There is, however, a method of calculating logarithms without extracting a higher root than the second.

Let n be the number whose logarithm is to be computed, n being between 1 and 10.

The geometrical mean between 1 and $10=\sqrt{1\times10}$, let this be denoted by A; the arithmetical mean between 0 and $1=\frac{0+1}{2}=\frac{1}{2}$, let this be denoted by a. Suppose that n falls between 1 and A, let the geometrical mean between 1 and $\text{A}=\text{B}$, and the arithmetical mean between 0 and $a=b$. Suppose next that n falls between A and B, find C, the geometrical mean between A, B, and c, the arithmetical mean between a, b.

This series of operations being continued until n is contained between two geometrical means, to which correspond two arithmetical means, of which the first six decimals are the same, it is certain that one of these means is the logarithm of n with the degree of approximation required.

To attain the maximum of approximation with six decimals it is necessary to calculate the seventh decimal figure, and in the case in which the seventh figure is 5 or greater than 5 to add 1 to the sixth decimal figure of the logarithm; by this means the error in excess or defect is rendered less than the half of one millionth part of 1.

The logarithms of prime numbers need alone to be computed, for the logarithms of other numbers are obtained by adding together the logarithms of the prime factors of which these numbers are composed.

The error in the logarithm of a composite number may by these additions be increased to more than the half of one millionth. It is, however, easy to fix the maximum of error which can be produced by this means.

In the present hypothesis the highest number in the tables is 10000; now the least prime number is 2, and $2^{17}=131072$; therefore a number which is less than 10000, being less than 2^{17}, cannot have more than seventeen factors, consequently the error of each logarithm being less than the half of one millionth of 1, the sum of the errors, even when they are either all in excess or all in defect, must be less than seventeen half millionths.

By calculating the logarithms of the prime numbers to eight decimal figures the maximum of error in the logarithm of the greatest composite number in the tables cannot exceed two units of the seventh order of decimals, and consequently it must be less than one unit of the sixth order.

218. When systems of logarithms are determined by their bases the transition from one system to another is easy; for, denoting by a a very small quantity, the series of numbers is given by the geometrical progression $\div\!\!\div 1 : 1+a : (1+a)^2 : (1+a)^3$, &c.; and supposing β and γ to be also very small quantities, the logarithms of the two systems which are compared may be taken respectively in the two arithmetical progressions:

$$\div 0 \,.\, \beta \,.\, 2\beta \,.\, 3\beta \,.\, 4\beta \,\ldots\ldots$$
$$\div 0 \,.\, \gamma \,.\, 2\gamma \,.\, 3\gamma \,.\, 4\gamma \,\ldots\ldots$$

It is evident that the logarithms of the same number in the second system and in the first have to each other a constant ratio, so that denoting by x', x the two logarithms, and by K the constant ratio,

$$x'=\mathrm{K}x.$$

With respect to the ratio K, to obtain it it is sufficient that the logarithms of a single number are known in the two systems. Now in the first system the logarithm of the base in the second is supposed to be known, and in the second system the logarithm of its own base is 1; making, therefore, a' the second base, and denoting by logarithm a' the logarithm of the second base in the first system,

$$1=\mathrm{K}\log. a', \text{ and } \mathrm{K}=\frac{1}{\log. a'}.$$

219. Resuming the two progressions:

$$\div\!\!\div \ldots \frac{1}{(1+a)^3} : \frac{1}{(1+a)^2} : \frac{1}{1+a} : 1 : (1+a) : (1+a)^2 : (1+a)^3 : \ldots\ldots,$$
$$\div \ldots \; -3\beta \;.\; -2\beta \;.\; -\beta \;.\; 0 \;.\; \beta \;.\; 2\beta \;.\; 3\beta \;\ldots\ldots;$$

if a, β are indefinitely small quantities the terms of the progressions may be considered to vary continuously, and consequently the second series must contain a multiple of β, which is equal to 1. Let $\mu\beta$ be this multiple, and let the corresponding term of the first series be equal to a,

$$\text{then simultaneously } \mu\beta=1,\ (1+a)^{\mu}=a.$$

$$\text{Wherefore } \beta=\frac{1}{\mu}, \text{ and } 1+a=a^{\frac{1}{\mu}}=a^{\beta};$$

$$\therefore (1+a)^2=a^{2\beta},\ (1+a)^3=a^{3\beta}, \text{ \&c.};$$

$$\frac{1}{1+a}=\frac{1}{a^{\beta}}=a^{-\beta},\ \frac{1}{(1+a)^2}=\frac{1}{a^{2\beta}}=a^{-2\beta}, \text{ \&c.}$$

Making these substitutions the two progressions become,

$$\div\!\div \ldots\ a^{-3\beta} : a^{-2\beta} : a^{-\beta} : 1 : a^{\beta} : a^{2\beta} : a^{3\beta} : \ldots.$$

$$\div \ldots\ -3\beta\,.\,-2\beta\,.\,-\beta\,.\,0\,.\,\beta\,.\,2\beta\,.\,3\beta\ \ldots.;$$

from the comparison of which it is evident that if any term of the second progression is denoted by x and the corresponding term of the first progression by y, the relation $y=a^x$ must subsist between these terms.

The logarithms of numbers may hence be defined as the exponents of the powers to which it is necessary to raise a constant quantity (called Base) in order to deduce from it all these numbers.

In adopting this definition it is to be understood that the base is a real and positive quantity. Under this limitation the base may be any number different from 1.

To establish this proposition, it is necessary to prove that if in the exponential equation, $y=a^x$, all possible values, negative as well as positive, are given to the exponent x, the corresponding values of y will comprehend all the numbers continuously between zero and infinity.

1st. Let a be greater than 1. If to x are given positive and increasing values (beginning from zero) such as $\frac{1}{10}$, $\frac{2}{10}$, &c., then,

$$y=a^0,\ a^{\frac{1}{10}},\ a^{\frac{2}{10}},\ a^{\frac{3}{10}},\ \ldots$$

or making $\sqrt[10]{a}=a'$,

$$y=1,\ a',\ a'^2,\ a'^3,\ \ldots..$$

a' being a quantity greater than 1, this series increases to infinity; and it is further evident that if x is made to increase by very small increments the consecutive values of y may be made to differ from each other by less than any given number.

$$\text{Making } x=-z,\ y=a^{-z}=\frac{1}{a^z}.$$

It has been proved that if z is made to pass through all positive values, beginning from 0, a^z increases continuously from 1 to ∞, therefore the quotient $\frac{1}{a^z}$ must decrease from 1 to 0; whence it is inferred that the negative values of x between 0 and $-\infty$ make y to take all descending values from 1 to 0.

2d. Let $a<1$, and make $a=\frac{1}{a'}$; then $y=\frac{1}{a'^x}$.

Since $a<1$, $a'>1$, therefore according as x is made to increase positively or negatively a'^x will increase from 1 to ∞ or decrease from 1 to 0, and consequently y will decrease from 1 to 0 or increase from 1 to ∞.

It is to be concluded, therefore, that every number except 1 may be taken for the base of a system of logarithms.

The preceding discussion also shows,

1st. That in every system of logarithms the logarithm of 1 is 0 and the logarithm of the base is 1.

2d. That in systems whose base is greater than 1, $\log.\infty=\infty$ and $\log.0=-\infty$.

3d. That in systems whose base is less than 1, $\log.\infty=-\infty$ and $\log.0=\infty$.

220. By the second definition of logarithms (Art. 219), the properties of logarithms are derived immediately from those of exponents.

For if $y, y', y'', \ldots$ are numbers, and $x, x'\ x'' \ldots$ the logarithms of these numbers, by the definition $y=a^x$, $y'=a^{x'}$, $y''=a^{x''}\ \ldots$

And by the rules for the calculus of exponents,

$$y.y'.y'' \ldots . = a^x.a^{x'}.a^{x''} \ldots = a^{x+x'+x''+} \ldots . ;$$

$$y \div y' = \frac{y}{y'} = \frac{a^x}{a^{x'}} = a^{x-x'},$$

$$y^n = (a^x)^n = a^{nx};$$

$$\sqrt[n]{y} = \sqrt[n]{a^x} = a^{\frac{x}{n}}.$$

But by the definition (Art. 219), the exponents of a in the expressions $a^{x+x'+x''\ldots}$, $a^{x-x'}$, a^{nx}, $a^{\frac{x}{n}}$, are the logarithms of $y.y'.y''\ldots$; $\frac{y}{y'}$, y^n, $\sqrt[n]{y}$ respectively; whence, as has been already proved (Art. 208—211), the logarithm of a product is equal to the sum of the logarithms of the factors of that product; the logarithm of a quotient is equal to the excess of the logarithm of the dividend over the logarithm of the divisor; the logarithm of a power is equal to the product of the logarithm of the root by the exponent of the power; and the logarithm of a root is equal to the quotient of the logarithm of the power by the index of the root.

221. Let it be required to calculate the logarithm of a given number b in the system whose base is a given number a.

This problem involves the resolution of the equation $a^x=b$, in which the unknown quantity is an exponent.

To resolve it. Suppose that the numbers a, b are both greater than 1; by giving to x the values 0, 1, 2, 3, &c. two powers of a, which may be represented by a^n and a^{n+1}, can be found between which b is comprehended; then x is between n and $n+1$.

Making $x=n+\frac{1}{x'}$, the equation $a^x=b$ becomes

$$a^{n+\frac{1}{x'}}=b, \text{ and therefore}$$

$$a^{\frac{1}{x'}}=\frac{b}{a^n}.$$

To abridge the expression make $\frac{b}{a^n}=c$; then,

$$a^{\frac{1}{x'}}=c; \text{ and therefore } c^{x'}=a.$$

In this expression $c>1$ and $c<a$, for $b>a^n$ and $b<a^{n+1}$, consequently in forming the powers c^1, c^2, c^3 . . . . it is found that a is comprehended between two consecutive powers $c^{n'}$ and $c^{n'+1}$, and that x' is therefore between n' and $n'+1$,

$$\text{Making } x'=n'+\frac{1}{x''},$$

$$c^{n'+\frac{1}{x''}}=a, \text{ and therefore } c^{\frac{1}{x''}}=\frac{a}{c^{n'}}.$$

$$\text{Again, making } \frac{a}{c^{n'}}=d,\ c^{\frac{1}{x''}}=d;$$

$$\text{and therefore } c=d^{x''}.$$

Operating in the same manner on this equation, it is found between what whole numbers n'' and $n''+1$, x'' is comprehended; then x'' being made equal to $n''+\frac{1}{x'''}$, the process may be repeated as often as necessary.

The results obtained are of the form,

$$x=n+\frac{1}{x'},\ x'=n'+\frac{1}{x''},\ \ x''=n''+\frac{1}{x'''}, \text{ \&c.}$$

Consequently the value of x may be expressed by the continued fraction,

$$x=n+\cfrac{1}{n'+\cfrac{1}{n''+\cfrac{1}{n'''+,\ \&c.}}}$$

From the theory of continued fractions it is known that the greater the number of terms taken the more exact the approximation, and that the error may be rendered less than any assigned number.

CALCULATION OF THE LOGARITHM OF A NUMBER.

Let $a=10$, $b=2$; therefore $a^x=b$ becomes $10^x=2$, and x is the logarithm of 2 in the system whose base is 10.

Since log. $1=0$, and log. $10=1$, x falls between 0 and 1.

Making $x=\frac{1}{a}$, $10^{\frac{1}{a}}=2$ and $10=2^a$.

Now $2^3=8$ and $2^4=16$, therefore the value of a is greater than 3 and less than 4.

Let $a=3+\beta$ $\therefore 10=2^3\times 2^\beta$ and $\frac{10}{8}=\frac{2^3\times 2^\beta}{2^3}$ or $1{\cdot}25=2^\beta$;

$$\therefore x=\frac{1}{3+\beta} \qquad - \quad - \quad - \quad - \qquad 1.$$

Next, to satisfy the equality $1{\cdot}25=2^\beta$, it is necessary that $\beta<1$, let $\beta=\frac{1}{\gamma}$; $\therefore 1{\cdot}25=2^{\frac{1}{\gamma}}$ or $1{\cdot}25^\gamma=2$.

Substituting successively 3 and 4 for γ, $1{\cdot}25^3<2$ and $1{\cdot}25^4>2$.

Whence the value of γ falls between 3 and 4; let $\gamma=3+\delta$.

$$\therefore 1{\cdot}25^3\times 1{\cdot}25^\delta=2, \text{ and } \frac{1{\cdot}25^3\times 1{\cdot}25^\delta}{1{\cdot}25^3}=\frac{2}{1{\cdot}25^3} \text{ or } 1{\cdot}25^\delta=1{\cdot}024.$$

Replacing β in the expression $x=\frac{1}{3+\beta}$ by $\frac{1}{\gamma}=\frac{1}{3+\delta}$,

$$x=\cfrac{1}{3+\cfrac{1}{3+\delta}} \qquad - \quad - \quad - \qquad 2.$$

In the same manner in the equality $1{\cdot}25^\delta=1{\cdot}024$, δ is fractional or of the form $\frac{1}{\epsilon}$; whence $1{\cdot}25=1{\cdot}024^\epsilon$; $\epsilon=9+\zeta$ and $1{\cdot}024^9\times 1{\cdot}024^\zeta=1{\cdot}25$ or $1{\cdot}024^\zeta=1{\cdot}0097$.

Replacing δ in equation 2 by $\frac{1}{\epsilon}=\frac{1}{9+\zeta}$

$$x=\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{9+\zeta}}} \qquad - \quad - \qquad 3.$$

Similarly the equality $1{\cdot}024^\zeta=1{\cdot}0097$, by the substitution of $\frac{1}{\eta}$ for ζ, gives $1{\cdot}024=1{\cdot}0097^\eta$; from which are obtained $\eta=2+\theta$ and $1{\cdot}0097^\theta=1{\cdot}0044$.

The equality $1{\cdot}0097^\theta=1{\cdot}0044$, by the substitution of $\frac{1}{\lambda}$ for θ, gives $1{\cdot}0097=1{\cdot}0044^\lambda$, $\lambda=2+\mu$ and $1{\cdot}0044^\mu=1{\cdot}00087$.

And the equality $1{\cdot}0044^\mu=1{\cdot}00087$, by the substitution of $\frac{1}{\nu}$ for μ, gives $\nu=5+\omega$; $1{\cdot}00087^\omega=1{\cdot}000042$; whence

$$x=\cfrac{1}{3+\cfrac{1}{3+\cfrac{1}{9+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{v}}}}}}}$$

The quotients being $\frac{1}{3}\,.\,3\,.\,9\,.\,2\,.\,2\,.\,5$, &c.

The convergents are $\frac{1}{3}\cdot\frac{3}{10}\cdot\frac{28}{93}\cdot\frac{59}{196}\cdot\frac{146}{485}\cdot\frac{789}{2641}$, &c.

Taking the convergent $\frac{789}{2641}$ as a sufficiently accurate approximation $x=\log.\ 2=\frac{789}{2641}=\cdot 0\cdot 3010301$, a result of which the six first decimals are correct.

222. When the base is 10, $10^1=10$, $10^2=100$, $10^3=1000$; . . . therefore the logarithms of the numbers 10, 100, 1000, . . . are 1, 2, 3 . . .; and in general the logarithm of 1000 . . . . . k (that is 1 followed by k zeros, or 10 raised to a power k) is k, or $\log.\ 10^k=k$.

To obtain the logarithms of the numbers 2, 3, 5, 7, ... it is necessary to resolve the equations $10^x=2$, $10^x=3$, $10^x=5$, $10^x=7$ . . . .; the logarithms of the numbers 4, 6, 8, and in general of composite numbers, are obtained from the logarithms of the prime factors of these numbers by addition (Art. 217).

The logarithms of numbers may be computed, with great labour, by the methods of Articles 216, 217, 221; more expeditious processes are derived from the integral calculus.

223. Logarithms being considered as exponents, and calculated for a particular base, it is easy to transform them into logarithms which shall have a different base.

Let a denote the base of an existing system of logarithms,
a', the base of a new system,
x' the log. of any number y, in system a',
then $a'^{x'}=y$.

Taking the logarithms of both members of this equation in the system a,

$$\text{Log. }(a'^{x'})=\log.\ y \text{ or } x' \log.\ a'=\log.\ y.$$

$$\therefore \frac{x'}{\log.\ y}=\frac{1}{\log.\ a'}=\text{a constant quantity.}$$

Now x' is the logarithm of y in the system a' and $\log.\ y$ is the logarithm of y in the system a; therefore whatever be the number y there exists a constant ratio between its two logarithms taken in the two systems.

When logarithms are deduced from progressions, this consequence is presented by the progressions themselves.

Let $\div\div\ h : hq : hq^2 : hq^3 : hq^4 : \ldots.$ be any geometrical progression; the logarithms of the different terms are,

$$\text{Log. } h.\ \log.\ h+\log.\ q,\ \log.\ h+2\log.\ q,\ \log.\ h+3\log.\ q, \text{ \&c.}$$

Therefore if numbers are in geometrical progression, their logarithms are in arithmetical progression.

Making $h=1$, the two progressions become,

$$\div\div\ 1 : q : q^2 : q^3 : q^4 : \ldots.$$
$$\div\ 0\ .\ \log.\ q\ .\ 2\log.\ q\ .\ 3\log.\ q\ .\ 4\log.\ q \ldots,$$

which correspond with the geometrical and arithmetical series of Article 219.

OF TABLES OF LOGARITHMS.

224. The base of the tabular logarithms is 10.

Some tables contain the logarithms of numbers from 1 to 10000, others from 1 to 108000 . . . ., the logarithms being variously given to 5, 6, or 7 decimal figures.

The problems to be worked with the tables are two:

1st. To find the logarithm of a given number.

2d. To find the number corresponding to a given logarithm.

The resolution of these problems depends on the arrangement of the tables employed, which is far from uniform. The arrangement, however, and manner of using any particular set of tables are generally explained in an introduction, to which it seems best to refer the reader.

225. Applications of logarithms.

a. Multiplication. Log. $(abc\ldots)=\log.\,a+\log.\,b+\log.\,c+\ldots.$

Ex. Let $a=497{\cdot}86$, $b=76{\cdot}897$, $c={\cdot}67948$; required the product abc?

$$\begin{array}{rl}
\left.\begin{array}{rl}
\text{Log. } 497{\cdot}86 & =2{\cdot}6971072 \\
\text{log. } 76{\cdot}897 & =1{\cdot}8859094 \\
\text{log. } {\cdot}67948 & =\bar{1}{\cdot}8321767
\end{array}\right\} & \text{add.} \\
\hline
\text{log. } 26013{\cdot}17=4{\cdot}4151933 &
\end{array}$$

The sum of the tenths is $24=2{\cdot}4$.

The sum of the characteristics is $2+1-1=2$. Whence, since the 2 carried from the sum of the decimal parts of the logarithms is positive, the characteristic of the log. of the product is $2+2=4$; the log. of the product is $4{\cdot}4151933$, and the product nearly $26013{\cdot}17$.

b. Division. Log. $(a\div b)=\log.\,a-\log.\,b$.

Let $a=87946$, $b=9842$.

$$\begin{array}{rl}
\left.\begin{array}{rl}
\text{Log. } 87946 & =4{\cdot}9442161 \\
\text{log. } 9842 & =3{\cdot}9930834
\end{array}\right\} & \text{subtract.} \\
\hline
\text{log. } 8{\cdot}935784=0{\cdot}9511327 &
\end{array}$$

The difference of the logs. of the numbers is $0{\cdot}9511327$, and the number corresponding to this log. is $8{\cdot}935784$, which is the quotient required.

The method of subtraction by arithmetical complements is explained, Part I. Art. 42. The process is the same with decimal fractions (and by consequence with logarithms) as with whole numbers.

Taking the last example as an instance,

$$\begin{array}{rl}
\left.\begin{array}{r}
\text{Log. } 87946=4{\cdot}9442161 \\
\text{The arithmetical complement of the log. of } 9842=6{\cdot}0069166
\end{array}\right\} & \text{add.} \\
\left.\begin{array}{r}
10{\cdot}9511327 \\
10{\cdot}0000000
\end{array}\right\} & \text{subtract.} \\
\text{The log. of the result, as before, is} \quad - \quad - \quad \bar{0}{\cdot}9511327 &
\end{array}$$

Since $\log.\,(a\div b)=\log.\left(\dfrac{a}{b}\right)=\log.\,a-\log.\,b$, the logarithm of a fraction is obtained by subtracting the log. of the denominator from the log. of the numerator.

As an example in which multiplication and division are combined, let it be required to find the product of the fractions $\frac{31}{75}$, $\frac{13}{12}$, and $\frac{47}{48}$?

$$\begin{array}{lr}
\text{Log. } 31 \quad - \quad - & =1{\cdot}4913617 \\
\text{log. } 13 \quad - \quad - & =1{\cdot}1139434 \\
\text{log. } 47 \quad - \quad - & =1{\cdot}6720979 \\
\text{log. } 75\text{, arith. comp.} & 8{\cdot}1249387 \\
\text{log. } 12 \quad - \quad - & 8{\cdot}9208188 \\
\text{log. } 48 \quad - \quad - & 8{\cdot}3187588 \\
\hline
 & 29{\cdot}6419193 \\
\text{for 3 arith. compts. subtract} & 30{\cdot} \\
\hline
\text{log. } {\cdot}438449 \quad - & =\bar{1}{\cdot}6419193
\end{array}$$

Otherwise the logarithms of the numerators may be added into one sum, the logarithms of the denominators into another; and the latter subtracted from the former.

c. Formation of powers. Log. $(a^m) = m \times \log. a$.

Let $a = 25{\cdot}4$, and $m = 3$; then $a^m = (25{\cdot}4)^3$, and $m \times \log. a = 3 \times \log. 25{\cdot}4$?

$$\left.\begin{array}{r} \text{Log. } 25{\cdot}4 = 1{\cdot}4048337 \\ 3 \end{array}\right\} \text{multiply.}$$

$$\text{log. } 16387{\cdot}07 = \underline{4{\cdot}2145011}$$

2d. Ex. Let $a^m = \left(\frac{3}{5}\right)^8$? $\therefore$ log. $(a^m) = 8(\log. 3 - \log. 5)$,

$$\begin{array}{r} \text{Log. } 3 = 0{\cdot}4771213 \\ \text{log. } 5\text{, arith. comp.} = \underline{9{\cdot}3010300} \\ \therefore \log. \frac{3}{5} = \bar{1}{\cdot}7781543 \\ \text{Multiply by exponent } 8, \qquad \underline{\qquad 8} \\ \log. \left(\frac{3}{5}\right)^8 = \underline{\bar{2}{\cdot}2252344} \end{array}$$

But $\bar{2}{\cdot}2252344$ is the log. of ·0167971, which number is, therefore, the 8th power of $\frac{3}{5}$.

When a is a large number, and $m > 2$, tables which give the logarithms of numbers to 6 or 7 figures afford only a rough approximation to the value of a^m. Gardiner's and Callet's tables contain supplementary tables of the logs. of numbers to 20 figures; with the help of these a near approximation can be obtained.

d. Extraction of roots. Log. $(\sqrt[n]{a}) = \frac{\log. a}{n}$.

Let $\sqrt[n]{a} = \sqrt[7]{1162049}$; then $\log. (\sqrt[7]{1162049}) = \frac{\log. 1162049}{7}$?

$$\begin{array}{l} \text{Log. } 1162049 = 6{\cdot}0652254. \\ \tfrac{1}{7} \text{ of this log. is } 0{\cdot}8664607. \end{array}$$

The number corresponding to this quotient is 7·352932;

therefore $\sqrt[7]{1162049} = 7{\cdot}352932$.

2d Ex. Required the value of the expression $x = \sqrt[11]{\frac{13}{27}}$?

Log. $x = \log. (\sqrt[11]{\frac{13}{27}}) = \frac{1}{11}(\log. 13 - \log. 27)$.

$$\left.\begin{array}{r} \text{Log. } 13 \quad - \quad - \quad = 1{\cdot}1139434 \\ \text{log. } 27\text{, arith. comp. } \underline{8{\cdot}5686362} \end{array}\right\} \text{add.}$$

$$\underline{\bar{1}{\cdot}6825796}$$

To render the negative characteristic of this log. divisible by 11 without altering the value of the expression; since $-10+10=0$; if -10 is added to the characteristic, and $+10$ to the mantissa,

$$\bar{1}{\cdot}6825796 = \overline{11} + 10{\cdot}6825796.$$

$$\begin{array}{r} \text{Dividing by 11,} \quad 11)\underline{\overline{11} + 10{\cdot}6825796} \\ \text{the log. of the root} = \bar{1}{\cdot}9711436 \\ \text{and the root, or } x = \underline{\quad {\cdot}9357149 \quad} \end{array}$$

e. Algebraic expressions:

Let $a : b :: c : x$, $ax = bc$, and $x = \frac{bc}{a}$?

$$\therefore \log. x = \log. \left(\frac{bc}{a}\right) = \log. b + \log. c - \log a.$$

Let $a=\cdot0963$, $b=\cdot24958$, $c=\cdot008967$.

$$\begin{array}{lcr}
\text{Log. } a(=\cdot0963),\text{ arith. comp.} & & 11\cdot0163737 \\
\text{log. } b(=\cdot24958) & - \quad - & \bar{1}\cdot3972098 \\
\text{log. } c(=\cdot008967) & - \quad - & \underline{\bar{3}\cdot9526472} \\
\text{log. } x=\cdot0232397 & - \quad - & \underline{\bar{2}\cdot3662307}
\end{array}$$

Hence the log. of a fourth proportional to three given numbers is obtained by adding together the logs. of the second and third terms, and subtracting the log. of the first term from the sum.

2d Ex. Let $x=\dfrac{\sqrt[3]{(a^2-b^2)3a}}{\sqrt{(a+b)\sqrt{cd}}}$?

$$\text{Log. } x=\text{log. } \sqrt[3]{(a^2-b^2)3a}-\text{log.}\sqrt{(a+b)\sqrt{cd}}.$$

But log. $\sqrt[3]{(a^2-b^2)3a}=\frac{1}{3}\text{ log. }[(a^2-b^2)3a]=\frac{1}{3}[\text{log. }(a+b)+\text{log. }(a-b)$ $+\text{log. }3+\text{log. }a]$,

$$\text{and log. }\sqrt{(a+b)\sqrt{cd}}=\tfrac{1}{2}[\text{log. }(a+b)+\tfrac{1}{2}\text{ log. }c+\tfrac{1}{2}\text{ log. }d]\,;$$

$\therefore$ log. $x=\frac{1}{3}[\text{log. }(a+b)+\text{log. }(a-b)+\text{log. }3+\text{log. }a]-\frac{1}{2}[\text{log. }(a+b)+$ $\frac{1}{2}\text{ log. }c+\frac{1}{2}\text{ log. }d]$.

Let $a=60$, $b=15$, $c=16$, $d=9$. The numerical value of x is

$$\text{log. } x=\tfrac{1}{3}[\text{log. }(60+15)+\text{log. }(60-15)+\text{log. }3+\text{log. }60]-\tfrac{1}{2}[\text{log. }(60+15)+\tfrac{1}{2}\text{ log. }16+\tfrac{1}{2}\text{ log. }9],$$

or log. $x=\frac{1}{3}[\text{log }75+\text{log. }45+\text{log. }3+\text{log. }60]-\frac{1}{2}[\text{log. }75+\frac{1}{2}\text{log. }16+\frac{1}{2}\text{log. }9]$.

$$\begin{array}{rcr}
\text{Log. }75=1\cdot8750613 & \qquad & \text{Log. }75=1\cdot8750613 \\
\text{log. }45=1\cdot6532125 & & \tfrac{1}{2}\text{ log. }16=0\cdot6020600 \\
\text{log. }3\ =0\cdot4771213 & & \tfrac{1}{2}\text{ log. }9=0\cdot4771212 \\
\text{log. }60=1\cdot7781513 & & 2)\overline{2\cdot9542425} \\
3)\overline{5\cdot7835464} & & \overline{\underline{1\cdot4771212}} \\
\overline{\underline{1\cdot9278488}} & &
\end{array}$$

Whence log. $x=1\cdot9278488-1\cdot4771212=0\cdot4507276$,
and $x=2\cdot823108$, the value required.

f. Exponential equations. The equation $a^x=b$, in which the unknown quantity is an exponent, is termed an Exponential equation.

By the resolution of the exponential equation $a^x=b$, the logarithm of the number b, in the system whose base is a, is obtained (Art. 221).

Conversely, tables of logarithms afford the means of resolving any exponential equation in which a, b represent numbers:

$$\text{For since } a^x=b \text{ log. }(a^x)=\text{log. } b,$$
$$\text{or } x \text{ log. } a=\text{log. } b,\ \therefore x=\frac{\text{log. } b}{\text{log. } a}.$$

If $a=2$ and $b=10$, $x=\dfrac{1}{\cdot3010300}=3\cdot3219313+$.

$a^x=b$ is an exponential equation of the first order.

$a^{b^x}=c$, $ab^{c^x}=d$, ... are exponential equations of the 2d, 3d, . . . order.

The expression a^{b^x} signifies that b is raised to the power x, and that a is then raised to the power b^x.

$$\text{Since } a^{b^x}=c,\ \text{log. }(a^{b^x})=\text{log. } c\,;$$
$$\text{or } b^x \text{ log. } a=\text{log. } c.$$
$$\text{Therefore log. }(b^x \text{ log. } a)=\text{log.(log. } c)\,;$$
$$\text{or log. }(b^x)+\text{log. log. } a=\text{log. log. } c.$$

Now log. $(b^x)=x$ log. b; and since log. a, log. c are decimal fractions, their logarithms can be taken like the logarithms of any other numbers.

$$\text{Therefore } x \text{ log. } b+\text{log. log. } a=\text{log. log } c\,;$$
$$\text{and } x=\frac{\text{log. log. } c-\text{log. log. } a}{\text{log. } b}.$$

If the value of an algebraic expression is negative, the calculation cannot be made by logarithms without changing the signs of the terms of the quantity, in order that it may be rendered positive, for a negative number has no logarithm.

This appears from these considerations:

1st. The base of a system of logarithms cannot be a negative number, for, the even powers of such a base being positive and the odd powers negative, the whole numbers equal to certain terms of the geometrical series must be some positive and others negative. These whole numbers cannot therefore be consecutive, as is required by the definition.

2d. No power of a positive number can be negative. Consequently the logarithm of a negative number, like the 2d, 4th, $2n$th root of a negative quantity, is an absurd or imaginary expression.

The value of an algebraic expression which is negative can, however, be rendered positive by changing the signs of all the terms. The logarithmic calculation being made on this hypothesis, the result is to be taken with the sign —.

Example. Let $x=\frac{a^2-b^2}{c}$. This expression being negative when b is greater than a, if the signs are changed, the value $x=\frac{b^2-a^2}{c}$ is positive. Taking the logs. of both members, $\log. x=\log. (b+a)+\log. (b-a)-\log. c$, the value of the second member of this equation being calculated, the result is to be taken with the sign —.

g. Geometrical progressions. The calculation of the formulæ of Article 191 is rendered easy by the employment of logarithms. Thus,

Formula 3, or $S=\frac{\sqrt[n-1]{l^n}-\sqrt[n-1]{a^n}}{\sqrt[n-1]{l}-\sqrt[n-1]{a}}$, being reduced to

$$S=a\times\frac{\sqrt[n-1]{\left(\frac{l}{a}\right)^n}-1}{\sqrt[n-1]{\frac{l}{a}}-1};$$

$$\text{then } \log.\left(\sqrt[n\ 1]{\left(\frac{l}{a}\right)^n}\right.=\frac{n}{n-1}(\log. l-\log. a)\text{; (let this }=x\text{);}$$

$$\log.\left(\sqrt[n-1]{\frac{l}{a}}\right)=\frac{1}{n-1}(\log. l-\log. a)\text{; (let this }=y\text{);}$$

Therefore $S=a\times\frac{(x-1)}{(y-1)}$, an expression which is equal to

$$\log. a+\log. (x-1)-\log. (y-1).$$

Next, in the formula $aq^{n-1}=l$, let the exponent n (which denotes the number of terms) be the unknown quantity.

Taking the logarithms of both members of this equation,

$$\log. a+(n-1)\log. q=\log. l.$$

$$\therefore n=\frac{\log. q+\log. l-\log. a}{\log. q};$$

$$\text{or } n=1+\frac{\log. l-\log. a}{\log. q}.$$

Example. Required the number of terms of a geometrical progression, of which the first term $a=3$, the ratio $q=2$, and the last term $l=6144$?

$$x=1+\frac{\log. 6144-\log. 3}{\log. 2}=1+\frac{3{\cdot}7884512-{\cdot}4771213}{{\cdot}3010300}=1+11=12.$$

The number of terms of the geometrical progression is 12.

h. Compound interest. A given principal is lent for a certain time at a fixed rate of interest, and subject to the condition that the interest of each year shall be combined with the principal of the preceding year; it is required to find the amount at the expiration of the given time?

Let p=the principal, t the number of years, r' the interest of 1 £. for 1 year, $\left(r'=\frac{r}{100}\right)$; and let the amount be denoted by A.

Since the interest of 1 £. for 1 year is r', the interest of p £. for 1 year is pr'; therefore at the end of the first year p becomes $p+pr'=p(1+r')$.

Let $p(1+r')=p'$. This new principal at the end of the second year becomes $p'(1+r')$. But $p'=p(1+r')$. Therefore at the end of the second year p has become $p(1+r')(1+r')=p(1+r')^2$.

In like manner, at the end of the third year, the amount is $p(1+r')^3$; and at the expiration of the time t it is $p(1+r')^t$.

$$\text{Therefore } A=p(1+r')^t.$$

Replacing r' by $\frac{r}{100}$, this formula is that found in Article 357, Part I.

When the value of t is considerable, the calculation of A from this formula without the aid of logarithms is laborious. Employing logarithms,

$$\log. A=\log. p+t\log. (1+r').$$

Example. Let $p=500$ £., $r'=\cdot04$ £., $t=3$ years; required A?

$$1+r'=1+\cdot04=1\cdot04;\ t\log. (1+r')=3\times\log. 1\cdot04.$$

$$\begin{aligned}\text{Log. } 500&=2\cdot6989700\\ 3\times\log. 1\cdot04=\cdot0170333\times3&=\underline{0\cdot0510999}\\ \log. 562\cdot432&=\underline{\underline{2\cdot7500699}}\end{aligned}$$

562·432 £. =562 £. 8 sh. $7\frac{2}{3}$ d.=the amount required.

The expression $A=p(1+r')^t$ involves four quantities, of which each may be considered in turn as an unknown quantity, and its value deduced from the other three.

1st. p, r', t, may be given to find A.
2d. A, r', t, may be given to find p.
3d. A, p, t, may be given to find r'.
4th. A, p, r', may be given to find t.

1st. The first case has been already resolved.

2d. Given A, r', t, to find p?

$$\text{Since } A=p(1+r')^t,\ p=\frac{A}{(1+r')^t},$$

$$\text{and } \log. p=\log. A-t\times\log (1+r').$$

Example. What is the present worth of 1000 £. payable in 4 years, allowing compound discount at 5 per cent. per annum?

$$A=1000,\ t=4,\ 1+r'=1\cdot05.$$

$$\begin{aligned}\left.\begin{aligned}\text{Log. } 1000&=3\cdot0000000\\ 4\times\log. 1\cdot05=\cdot0211893\times4&=0\cdot0847572\end{aligned}\right\}&\text{ subtract.}\\ \log. 822\cdot7025=\underline{\underline{2\cdot9152428}}&\end{aligned}$$

The present worth is 822 £. 14 sh. 0 d. 2·4 qr.

3d. Given p, A, t, to find r'?

$$\text{Since } A=p(1+r')^t,\ 1+r'=\sqrt[t]{\frac{A}{p}},$$

$$\text{therefore } \log. (1+r')=\frac{\log. A-\log. p}{t}.$$

This equation gives $1+r'$, and therefore r' and r.

Example. At what rate per cent. compound interest will 500 £. augment to 1000 £. in 20 years?

$$A=1000,\ p=500,\ t=20.$$

$$\begin{aligned}\left.\begin{aligned}\text{Log. } 1000&=3\cdot0000000\\ \log. 500&=2\cdot6989700\end{aligned}\right\}&\text{ subtract.}\\ 20)\underline{0\cdot3010300}&\\ \log. 1\cdot0327=\underline{\underline{0\cdot0155150}}&\end{aligned}$$

Therefore $r'=\cdot0327$ and $r'\times100=r=3\cdot27$=rate per cent.

4th. Given A, p, r', to find t?

$$\text{Since } A=p(1+r')^t,$$
$$\log. A=\log. p+t\times\log(1+r');$$
$$\therefore t=\frac{\log. A-\log. p}{\log. (1+r')}.$$

Example. In what time will 500 £. be doubled at 5 per cent. per annum compound interest?

$$p=500,\quad A=1000,\quad 1+r'=1{\cdot}05.$$
$$\left.\begin{array}{r}\text{Log. } 1000=3{\cdot}0000000\\ \text{log. } 500=\underline{2{\cdot}6989700}\end{array}\right\}\text{subtract.}$$
$$\therefore \text{Log. } A-\log. p=\underline{\cdot 3010300}$$
$$\text{Log. } 1{\cdot}05=\ \cdot 0211893.$$
$$\therefore \frac{\log. A-\log. p}{\log. (1+r')}=\frac{3010300}{211893}=14{\cdot}2\text{ years}=t.$$

When it is required that A shall be double of p, as in the example, or triple of p, ... or equal to m times p, the formula $A=p(1+r')^t$ becomes $mp=p(1+r')^t$, or $m=(1+r')^t$.

$$\therefore t=\frac{\log. m.}{\log. (1+r')}.$$

Whence, in this case, the value of t is independent of the principal.

It is required to find what sum must be invested to entitle the lender to receive a fixed annuity, a, such that the capital (or sum invested), the interest of the capital, and the interest of the interest shall be repaid at the expiration of t years, the rate of interest being r' per annum for 1 £. or r £. per cent?

Let C denote the capital invested; at the end of t years the amount is $C(1+r')^t$.

Again, the amount of a £. paid is

at the end of the 1st year $a(1+r')^{t-1}$;
at the end of the 2d year $a(1+r')^{t-2}$;
. . . .
at the end of the t years a.

Therefore the sum of the values of the yearly payments is

$$a(1+r')^{t-1}+a(1+r')^{t-2}+\ldots\ldots+a.$$

This sum (reversing the order of the terms) is that of a geometrical progression whose first term is a, ratio $1+r'$, and number of terms t; and which is equal to

$$\frac{a(1+r')^t-a}{(1+r')-1}=\frac{a[(1+r')^t-1]}{r'}\quad\text{(Art. 189).}$$

But, by the question, this sum is also equal to $C(1+r')^t$

$$\text{therefore } C(1+r')^t=\frac{a[(1+r')^t-1]}{r'}.$$

This formula, like the preceding, contains four quantities, C, a, r', t, any three of which being given the fourth can be found.

1st. a, r', t may be given to find C.
2d. C, r', t - - - - a.
3d. C, a, t - - - - r'.
4th. C, a, r' - - - - t.

1st. If a, r', t are given to find C,

$$C=\frac{a[(1+r')^t-1]}{r'(1+r')^t}.$$

Taking the logarithms of both members of this equation,

$$\text{Log. } C=\log. a+\log. [(1+r')^t-1]-\log. r'-t\log. (1+r').$$

Example: What sum must be paid to purchase an annuity of 100£. for twenty years, the interest of money being four per cent.?

In this example $a=100$, $1+r'=1\cdot04$, $t=20$.

$$\begin{array}{rr}
\text{Log. } 1\cdot04= & 0\cdot0170333 \\
 & 20 \\ \hline
20\times\text{log. } 1\cdot04=2\cdot19112= & \underline{\underline{0\cdot3406660}} \\
\therefore\ (1+r')^t-1=2\cdot19112-1= & 1\cdot19112 \\
\text{Log. } 100= & 2\cdot0000000 \\
\text{log. } 1\cdot19112=\text{log. } [(1+r')^t-1]= & 0\cdot0759555 \\
\text{log. } \cdot04 \text{ arith. comp.}= & 11\cdot6020600 \\
\text{log. } 2\cdot19112=\text{log. } (1+r')^t \text{ arith. comp.}= & 9\cdot6593340 \\ \hline
\text{log. } 2174\cdot45= & \underline{\underline{3\cdot3373495}}
\end{array}$$

The sum required is 2174£. 9 sh.

2d. Given C, r', t, to find a?

$$\text{Since } \frac{a[(1+r')^t-1]}{r'}=C(1+r')^t;$$

$$\therefore\ a=\frac{Cr'(1+r')^t}{(1+r')^t-1};$$

$$\text{and log. } a=\text{log. } C+\text{log. } r'+t\times\text{log. } (1+r')-\text{log. } [(1+r')^t-1].$$

Example: Let $C=1000$£., $r'=\cdot05$£., $t=10$ years. Required a?

$$1+r'=1\cdot05.$$

$$\text{Log. } (1+r')^t=10\times\text{log. } 1\cdot05=0\cdot2118930.$$

Now $0\cdot2118930$ is the logarithm of $1\cdot6289$;

therefore $(1+r')^t=(1\cdot05)^{10}=1\cdot6289$, and $(1+r')^t-1=0\cdot6289$.

$$\begin{array}{rr}
\text{Log. } 1000 \text{ or } C= & 3\cdot0000000 \\
\text{log. } \cdot05 \text{ or } r'= & \bar{2}\cdot6989700 \\
\text{log. } 1\cdot6289 \text{ or } t\times\text{log. } (1+r')= & 0\cdot2118930 \\
\text{log. } \cdot6289 \text{ or log. } [(1+r')^t-1] \text{ arith. comp.}= & 10\cdot2014184 \\ \hline
\text{log. } 129\cdot554= & \underline{\underline{2\cdot1122814}}
\end{array}$$

Whence the annuity is 129£. 11·08 sh.

3d. Given C, a, t, to find r'?

The equation $C(1+r')^t=\dfrac{a[(1+r')^t-1]}{r'}$ is reducible to

$$a(1+r')^t-Cr'(1+r')^t=a, \text{ or to } (1+r)^t(a-Cr)=a,$$

from which the value of r' cannot be determined otherwise than by the resolution of an equation of the order $t+1$.

As a particular case, let $t=2$; then

$$(1+r')^2(a-Cr')=a, \text{ or } (1+2r'+r'2)(a-Cr')=a;$$

this is a cubic equation.

4th. Given C, a, r', to find t?

From $C(1+r')^t=\dfrac{a[(1+r')^t-1]}{r'}$ is obtained $(1+r')^t(a-Cr')=a$;

$$\text{therefore } t\times\text{log. } (1+r')+\text{log. } (a-Cr')=\text{log. } a,$$

$$\text{and } t=\frac{\text{log. } a-\text{log. } (a-Cr')}{\text{log. } (1+r')}.$$

If the sum a£. is invested at the beginning of each year for t consecutive years, and all the investments bear compound interest at the rate of r'£. per pound or r£. per cent., what is the amount of the accumulation in t years?

The amount of a£. for t years at r' per pound compound interest is $a(1+r')^t$; the amount of a£. for $t-1$ years at r' per pound compound interest is $a(1+r')^{t-1}$, &c.; and so on to the last investment a, which bears interest for one year only, and of which the interest therefore is $a(1+r')$.

Denoting the sum of these accumulations by S,

$$S=a(1+r')^t+a(1+r')^{t-1}+a(1+r')^{t-2}+\ldots\ldots+a(1+r')\text{; or}$$

$$S=a(1+r')[1+(1+r')+(1+r')^2+\ldots\ldots+(1+r')^{t-1}]\text{; or}$$

$$S=\frac{a(1+r')[(1+r')^t-1]}{r'}.$$

From this relation any three of the quantities S, a, r', t being given, the fourth can be found.

Example 1st. Suppose that the yearly investment $a=50$ £., that $r'=\cdot03$, and that $t=20$ years; required S?

$$\text{Log. } S=\text{log. } a+\text{log. } (1+r')+\text{log. } [(1+r')^t-1]-\text{log. } r'.$$
$$=\text{log. } 50+\text{log. } 1\cdot03+\text{log. } 0\cdot8011-\text{log. } 0\cdot03.$$
$$=\text{log. } 1375\cdot221. \quad \therefore S=1375\text{ £. } 4\text{ sh. } 5\cdot04\text{ d.}$$

Example 2d. What must a person whose expectation of life is ten years pay yearly to secure 1000 £. to his heirs at his death, the rate of interest being five per cent.?

$$\text{Log. } a=\text{log. } S-\text{log. } (1+r')-\text{log. } [(1+r')^t-1]+\text{log. } r'.$$
$$=\text{log. } 1000-\text{log. } 1\cdot05-\text{log. } 0\cdot62885+\text{log. } 0\cdot05.$$
$$=\text{log. } 75\cdot724. \quad \therefore a=75\text{ £. } 14\text{ sh. } 5\text{ d. } 3\cdot04\text{ qrs.}$$

Several sums of money being payable at different times, it is required to find what sum must be paid at a given time so as to be equivalent to these other sums?

Let a £. be payable at the end of m years,
b be payable at the end of n years.
x the unknown sum which is to be paid in t years to discharge the debts a, b.

By a preceding part of this Article the present worth of a £. payable in m years at r' per cent. is $\frac{a}{(1+r')^m}$;

the present worth of b, in like manner, is $\frac{b}{(1+r')^n}$;

and the present worth of x is $\frac{x}{(1+r')^t}$.

Whence, by the question, $\frac{x}{(1+r')^t}=\frac{a}{(1+r')^m}+\frac{b}{(1+r')^n}$,

$$\text{or } x(1+r')^{-t}=a(1+r')^{-m}+b(1+r')^{-n}.$$
$$\therefore x=a(1+r')^{t-m}+b(1+r')^{t-n}.$$

Additional examples in compound interest, &c.:

1st. What is the amount of 721 £. for 21 years at 4 per cent. per annum compound interest?..........................*Ans.* 1642 £. 19 sh. 9½ d.

2d. What is the compound interest of 410 £. for 2½ years at 4½ per cent. per annum, the interest being payable half-yearly?
Ans. 48 £. 4 sh. 11¼ d.

3d. The sum of 400 £. was lent at compound interest; in 9 years the amount was 569 £. 6 sh. 8 d. Required the rate per cent.?
Ans. 4 per cent.

4th. In what time will a sum of money triple itself at 4 per cent. compound interest?....................................*Ans.* In 28·011 years.

5th. What is the amount of 1000 £. for 1 year at 5 per cent. per annum compound interest, the interest being payable daily?
Ans. 1051 £. 19 sh.

6th. What is the present value of an annuity of 500 £. to last 40 years, compound interest being allowed at the rate of 2½ per cent. per annum?...*Ans.* 12540 £.

7th. What is the present worth of 3750 £. payable in 6½ years, reckoning 4 per cent. discount and also compound interest?
Ans. 2963 £. 13 sh. 4 d.

8th. What is the present value of an annuity of 350£. assigned for 8 years at 4 per cent. interest ?..................*Ans.* 2356£. 9sh. 2d.

9th. The number of settlers in a colony being 20000; if the yearly rate of increase of the population is $\frac{1}{15}$ part, how many inhabitants will the colony contain in 100 years ?................*Ans.* 14489276.

10th. A town contains 20000 inhabitants; the population having regularly increased $\frac{3}{100}$ yearly, what was the number of inhabitants 10 years ago ?..*Ans.* 14882.

SECTION XI.

OF THE COMPOSITION OF EQUATIONS.

226. The only equations of which complete solutions have been obtained are those of the first and second degree. But general properties which lead to the resolution of numerical equations of any degree whatever have been discovered. These properties are connected with a particular form to which every equation can be reduced.

An equation of any degree, m, involving one unknown quantity, ought, when complete, to contain all the powers of the unknown quantity from that of the degree m to the first inclusive, each being multiplied by known quantities, and, besides these, a term containing only known quantities. The equation of the fourth degree, for example, ought to contain all the powers of x from the fourth to the first inclusive. If the same power of x is found in several terms, they must be reduced into one. Then all the terms of the equation being transposed to the left member, and the sign of the first term being rendered + by the change, if necessary, of the signs of all the other terms, the equation has the form,

$$nx^4+px^3+qx^2+rx+s=0,$$

in which the letters p, q, r, s may represent negative as well as positive numbers.

Dividing all the terms by n, in order that the coefficient of the first term may be 1, and making $\frac{p}{n}=\mathrm{P}$, $\frac{q}{n}=\mathrm{Q}$, $\frac{r}{n}=\mathrm{R}$, $\frac{s}{n}=\mathrm{S}$, the preceding equation becomes

$$x^4+\mathrm{P}x^3+\mathrm{Q}x^2+\mathrm{R}x+\mathrm{S}=0.$$

In the following pages it is always supposed that equations are reduced to this form.

Hence the general equation of any degree, m, may be represented as follows:

$$x^m+\mathrm{P}x^{m-1}+\mathrm{Q}x^{m-2}+\ .\ .\ .\ .\ .\ +\mathrm{T}x+\mathrm{U}=0,$$

in which the interval marked by points is filled up when a particular value is given to the exponent m.

For the sake of brevity, the first member of the equation $x^m+\mathrm{P}x^{m-1}+$ &c.$=0$ is sometimes denoted by a capital X.

Every expression, whether real or imaginary, which, being substituted for x in an equation prepared in this manner, renders its first member equal to zero, and by consequence satisfies the equation, is said to be a root of the equation. The meaning of the word root, in this definition, is more general than when it is employed to describe the result of the extraction of some root of a quantity, a.

227. The root of any equation of the form $\mathrm{X}=0$ being represented by a, the first member of this equation is exactly divisible by the binomial $x-a$.

For, since a is a value of x, if a is substituted for x in the general equation $\mathrm{X}=0$,

$$a^m+\mathrm{P}a^{m-1}+\mathrm{Q}a^{m-2}\ .\ .\ .\ .\ .\ +\mathrm{T}a+\mathrm{U}=0,$$

and, consequently, $\mathrm{U}=-a^m-\mathrm{P}a^{m-1}-\mathrm{Q}a^{m-2}\ .\ .\ .\ .\ -\mathrm{T}a,$

so that the proposed equation is identically the same as

$$\left.\begin{array}{l} x^m+Px^{m-1}+Qx^{m-2}\ \ldots\ldots\ldots\ +Tx\\ -a^m-Pa^{m-1}-Qa^{m-2}\ \ldots\ldots\ldots\ -Ta\end{array}\right\}=0,$$

which is reducible to

$$x^m-a^m+P(x^{m-1}-a^{m-1})+Q(x^{m-2}-a^{m-2})\ \ldots\ldots\ +T(x-a)=0.$$

The quantities x^m-a^m, $x^{m-1}-a^{m-1}$, $x^{m-2}-a^{m-2}$, ... $x-a$, being all divisible by $x-a$ (Art. 22), it is evident that the first member of the equation $x^m-a^m+P(x^{m-1}-a^{m-1})+\ \ldots\ldots\ +T(x-a)=0$ is divisible by this quantity; and that the first member of the proposed equation $X=0$ is therefore divisible by $x-a$, which was to be proved.

To form the quotient it is sufficient to substitute for the quantities x^m-a^m, $x^{m-1}-a^{m-1}$, ... $x-a$, the quotients which are obtained from the division of these quantities by $x-a$, and which are, respectively,

$$\begin{array}{r} x^{m-1}+ax^{m-2}+a^2x^{m-3}\ \ldots\ldots\ +a^{m-1},\\ x^{m-2}+ax^{m-3}\ \ldots\ldots\ +a^{m-2},\\ x^{m-3}\ \ldots\ldots\ +a^{m-3},\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ +1.\end{array}$$

Arranging the results according to the powers of x,

$$\left.\begin{array}{r|r|lr} x^{m-1}+a & x^{m-2}+a^2 & x^{m-3}\ldots\ldots\ldots & +a^{m-1}\\ +P & +Pa & & +Pa^{m-2}\\ & +Q & & +Qa^{m-3}\\ & & & \ldots\ldots\\ & & & +T\end{array}\right\}=0.$$

Denoting $a+P$ by P', a^2+Pa+Q by Q', &c. this equation (which is of the degree $m-1$) can be put under the form $x^{m-1}+P'x^{m-2}+Q'x^{m-3}+$, &c. $=0$, or $X'=0$.

Whence $x^m+Px^{m-1}+Qx^{m-2}+$, &c. $=(x-a)(x^{m-1}+P'x^{m-2}+Q'x^{m-3}+$, &c.$)$, and the proposed equation can be verified in two ways, namely, by making $x-a=0$, or by making $x^{m-1}+P'x^{m-2}+Q'x^{m-3}+$, &c. $=0$.

If the equation $x^{m-1}+P'x^{m-2}+Q'x^{m-3}+$, &c. $=0$ has a root, b, its first member is divisible by $x-b$, and there is found as before,

$$x^{m-1}+P'x^{m-2}+Q'x^{m-3}+,\ \&c.\ =(x-b)(x^{m-2}+P''x^{m-3}+Q''x^{m-4}+,\ \&c.)$$

Consequently $x^m+Px^{m-1}+Qx^{m-2}+$, &c. $=(x-a)(x-b)(x^{m-2}+P''x^{m-3}+Q''x^{m-4}+$, &c.$)$

The equation $X=0$ can therefore be verified by

$$x-a=0,\ x-b=0,\ x^{m-2}+P''x^{m-3}+Q''x^{m-4}+,\ \&c.\ =0.$$

If the last of these equations has a root, c, its first member can be again decomposed into two factors, $x-c$, $x^{m-3}+P'''x^{m-4}+Q'''x^{m-5}+$, &c. and

$$x^m+Px^{m-1}+Qx^{m-2}+,\ \&c.\ =(x-a)(x-b)(x-c)(x^{m-3}+P'''x^{m-4}+Q'''x^{m-5},\ +\&c.)$$

Whence the proposed equation can be verified in four manners; namely, by making in turn $x-a=0$, $x-b=0$, $x-c=0$, $x^{m-3}+P'''x^{m-4}+Q'''x^{m-5}+$, &c. $=0$.

By continuing this process of decomposition, factors of the degrees $m-4$, $m-5$, $m-6$, &c. are obtained in succession; and if each of these factors, when made equal to zero, is susceptible of a root, the first member of the equation $X=0$ is reduced to the form $(x-a)(x-b)(x-c)(x-d)\ldots(x-l)$; that is, it is decomposed into as many factors of the first degree as there are *ones* in the exponent m of its degree.

The equation $x^m+Px^{m-1}+Qx^{m-2}+\ \ldots\ +Tx+U=0$ is, in this case, capable of verification in m ways, namely, by making $x-a=0$, $x-b=0$, $x-c=0$, $x-d=0$, ... $x-l=0$.

It is necessary to remark, that these equations ought not to be regarded as true, except in turn, and that to suppose them to have place at the same time leads to manifest contradictions.

For from $x-a=0$ is obtained $x=a$, and from $x-b=0$ is obtained $x=b$, consequences which cannot be reconciled when a, b are unequal quantities.

In effect x, in each of these equations, represents the unknown quantity, as it may do in any two unrelated equations whose roots, although equal to x, are not equal to each other.

228. The first member of the equation $x^m+Px^{m-1}+Qx^{m-2}+\ldots Tx+U=0$, being decomposed into m factors of the first degree, $x-a$, $x-b$, $x-c$, $x-d$, $\ldots\ldots x-l$, it cannot be divided by any other expression of the first degree; for if its division by a binomial $x-\alpha$, different from all of these, were possible, then, as in the case of division by $x-a$,

$$x^m+Px^{m-1}+Qx^{m-2}+, \text{\&c.} =(x-\alpha)(x^{m-1}+px^{m-2}+qx^{m-3}+, \text{\&c.})$$

and consequently

$$(x-a)(x-b)(x-c)(x-d)\ldots(x-l)=(x-\alpha)(x^{m-1}+px^{m-2}+qx^{m-3}+, \text{\&c.}).$$

Changing x into α in this expression,

$$(\alpha-a)(\alpha-b)(\alpha-c)(\alpha-d)\ldots(\alpha-l)=(\alpha-\alpha)(\alpha^{m-1}+p\alpha^{m-2}+q\alpha^{m-3}+, \text{\&c.})$$

Now the factor $\alpha-\alpha=0$, and therefore the second member of this equation vanishes; but this is not the case with the first member, which is the product of factors all different from zero, so long as α differs from each of the roots a, b, c, d, $\ldots\ldots l$; the supposition, therefore, is not true. Therefore an equation of any degree whatever cannot have more binomial factors of the first degree than there are *ones* in the exponent of its degree, and, consequently, it cannot have a greater number of roots.

In this investigation it is assumed that every algebraic equation has at least one root, real or imaginary. If some of the quantities a, b, $c\ldots\ldots$ are equal to each other, the number of different values of x is less than m. The degree of the equation is, however, always taken to indicate the number of roots, it being at the same time understood that two or more of these roots may be equal.

229. If an equation is considered as the product of factors $x-a$, $x-b$, $x-c$, $x-d\ldots\ldots$, in number equal to the exponent of its degree, it has the form of the product of Article 139, with this difference, that its terms are alternately positive and negative.

Taking four factors, $x-a$, $x-b$, $x-c$, $x-d$, the equation $(x-a)(x-b)(x-c)(x-d)=0$ is expressed by,

$$\begin{array}{r|r|r|l}
x^4-a & x^3+ab & x^2-abc & x+abcd=0. \\
-b & +ac & -abd & \\
-c & +ad & -acd & \\
-d & +bc & -bcd & \\
 & +bd & & \\
 & +cd & &
\end{array}$$

Hence, the second terms of the binomials $x-a$, $x-b$, $x-c$, $x-d$, &c. being the roots of the equation taken each with a contrary sign, the properties proved in the case of the equation of the second degree subsist in the present instance, as follows:

The coefficient of the second term, taken with a contrary sign, is equal to the sum of the roots.

The coefficient of the third term is equal to the sum of the products of the roots multiplied together two by two.

The coefficient of the fourth term, taken with a contrary sign, is equal to the sum of the products of the roots multiplied together three by three; and so on for the other coefficients, care being taken to change the signs of the coefficients of the even rank.

The last term (subject to the law which attaches to the sign of the terms of an even rank) is equal to the product of all the roots.

By making equal to zero the product of the three factors $x-5$, $x+4$, $x+3$, the equation $x^3+2x^2-23x-60=0$ is formed, the roots of which are $+5$, -4, -3.

The sum of these roots is $5+(-4)+(-3)=-2$.

The sum of their products, two by two, is

$$(+5)\times(-4)+(+5)\times(-3)+(-4)\times(-3)=-20+(-15)+12=-23,$$

and their product is $(+5)\times(-4)\times(-3)=+60$.

These results, -2, -23, $+60$, can be deduced from the coefficients

$$2,\ -23,\ -60$$

by changing the signs of the 2d and 4th terms.

If the product of the factors $x-2$, $x-3$, $x+5$, is made equal to zero, the equation obtained is $x^3-19x+30=0$, which wants the second term or term in x^2, and this because the sum of the roots, which taken with contrary signs form the coefficient of the 2d term, is $2+3-5=0$; that is, because the sum of the positive is equal to the sum of the negative roots.

230. An equation being considered as formed of the product of many factors of the first degree, it has been proved that the equation cannot have more simple factors than the number marked by the exponent of its degree. But if the simple factors $x-a$, $x-b$, &c. are combined two by two, quantities of the second degree are formed, which are also factors of the equation, and of which the number is expressed by the formula $\frac{m(m-1)}{1.2}$, so that if the equation is of the fourth degree the number of factors of the second degree is $\frac{4\times3}{2}$, or 6.

If the simple factors are combined three by three, the number of factors of the third degree is expressed by the formula $\frac{m(m-1)(m-2)}{1.2.3}$, which, for $m=4$, is $\frac{4\times3\times2}{1\times2\times3}=4$.

231. $$x^m+Px^{m-1}+Qx^{m-2}+\ldots\ldots+Tx+U=0,$$

being an equation which has a root a, if the equation is divided by $x-a$, the quotient is

$$\left.\begin{array}{r|r|lr} x^{m-1}+a & x^{m-2}+a^2 & x^{m-3}\ldots\ldots & +a^{m-1} \\ +P & +Pa & & +Pa^{m-2} \\ & +Q & & +Qa^{m-3} \\ & & & \cdot\quad\cdot\quad\cdot \\ & & & +T \end{array}\right\}=0.$$

This equation, which is depressed to the degree $m-1$, must contain the remaining roots of the proposed equation.

The coefficients of the depressed are formed from the coefficients of the proposed equation, as follows:

The coefficient of the first term in the depressed is equal to the coefficient of the first term in the original equation.

The coefficient of the second term is equal to the coefficient of the first term multiplied by the root a, plus the coefficient of the second term of the original equation.

The coefficient of the third term is equal to the coefficient of the second term of the depressed equation multiplied by the root a, plus the coefficient of the third term of the original equation.

The coefficient of the mth term of the depressed equation is equal to a times the coefficient of the preceding or $(m-1)$th term of the depressed equation, plus the coefficient of the mth term of the original equation.

One root of the proposed equation must be given when the depressed equation is required.

1st example. Let a root of the equation, $15x^5-19x^4+6x^3+15x^2-19x+6=0$, be $\cdot6$; required the depressed equation, or the equation containing the other roots?

The coefficient of the first term being 15, $15x^4$ is the first term of the depressed equation.

$15\times\cdot6+(-19)=9-19=-10$, $\therefore$ $-10x^3$ is the second term,
$-10\times\cdot6+6=-6+6=0$, $\therefore$ $0\times x^2$, or 0, is the third term,
$0\times\cdot6+15=0+15=15$, $\therefore$ $15x$ is the fourth term,
$15\times\cdot6+(-19)=9-19=-10$, $\therefore$ -10 is the last term,
and the depressed equation is $15x^4-10x^3+15x-10$.

The formation of the coefficients of the depressed from the coefficients of the proposed may be represented concisely, thus:

Coefficients of the proposed equation	15,	−19,	+6,	+15,	−19
Corrections - -		+9,	−6,	+0,	+9
Coefficients of the depressed equation	15,	−10,	0,	+15,	−10

2d example. One root of the equation

$$x^3-3x^2-10x+24=0$$

being 2, it is required to find the equation in x^2 involving the other roots?

Ans. $x^2-x-12=0$.

3d example. If 3 is a root of the equation

$$x^4-11x^2+14x-24=0,$$

what is the depressed equation containing the other roots?

Ans. $x^3+3x^2-2x+8=0$.

4th example. One root of the cubic equation

$$x^3-7x^2+36=0$$

is 3, what are the other roots?

Ans. 6 and −2.

SECTION XII.

OF THE TRANSFORMATION OF EQUATIONS.

232. In the transformation of equations the object is to change one equation into another whose roots shall have a given relation to the roots of the first equation.

The principal cases of transformation are,

1st. To change the signs of the roots of a given equation.

This amounts to the finding of an equation whose roots are those which would be obtained by changing the signs of the roots of the given equation; therefore if x is the unknown quantity of the given equation and y that of the transformed equation, the relation $x=-y$ must subsist between them. Whence it is necessary to replace x by $-y$ in the given equation; for it is evident that if a quantity, a, is a root of one of the equations, the quantity $-a$ must be a root of the other.

Let the given equation be

$$x^m+Px^{m-1}+Qx^{m-2}+\text{, \&c.}=0 \quad - \quad - \quad - \quad - \quad \text{(A).}$$

If $-y$ is substituted for x the absolute values of the coefficients of the equation in y are evidently the same as those of the equation in x, but the signs of the coefficients of the odd powers are contrary. If m is even the first term of the transformed equation is y^m, but if m is odd the first term is $-y^m$. By changing the signs of all the terms of the transformed equation (which is permitted) the first term is rendered positive; in this case the coefficients of the even powers of the proposed and transformed equations have contrary signs. For example, if the proposed equation is

$$x^3-5x^2-5x+1=0,$$

the transformed equation, which has the same roots, but of contrary signs, is

$$x^3+5x^2-5x-1=0.$$

233. 2d. To form an equation whose roots shall be the reciprocals of the roots of a given equation.

$$x^m+Px^{m-1}+Qx^{m-2}\ldots\ldots+Tx+U=0 \quad - \quad - \quad - \quad \text{(A).}$$

Two quantities whose product is 1 are reciprocals each of the other; hence, if the roots of equation A are a, b, c, ... the roots of the transformed equation must be $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ ..... Consequently it is sufficient in A to change x

into $\frac{1}{x}$. By this means the equation becomes

$$\frac{1}{x^m}+\frac{P}{x^{m-1}}+\frac{Q}{x^{m-2}}+\ldots\ldots+\frac{T}{x}+U=0;$$

or, multiplying by x^m, dividing by U, and reversing the order of the terms,

$$x^m+\frac{T}{U}x^{m-1}+\ldots.+\frac{Q}{U}x^2+\frac{P}{U}x+\frac{1}{U}=0.$$

234. 3d. To multiply the roots of a given equation by any quantity whatever.

Let the quantity be n, and the given equation

$$x^m+Px^{m-1}+Qx^{m-2}\ldots..+\frac{T}{x}+U=0;$$

the problem requires that the roots of the transformed equation shall be the products which are obtained by multiplying all the roots of equation A by n. Therefore, y being the new unknown quantity, $y=nx$ or $x=\frac{y}{n}$.

Consequently equation A is changed into

$$\frac{y^m}{n^m}+\frac{Py^{m-1}}{n^{m-1}}+\frac{Qy^{m-2}}{n^{m-2}}\ldots.\frac{Ty}{n}+U=0.$$

It is evident that the substitution of na for y in this equation gives the same result as the substitution of a for x in equation A; so that a cannot be a root of the equation in x without the product na being a root of the equation in y.

Multiplying all the terms of the transformed equation by n^m,

$$y^m+Pny^{m-1}+Qn^2y^{m-2}\ldots.+Tn^{m-1}y+Un^m=0 \quad - \quad - \quad (B).$$

If the coefficients of the proposed and transformed equations are compared, it becomes evident that the latter are formed by multiplying the former by n^0, n^1, n^2, &c. respectively, so that in each term the sum of the exponents of n and y is equal to m.

This transformation comprehends that in which it is proposed to divide the roots by a number, n; for this amounts to the multiplication of the roots by $\frac{1}{n}$, and by consequence to the division of the coefficients of the given equation by n^0, n^1, n^2, &c.

1st example: Multiply the roots of the equation $x^4-12x^3+17x^2-9x+7=0$ by 2.

In this example $n^0=1$, $n^1=2$, $n^2=4$, $n^3=8$, $n^4=16$;

and the transformed equation is $x^4-24x^3+68x^2-72x+112=0$.

2d example: Divide the roots of the equation $x^3+3x^3-12x+432$ by 6?

$n^0=1$, $n^1=\frac{1}{6}$, $n^2=\frac{1}{36}$, $n^3=\frac{1}{216}$;

and the transformed equation is $x^3+\frac{1}{2}x^2-\frac{1}{3}x+2=0$.

235. 4th. To transform an equation which has fractional coefficients into another of which the coefficients shall be whole numbers, and that of the first term 1.

Leaving the quantity n indeterminate, let $nx=y$, and let the transformation of the last Article be effected.

The coefficients P, Q, &c. being fractional, let a common multiple, M, of the denominators be formed; then in equation B take $n=M$; the coefficients Pn, Qn^2, &c. will thus be rendered integer.

In practice it is sufficient to take the least common multiple of the denominators.

Example: It is required to transform the equation $x^3-\frac{3x^2}{2}+\frac{5x}{4}-\frac{2}{9}=0$ into an equation whose coefficients are integer, and that of the first term 1.

The least common multiple of the denominators is $4\times9=36$. Making $n=36$ in formula B,

$$x^3-\frac{3\times36\times x^2}{2}+\frac{5\times36^2\times x}{4}-\frac{2\times36^3}{9}=0;$$

$$\text{or } x^3-54x^2+1620x-10368=0.$$

236. 5th. To augment or diminish the roots of a given equation by a given quantity.

Let the given quantity be a, and the given equation

$$x^m+Px^{m-1}+Qx^{m-2}\ldots\ldots+Tx+U=0 \quad - \quad - \quad (A).$$

Making $x=y+a$, $y=x-a$; consequently, by substituting $y+a$ for x in equation A, a transformed equation is obtained, of which the roots are those of equation A diminished by a.

In order that the roots of equation A may be augmented by the quantity a, it is necessary to change a into $-a$.

Effecting the substitution of $y+a$ for x, equation A becomes

$$(y+a)^m+P(y+a)^{m-1}+Q(y+a)^{m-2}\ldots\ldots+T(y+a)+U=0.$$

Developing the expressions $(y+a)^m$, $(y+a)^{m-1}$, &c.

$$\left.\begin{array}{r|l|l}
y^m+ma & y^{m-1}+\frac{m(m-1)}{2}a^2 & y^{m-2}\ldots\ldots+a^m \\
+P & \quad +(m-1)Pa & \qquad\qquad\quad +Pa^{m-1} \\
 & \quad +Q & \qquad\qquad\quad +Qa^{m-2} \\
 & & \qquad\qquad\quad \ldots\ldots \\
 & & \qquad\qquad\quad \ldots\ldots \\
 & & \qquad\qquad\quad +Ta \\
 & & \qquad\qquad\quad +U
\end{array}\right\}=0 \quad - \quad - \quad (C).$$

If it is required that the exponents of y shall increase from left to right, it is necessary to substitute $a+y$ instead of $y+a$ in equation A. The result is

$$A+A'y+\tfrac{1}{2}A''y^2+\frac{1}{2.3}A'''y^3\ldots\ldots+y^m=0.$$

Making, for abridgment,

$$A=a^m+Pa^{m-1}+Qa^{m-2}\ldots\ldots+Ta+U.$$
$$A'=ma^{m-1}+(m-1)Pa^{m-2}+(m-2)Qa^{m-3}\ldots\ldots+T.$$
$$A''=m(m-1)a^{m-2}+(m-1)(m-2)Pa^{m-3}+(m-2)(m-3)Qa^{m-4}\ldots\ldots$$
$$A'''=m(m-1)(m-2)a^{m-3}+(m-1)(m-2)(m-3)Pa^{m-4}+\ldots\ldots$$
$$\text{\&c. \&c.}$$

The quantities A, A', A'' .... are connected with the proposed equation $X=0$ by a very simple law.

A is formed by writing a instead of x, in the first member X.

A' is formed from A by multiplying each term of A by the exponent of a in that term, and diminishing the exponent of a by 1. As the last term is equivalent to Ua^0, it must vanish in the operation, being multiplied by zero.

A'' is formed in the same manner from A', A''' from A'', &c.

The quantities A', A'', A''' .... are termed Derived Polynomials. A' is said to be the derived polynomial of A, A'' of A'..... A' is also designated the first derived polynomial of A, A'' the second, &c.

Example: Let it be required to substitute $a+y$ for x in the equation

$$x^3-2x^2-4x+5=0.$$
$$A=a^3-2a^2-4a+5.$$
$$A'=3a^2-4a-4.$$
$$A''=6a-4.$$
$$A'''=6.$$

Consequently, taking care to divide A'' by 2, and A''' by 2×3, the transformed equation is

$$(a^3-2a^2-4a+5)+(3a^2-4a-4)y+(3a-2)y^2+y^3=0.$$

If a is a root of the equation $X=0$, this equation is divisible by $x-a$, and the depressed equation is of the form

$$x^{m-1}+P'x^{m-2}+Q'x^{m-3}\ldots\ldots+T'=0.$$

The coefficients P′, Q′, &c. can be formed from the coefficients P, Q, &c. of the proposed equation, and from each other, by a process explained in Article 231.

If a is not a root, the proposed equation is not exactly divisible by $x-a$. Effecting the division as far as possible, the result, which is of the same degree (m) as the proposed equation, is of the form

$$x'^m+\mathrm{A}^{\mathrm{M}}x'^{m-1}+\mathrm{A}^{\mathrm{M}-1}x'^{m-2}\ .\ .\ .\ .\ .\ +\mathrm{A}'x'+\mathrm{A}=0,$$

in which $x'=x-a$; that is, in which the roots are those of the proposed equation diminished by a, and the coefficients are the Mth, M−1th, ... second, first derived polynomials of A.

If the proposed equation is numerical, the coefficients of the equation in x' can be obtained from the formulæ

$$\mathrm{A}=a^m+\mathrm{P}a^{m-1},\ \&c.;\ \mathrm{A}'=ma^{m-1}+(m-1)\mathrm{P}a^{m-2},\ \&c.\ \&c.$$

For example, let it be required to transform the equation

$$x^4+5x^3+4x^2+3x-105=0$$

into another equation whose roots are less by 2 than the roots of the proposed equation.

The transformed equation is of the form

$$x'^4+\frac{\mathrm{A}'''}{2.3}x'^3+\frac{\mathrm{A}''}{2}x'^2+\mathrm{A}'x'+\mathrm{A}=0.$$

$$\mathrm{A}=2^4+5\times2^3+4\times2^2+3\times2-105=16+40+16+6-105=-27.$$

$$\mathrm{A}'=4\times2^3+3\times5\times2^2+2\times4\times2+3\times1=32+60+16+3=111.$$

$$\frac{\mathrm{A}''}{2}=\frac{3\times4\times2^2+2\times3\times5\times2+2\times4}{2}=\frac{48+60+8}{2}=\frac{116}{2}=58.$$

$$\frac{\mathrm{A}'''}{2.3}=\frac{2\times3\times4\times2+2\times3\times5}{2\times3}=\frac{48+30}{6}=\frac{78}{6}=13.$$

Therefore the transformed equation is

$$x'^4+13x'^3+58x'^2+111x'-27=0.$$

The following rule, founded on observation of the law according to which the coefficients of the transformed equation are composed of the coefficients of the proposed equation and the quantity a which expresses the difference between x and y, is given by Mr. Horner in his Method for the General Resolution of Numerical Equations:

"1st. To the first coefficient add the correction 0, and to each succeeding coefficient add a times the amount of the preceding coefficient and its correction; proceed thus with all the coefficients in order till the absolute term is corrected.

"2d. Apply a similar process to the series of *corrections*, and then to those which are deduced from them; and so on, as often as it can be done, stopping each process one term short of the preceding.

"3d. Add up each column of additional corrections, together with the once-corrected coefficient under which it stands; and the sum will be the new coefficients, standing vertically under those given ones for which they are to be substituted."

The transformation of the last example is effected by this rule as follows:

$$\begin{array}{rrrrr}
1, & +\ 5, & +\ 4, & +\ \ 3, & -105 \\
0, & +\ 2, & +14, & +\ 36, & +\ 78 \\
\hline
1, & +\ 7, & +18, & +\ 39, & -\ 27 \\
 & +\ 2, & +18, & +\ 72 & \\
 & +\ 2, & +22 & & \\
 & +\ 2, & & & \\
\hline
1, & +13, & +58, & +111, & -\ 27 \\
\hline\hline
\end{array}$$

The last results taken in order are the coefficients of the transformed equation, which, as already found, is,

$$x'^4+13x'^3+58x'^2+111x'-27=0.$$

Explanation. In this example, $a=2$.

The first line of the calculation contains the coefficients of the proposed equation and the term wholly known.

The second and third lines are formed in this manner.

0 is the correction of the first coefficient, $1+0=1$. This 1 is the first term of the third line.

This result 1, multiplied by $2=2$. This 2 is written under the second coefficient, 5, to which it is added. The sum $+7$ is written in the third line, of which it is the second term.

This 7 is multiplied by 2; the product 14 is written under the third coefficient, $+4$, to which it is added. The sum $+18$ is the third term of the third line.

This 18 is multiplied by 2; the product 36 is written under the fourth coefficient, $+3$, to which it is added. The sum 39 is the fourth term of the third line.

This 39 is multiplied by 2; and the product 78 is written under the absolute term. The sum -27 is the fifth and last term of the third line.

The fourth line is formed from the third and second thus; 1, the first term of the third line, multiplied by $2=2$. This 2 is written under the second term of the third line.

This 2 multiplied by $2=4$; the product 4 is added to 14, the third term of the second line; and the sum 18 is written under the third term of the third line.

This 18 is multiplied by 2; and the product 36 added to 36, the fourth term of the second line. The sum 72 is written under the fourth term of the third line.

The fifth line is formed from the fourth; thus,

1 (which in this example may be considered as the first term of each line) multiplied by $2=2$. The 2 is written under the second term of the fourth line.

This 2 is multiplied by 2; and the product 4, being augmented by 18, the third term of the fourth line, the sum 22 is written under the third term of the fourth line.

The sixth line is formed from the fifth, as the fifth from the fourth, and so on. The sum of the third, fourth, fifth, sixth . . . lines is contained in the last line.

The numbers in the last line are the coefficients of the transformed equation.

2d Example. It is required to transform the equation,

$$x^5-8x^4+11x^3-39x^2+31x-101=0,$$

into an equation whose roots shall be less by 4 than the roots of the proposed equation?

$$\begin{array}{rrrrrr}
1, & -8, & +11, & -39, & +31, & -101 \\
0, & +4, & -16, & -20, & -236, & -820 \\
\hline
1, & -4, & -5, & -59, & -205, & -921 \\
 & 4, & +0, & -20, & -316 & \\
 & 4, & +16, & +44 & & \\
 & 4, & +32 & & & \\
 & 4 & & & & \\
\hline
1, & +12, & +43, & -35, & -521, & -921 \\
\hline
\end{array}$$

The transformed equation is

$$x'^5+12x'^4+43x'^3-35x'^2-521x'-921.$$

3d Example. It is required to transform the equation,

$$2x^4-13x^2+10x-19=0,$$

into an equation whose roots shall be less by 3 than the roots of the proposed equation?

In this example the coefficient of the term in x^3 is 0, and $a=3$.

$$\begin{array}{rrrrr}
2, & +0, & -13, & +10, & -19 \\
0, & +6, & +18, & +15, & +75 \\
\hline
2, & +6, & +5, & +25, & +56 \\
 & 6, & +36, & +123 & \\
 & 6, & +54 & & \\
 & 6 & & & \\
\hline
2, & +24, & +95, & +148, & +56
\end{array}$$

The transformed equation is,

$$2x'^4+24x'^3+95x'^2+148x'+56=0.$$

4th Example. It is required to transform the equation,

$$6x^3-3x^2+4x-1=0,$$

into another equation, whose roots shall be greater by 3 than the roots of the proposed equation?

In this example, $a=-3$.

$$\begin{array}{rrrr}
6, & -3, & +4, & -1 \\
0, & -18, & +63, & -201 \\
\hline
6, & -21, & +67, & -202 \\
 & -18, & +117 & \\
 & -18 & & \\
\hline
6, & -57, & +184, & -202
\end{array}$$

The transformed equation is,

$$6x'^3-57x'^2+184x'-202=0.$$

By making the additions as well as the multiplications mentally, and recording only the results, the calculation of the coefficients of a transformed equation, by this method, may be abridged in appearance. The process for the second example (which is the longest) may be abridged as follows:

$$\begin{array}{rrrrrr}
1, & -8, & +11, & -39, & +31, & -101 \\
 & -4, & -5, & -59, & -205, & -921 \\
 & 0, & -5, & -79, & -521 & \\
 & 4, & +11, & -35 & & \\
 & 8, & +43 & & & \\
 & 12 & & & &
\end{array}$$

The first line as before contains the coefficients of the proposed equation; the second line is formed as follows from the first.

$1\times4=4$; $4+-8=-4$; this -4 is written under the second term of the first line.

$-4\times4=-16$; $-16+11=-5$; this -5 is written under the third term of the first line.

$-5\times4=-20$; $-20+-39=-59$; this -59 is written under the fourth term.

$-59\times4=-236$; $-236+31=-205$; this -205 is written under the fifth term.

$-205\times4=-820$; $-820+-101=-921$; this -921 is written under the last term.

The third line is formed from the second, the fourth from the third, &c., in the same manner.

When the calculation is made in this manner, the last terms of the horizontal lines are the coefficients of the transformed equation.

237. 6th, to transform an equation into another which shall want a certain term.

Let the equation be $x^m+Px^{m-1}+Qx^{m-2}\ldots\ldots+Tx+U=0$ (A).

Make $x=y+a$; y being a new unknown quantity, and a an indeterminate quantity which may be disposed of at pleasure.

By this substitution is obtained the equation distinguished by the mark C (Art. 236); if it is required to make the second term disappear, it is necessary to determine a by the equation $ma+P=0$, or $a=-\frac{P}{m}$.

Then the value $x=y+a$ becomes $x=y-\frac{P}{m}$.

Observing that, in this expression, P is the coefficient of the second term of equation A, and m is the exponent of the highest power of x, it follows that the second term of an equation is made to vanish by substituting, for the unknown quantity of that equation, a new unknown quantity, to which is added the coefficient of the second term of the equation, taken with a contrary sign, and divided by the degree of the equation.

The third term of equation A is made to disappear by assuming

$$\tfrac{1}{2}m(m-1)a^2+(m-1)Pa+Q=0.$$

From this equation two values of a can be obtained; consequently there are two different quantities by the substitution of which an equation can be transformed into another which shall be deprived of its third term.

To make the fourth term vanish, it is necessary to resolve an equation of the third degree, and so on to the last term, which (to make it disappear) requires the determination of a from an equation like that proposed; namely,

$$a^m+Pa^{m-1}+Qa^{m-2} \;\ldots\ldots\; +Ta+U=0.$$

The reason is this: to equal to zero the last term of the equation in y is to suppose that one of the values of y is zero; for when this equation has no last term, it is evidently verified by making $y=0$; consequently the relation $x=y+a$ becomes $x=a$; therefore for a may be taken any of the values of x, therefore a ought to be determined by the same equation as x.

When one term of an equation is made to vanish it may happen that other terms may disappear at the same time; but that this may happen it is necessary that there be certain relations between the coefficients of the proposed equation.

Let it, for example, be required to find the relation which must subsist among the coefficients of an equation in order that the second and third terms may vanish together?

It is evidently necessary (Eq. C) that

$$ma+P=0 \text{ and } \tfrac{1}{2}m(m-1)a^2+(m-1Pa)+Q=0.$$

The first equation gives $a=-\frac{P}{m}$. It is necessary that this value satisfy the second equation, or that $\frac{m(m-1)P^2}{2m^2}-\frac{(m-1)P^2}{m}+Q=0$, or that $Q-\frac{(m-1)P^2}{2m}=0$, which is the relation required.

If the value of Q, deduced from this expression, is substituted in equation A, the general form of equation, whose second and third terms ought to disappear together, will be obtained.

It is not possible, by successive transformations, to make first one term vanish, then a second, then a third, &c., for each operation would reproduce the term destroyed by the preceding.

Taking the equation $x^m+Qx^{m-2}+$, &c.$=0$, which wants the second term, the substitution of $y+a$ for x in this equation gives

$$(y+a)^m+Q(y+a)^{m-2}+, \&c. =0.$$

Developing this equation, the result is

$$y^m+may^{m-1}+[\tfrac{1}{2}m(m-1)a^2+Q]y^{m-2}+, \&c. =0.$$

Now it is evident that the value of a, which in this equation makes $\frac{1}{2}m(m-1)a^2+Q=0$, must be different from 0, consequently the term in y^{m-1} must be reproduced by the process employed for the removal of the third term.

1st Example. It is required to transform the equation,

$$x^2+px=q, \text{ or } x^2+px-q=0,$$

into another which shall want the second term, assuming $x=y-\frac{p}{2}$?

$$\begin{array}{rl} x^2=\left(y-\frac{p}{2}\right)^2= & y^2-py+\frac{p^2}{4}, \\ px=p\left(y-\frac{p}{2}\right)= & \qquad py-\frac{p^2}{2}, \\ -q= & \underline{\qquad\qquad\quad -q} \end{array}$$

Whence the transformed equation is $\underline{y^2 \quad * \quad \frac{-p^2}{4}-q}$.

Resolving this equation, the values of y are,

$$y=\pm\sqrt{\frac{p^2}{4}+q}.$$

Consequently the corresponding values of x are

$$x=-\frac{p}{2}\pm\sqrt{\frac{p^2}{4}+q}.$$

2d Example. It is required to take away the second term of the equation,

$$x^3-9x^2+26x-34=0,$$

assuming $x-3=y$, or $y+3=x$?

$$\begin{array}{rr} x^3=(y+3)^3= & y^3+9y^2+27y+27 \\ -9x^2=-9(y+3)^2= & -9y^2-54y-81 \\ 26x=26(y+3)= & +26y+78 \\ -34= & \underline{\qquad\qquad\qquad -34} \\ & \underline{y^3 \quad * \quad - \quad y-10=0} \end{array}$$

Whence $y^3-y-10=0$ is the transformed equation required.

3d Example. It is required to take away the second term of the equation, $x^3+6x^2-3x+4=0$?

Ans. The transformed equation is $y^3-15y+26=0$.

4th Example. It is required to take away the second term of the equation, $x^4-12x^3+17x^2-9x+7=0$?

Ans. The transformed equation is $y^4-37y^2-133y-110=0$.

238. The success of certain transformations can be made evident by means of the composition of equations.

Admitting that the equation $X=0$ decomposed into factors is

$$(x-a)(x-b)(x-c) \;.\;.\;.\;.\; =0,$$

if x is replaced by $-x$, and the signs of all the factors are afterwards changed, the result is

$$(x+a)(x+b)(x+c) \;.\;.\;.\; =0;$$

an equation, the roots of which are evidently equal to those of the proposed equation, but of contrary signs.

This is the transformation of Article 232.

If in the equation $(x-a)(x-b)(x-c) \;.\;.\;.\;.\; =0$, x is changed into $\frac{1}{x}$,

$$\left(\frac{1}{x}-a\right)\left(\frac{1}{x}-b\right)\left(\frac{1}{x}-c\right) \;.\;.\;.\;.\; =0,$$

multiplying the factors respectively by $\frac{x}{a}$, $\frac{x}{b}$, $\frac{x}{c}$, &c., and changing the sign of each factor, the result is

$$\left(x-\frac{1}{a}\right)\left(x-\frac{1}{b}\right)\left(x-\frac{1}{c}\right) \;.\;.\;.\;.\; =0;$$

an equation which has the roots $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, . . . as required in Article 233.

If again x is made equal to $\frac{y}{n}$, and each factor is multiplied by n (which amounts to the multiplication of every term by n^m) the result is

$$(y-na)(y-nb)(y-nc)\ .\ .\ .\ .=0,$$

an equation of which the roots are na, nb, nc, . . .

This transformation is that of Article 234.

If $x=x'+a'$ the equation $x(-a)(x-b)(x-c)\ .\ .\ .\ .=0$ becomes $(x'+a'-a)(x'+a'-b)(x'+a'-c)\ .\ .\ .\ .=0$, which has evidently for roots $a-a'$, $b-a'$, $c-a'$, ... that is to say, the roots of the proposed equation, each diminished by a'.

When, to make the term Px^{m-1} of the equation $X=0$ to vanish, x is made equal to $y-\frac{P}{m}$, the roots of the transformed equation are

$$a+\frac{P}{m},\ b+\frac{P}{m},\ c+\frac{P}{m}\ .\ .\ .\ .$$

Now $a+b+c$ . . . to m terms $=-P$.

$\therefore\ a+\frac{P}{m}+b+\frac{P}{m}+c+\frac{P}{m}$ . . . . to m terms$=a+b+c$ . . . to m terms $+$ $\frac{P}{m}+\frac{P}{m}+\frac{P}{m}$ . . . . to m terms,

$$=-P+m\times\frac{P}{m}=-P+P=0;$$

therefore the coefficient of the second term of the transformed equation is equal to zero, or the equation in y has no second term.

In these instances a very simple relation exists between the original unknown quantity x and the new unknown quantity y or x'; but whatever the nature of this relation the process to be followed is always the same; the relation is first to be expressed by an equation; then the value of x is to be deduced in terms of y; then, lastly, this value is to be substituted in the proposed equation.

The equation of relation between x and y may be too complicated to be resolved with regard to x. In this case it becomes necessary to eliminate x between the equation of relation and the proposed equation $X=0$.

SECTION XIII.

OF ELIMINATION AND ITS APPLICATIONS.

239. Two equations of the first degree, containing two unknown quantities, being given, the elimination of one of the unknown quantities is easily accomplished; and the final equation (that is, the equation which gives the other unknown quantity) is also of the first degree. But such is not the case when the equations are of a degree superior to the first.

Let there be between the unknown quantities x, y two complete equations, one of the degree m, the other of the degree n.

These two equations can always be reduced to the form

$$x^m+Px^{m-1}+Qx^{m-2}\ .\ .\ .\ .+Tx+V=0\quad -\quad -\quad A,$$
$$x^n+P'x^{n-1}+Q'x^{n-2}\ .\ .\ .\ .+T'x+V'=0\quad -\quad -\quad B.$$

In which P, P′ are respectively of the form $a+by$, $a'+b'y$;

Q, Q′ . . . . . $c+dy+ey^2$, $c'+d'y+e'y^2$;

R, R′ . . . . . $f+gy+hy^2+ky^3$, $f'+g'y+h'y^2+k'y^3$;

&c. &c.;

and the terms V, V′ contain y to the powers m, n respectively.

To resolve equations A, B is to assign all the pairs of numbers which, being substituted for x, y, reduce A, B at the same time to the form $0=0$. These numbers are termed roots or solutions of the proposed equation.

The principle of the resolution of these equations may be presented under two different points of view.

1st. Let α, β be values of x, y, such that being substituted simultaneously in the equation A, B, these equations are reduced to $0=0$; then it is evident, that if β is first substituted for y, and in the results (which contain only x), x is replaced by α, the same result $0=0$ is obtained.

But since the polynomials which form the first members of the equations A, B ought, after the substitution of β for y, to become zero at the same time for $x=\alpha$, they must have the form,

Equation A of $Q(x-\alpha)$,
Equation B of $Q'(x-\alpha)$.

Therefore the substitution of β for y introduces into the equations A, B a common factor, $x-\alpha$, the equalling of which to 0 gives the solution $x=\alpha$ corresponding to the solution $y=\beta$.

This is a property of every root y, and every value of y which does not possess it ought to be rejected as incapable of forming part of the solutions.

2d. Let it be supposed that the equations A, B are resolved separately, with regard to x (y being taken, for the time, as a known quantity); let the roots of equation A be represented by a, a', a'', &c.; the roots of equation B by α, α', α'', ...; and let the equations be put under the form

$$\text{A of } (x-a)(x-a')(x-a'') \;.\;.\;.\;.\; =0,$$
$$\text{B of } (x-\alpha)(x-\alpha')(x-\alpha'') \;.\;.\;.\;.\; =0.$$

If now any one of the roots a, a', a'' . . . is made equal to any one of the roots α, α', α'', . . . (as, for example, a to α,) the result is an equation in y which will give for y one or more values β, β', β'', &c., such that, being substituted in A, B, these equations will acquire a common divisor, $x-a$, or $x-\alpha$; and the value $y=\beta$ deduced from $a=\alpha$, with the corresponding value $x=a=\alpha$, (which becomes a number by the substitution of β for y) will simultaneously reduce the equations A, B to the form $0=0$.

Then, if β, one of the values of y, is substituted in the two polynomials A, B, the coefficients P, Q, R . . . P′, Q′, R′ . . . will be changed into numbers, which may be expressed by (P), (Q), (R) . . . (P′), (Q′), (R′) . . ., and the resulting equations,

$$x^m+(P)x^{m-1}+(Q)x^{m-2} \;.\;.\;.\;.\; +(T)x+(V)=0 \;.\;.\;.\;.\;.\; C,$$
$$x^n+(P')x^{n-1}+(Q')x^{n-2} \;.\;.\;.\;.\; +(T')x+(V')=0 \;.\;.\;.\;.\;.\; D.$$

containing only x, will be satisfied by $x=a$, or $x=\alpha$; that is to say, they will acquire a common divisor of the first degree $x-a$. Seeking, therefore, the common measure of the polynomials which form the first members of equations C, D, and making it equal to zero, the result obtained is $x=a=\alpha$, a being a number.

In the same manner is found another value of x, corresponding to the second value β' of y; and so on for other values.

240. When, therefore, two proposed polynomials are operated upon as if to find their common measure $x-a$ (a being a quantity which depends on y, or a function of y) and a remainder is found which involves y only, it is this remainder which, for all the values β, β', β'' . . . of y, becomes zero. Making the remainder equal to zero, the last divisor becomes the greatest common measure of the proposed polynomials. Let $R=0$ be the equation formed by making the remainder in y equal to zero; the values $y=\beta$, β', β'' . . . are obtained by resolving this equation, and the corresponding values of x by substituting successively the values β, β', β'' . . . of y in the last divisor (or remainder which is of the first degree in x, and which may be represented by $Ax-B$, A, B being functions of y), and making, after each substitution, $Ax-B=0$ a condition from which the proposed polynomials become zero. By this means numbers α, α', α'' . . . are obtained as values of

x, which, taken with $\beta, \beta', \beta'', \ldots$ the values of y, satisfy the proposed equations.

It is to be observed that for the two proposed equations between x and y two others are substituted, the one in y only, the other in x and y.

Example 1. Let $x^3+3yx^2+3y^2x-98=0$ - - - - 1,

$x^2+4yx-2y^2-10=0$ - - - - - 2.

$$x^2+4yx-2y^2-10)\ x^3+3yx^2+3y^2x-98\ (x-y$$

$$\underline{x^3+4yx^2-2y^2x-10x}$$

$$-yx^2+5y^2x+10x-98$$

$$\underline{-yx^2-4y^2x+2y^3+10y}$$

1st rem. - - - $9y^2x+10x-2y^3-10y-98$

or $(9y^2+10)x-2y^3-10y-98$

$$x^2+4yx-2y^2-10$$
$$\underline{9y^2+10}$$

$$(9y^2+10)x-2y^3-10y-98)\ (9y^2+10)x^2+(36y^3+40y)x-18y^4-110y^2-100\ (x$$
$$\underline{(9y^2+10)x^2-(2y^3+10y+98)x}$$

$$(38y^3+50y+98)x-18y^4-110y^2-100$$
$$\text{or } (19y^3+25y+49)x-9y^4-55y^2-50$$
$$\underline{9y^2+10}$$

$$(9y^2+10)x-2y^3-10y-98)\ (9y^2+10)(19y^3+25y+49)x-81y^6-585y^4-1000y^2-500\ (19y^3+25y+49$$
$$\underline{(9y^2+10)(19y^3+25y+49)x-38y^6-240y^4-1960y^3-250y^2-2940y-4802}$$

$$-43y^6-345y^4+1960y^3-750y^2+2940y+4302$$

Making this remainder equal to zero, and rendering the first term positive, the final equation is

$$43y^6+345y^4-1960y^3+750y^2-2940y-4302=0,$$

and the common measure of the first degree, or the equation which gives the values of x, is

$$(9y^2+10)x-2y^3-10y-98=0.$$

241. The same final equation and the same common divisor may belong to an indefinite number of systems of two equations of the same degree or of different degrees. If it is proposed to form two equations which shall have $x-y$ for common divisor of the first degree, and $y^2-1=0$ for final equation. To the product of $x-y$ by any factor which is a function of x, or of x and y (and, for example, of the first degree in x), let y^2-1 be added, the result is a polynomial of the second degree; to the product of this polynomial by a function of x or of x and y let the polynomial divisor of the first degree be added, and so on. Any two whatever of these consecutive polynomials are of different degrees; and, under the same degree, their composition may be varied at pleasure. But any two of them, being made equal to zero, will admit the same solutions, and in the same number, as a necessary consequence of the law of their formation.

242. The accuracy of the preceding method may be established by this demonstration:

Let A, B be the proposed polynomials,
R, R′ . . . . the successive remainders,
Q, Q′ . . . . the successive quotients.

Then $A=BQ+R$ - - - - - 1,

$B=RQ'+R'$ - - - - - 2,

$R=R'Q''+R''$ - - - - 3.

. . . .

It follows from the first equality that each of the systems of values of x, y which gives $A=0, B=0$, must also give $R=0$, and, reciprocally, that the solutions in x, y of $B=0, R=0$, are also solutions of $A=0, B=0$. Therefore the question, to find the roots common to $A=0, B=0$, is reduced to this other; to find the roots which satisfy the system of the two equations $B=0$, $R=0$, in one of which x or y is raised to a degree less by one than in the polynomials A, B.

By the same reasoning it is proved that the roots common to $B=0$, $R=0$ are the same as those which satisfy the pairs $R=0$, $R'=0$; $R'=0$, $R''=0$, &c. As the degree of the divisor is diminished at each operation, two equations must of necessity be obtained; the one in x, y, the other in x only or y only, and of which the pairs of common values are precisely the same, and in the same number as those of the proposed polynomials. The last, containing one unknown quantity, will, therefore, be the final equation required.

The last remainder but one being the common divisor of not only the polynomials A, B, but of all the preceding remainders also, it follows that the systems of roots which result from the equality to zero of the two last remainders satisfy all the other remainders as well as the proposed polynomials A, B. Whence it also follows that the same final equation and the same common divisor may belong to an indefinite number of equations; since with A, B others may be formed composed of A, B, as A, B are themselves by means of the remainders and quotients.

243. If this method of elimination is compared with that which is followed in the resolution of equations of the first degree, containing several unknown quantities, the analogy of the two is evident.

Let there be, for example, two equations of the first degree, these may be always put under the form $x+p=0$, $x+q=0$.

But substituting in the one the value of x taken from the other, $p-q$ is obtained for final equation. Now this is the condition under which the two quantities $x+p$ and $x+q$ admit a common divisor in x, as may be readily shown by division.

In general the different processes of elimination, which may be employed to effect the resolution of equations involving many unknown quantities, have this common object, to substitute for the proposed system of equations another system perfectly equivalent, and to decompose the difficulty which results from the simultaneous existence of many unknown quantities in all the equations compared together, by reducing successively their resolution to that of more simple equations, of which the last involves only one unknown quantity, and which may be resolved exactly, or at least by the known methods of approximation.

244. If the value of y, deduced from the final equation, causes the remainder of the first degree in x to vanish, the preceding remainder in x^2 becomes the common measure of the proposed polynomials. Substituting the value of y in this remainder, and making it equal to zero, two values of x are obtained corresponding to the same value of y.

But if the value of y causes the remainder in x^2 also to vanish, the remainder in x^3 is the common measure of the proposed polynomials, and three values of x correspond to the same value of y.

And, in general, if the value of y causes the remainder in x^{n-1} to vanish, the remainder in x^n is the common measure required, and n values of x correspond to the same value of y.

If the last division is exact the two equations have a common measure, without any determination of y; they are therefore of the form $Q\times D=0$; $Q'\times D=0$; D being the common measure.

In this case both equations are satisfied by making $D=0$; and if D contains x and y the equation $D=0$ gives the means of determining one of the quantities x, y in terms of the other.

But if D contain only x or y, then x or y is determinate, and y or x altogether indeterminate.

By making, next, $Q=0$, $Q'=0$, two equations are obtained, which may give determinate solutions of the proposed equations. If, for example,

$$(ax+by-c)(mx+ny-d)=0\text{; and }(a'x+b'y-c')(mx+ny-d)=0,$$

by making the common factor $mx+ny-d=0$, one equation only is given to determine the two quantities x, y, and the question seems indeterminate; but if the common factor is suppressed the equations

$$ax+by-c=0\,;\ a'x+b'y-c'=0$$

are obtained, which must give a determinate solution.

245. If the final equation is independent of y, or contains only known quantities, which do not destroy each other, the proposed equations involve contradictory conditions.

To render this more evident let two equations in x, y be,

$$yx^3-(y^3-3y-1)x+y=0\,;\ x^2-y^2+3=0.$$

Dividing the first by the second, and the divisor by the remainder as follows:

$$\begin{array}{ll}
 & x^2-y^2+3)yx^3-(y^3-3y-1)x+y(xy \\
 & \qquad\qquad \underline{yx^3-(y^3-3y)x\qquad\quad} \\
\text{1st remainder} \quad - & \qquad - \quad - \quad - \quad \underline{\underline{x+y}} \\
 & \qquad x+y)x^2-y^2+3(x-y \\
 & \qquad\qquad \underline{x^2+xy\qquad\quad} \\
 & \qquad\qquad\quad -xy-y^2+3 \\
 & \qquad\qquad\quad \underline{-xy-y^2\qquad} \\
\text{2d remainder} \quad - & \qquad - \qquad\quad - \quad \underline{\underline{+3}}
\end{array}$$

The final equation is $+3$, and the common measure $x+y$.

Denoting the first quotient xy by Q, and the second $x-y$ by Q′; therefore,

$$x^2-y^2+3=(x+y)(x-y)+3=(x+y)Q'+3.$$

$$yx^3-(y^3-3y-1)x+y=[x^2-y^2+3]xy+x+y=[(x+y)Q'+3]Q+x+y.$$

Hence it appears that one of the proposed equations is formed from the other by multiplying that other by xy or Q, and adding $x+y$ to the product, and consequently that the two equations involve contradictory conditions; also the final equation indicates that if the number 3 were not contained in the second equation, the sum $x+y$ would be the common measure of the two polynomials.

Consequently, a system of two equations in x, y cannot give values of x, y, when the final equation is numerical.

246. It is necessary to observe that the quantities by which the partial dividends are multiplied to render the divisions possible often introduce into the last remainder factors which are foreign to the question, and which render this remainder not the true final equation.

To avoid the error arising from these factors the most obvious precaution is to substitute in the proposed equations each of the values given by the equation in y, then all the values which communicate a common measure to these equations necessarily belong to the question, and the others ought to be excluded.

It is also evident that the final equation may be rendered incomplete by the suppression, during the process, of any factor in y.

The following method is free from these inconveniences.

Let equation A or $x^m+Px^{m-1}+Qx^{m-2}\ \ldots\ldots\ +Tx+V$ be supposed equal to

$$(x-a)(x^{m-1}+Ax^{m-2}+Bx^{m-3}+,\ \&c.)\qquad - \quad - \quad C;$$

and equation B or $x^n+P'x^{n-1}+Q'x^{n-2}\ \ldots\ +T'x+V'$ to

$$(x-a)(x^{n-1}+A'x^{n-2}+B'x^{n-3}+,\ \&c.)\qquad - \quad - \quad D;$$

also let equation A be multiplied by $x^{n-1}+A'x^{n-2}+B'x^{n-3}$, &c.

and equation B be multiplied by $x^{m-1}+Ax^{m-2}+Bx^{m-3}$, &c.

it is evident that the products must be equal; therefore,

$$(x^m+Px^{m-1}+Qx^{m-2}+,\ \&c.)(x^{n-1}+A'x^{n-2}+B'x^{n-3}+,\ \&c.)=(x^n+P'x^{n-1}+Q'x^{n-2}+,\ \&c.)(x^{m-1}+Ax^{m-2}+Bx^{m-3}+,\ \&c.)\qquad - \quad - \quad E.$$

Performing the multiplications and making equal to each other, the coefficients of the same powers of x (Art. 174) $m+n-1$ equations are obtained between the indeterminate quantities A, B, C, ... A′, B′, C′, ... Now the number of indeterminate quantities in equation C is $m-1$, and in equation D, $n-1$; therefore the number in equation E is $m+n-2$. Of the $m+n-1$ equations $m+n-2$ suffice to determine A, B, C, . . . A′, B′, C′, ... ; and one equation remains between P, Q, R . . . . P′, Q′, R′ . . . . which it is necessary to satisfy in such a manner that the equations C, D may have a common divisor $x-a$; this equation of condition is the final equation required.

Since none of the indeterminate quantities A, B, C . . . A′, B′, C′ . . . . is multiplied by itself, the equations by means of which these quantities are determined are of the first degree.

The final equation being resolved, and the values of y successively substituted in A, B, C, ... A′, B′, C′, . . . , the results are obtained from the division of the polynomials C, D by the common divisor $x-a$.

If the equations A, B are incomplete, the two products E cannot be complete polynomials of the degree $m+n-1$; but the terms which are deficient in the one are found in the other. For taking the least favourable case, viz.,

$$x^m+P=0;\ x^n+P'=0;$$

the identity which results from the equality of the two products is

$$(x^m+P)(x^{n-1}+A'x^{n-2}+,\ \&c.)=(x^n+P')(x^{m-1}+Ax^{m-2}+,\ \&c.)$$

Example. Let $x^2+Px+Q=0$;

$$x^2+P'x+Q'=0.$$

Denoting by $x-a$ the factor which is to be rendered common to these equations by the suitable determination of y, the first equation may be considered the product of $x-a$ by a factor $x+A$ of the first degree; and the second the product of $x-a$, by a factor $x+A'$ also of the first degree.

$$\therefore x^2+Px+Q=(x-a)(x+A),$$

$$x^2+P'x+Q'=(x-a)(x+A'),$$

$$\text{and } (x^2+Px+Q)(x+A')=(x^2+P'x+Q')(x+A),$$

$$\text{or } x^3\begin{array}{l|l|l}+P & x^2+Q & x+QA'\\ +A' & +PA' & \end{array}=x^3\begin{array}{l|l|l}+P' & x^2+Q' & x+AQ'.\\ +A & +AP' & \end{array}$$

Making the coefficients of the same powers of x equal to each other,

$$P+A'=P'+A' \text{ or } A-A'=P-P' \quad - \quad - \quad - \quad 1,$$

$$Q+PA'=Q'+AP' \text{ or } AP'-PA'=Q-Q' \quad - \quad - \quad - \quad 2,$$

$$QA'=AQ' \text{ or } AQ'-QA'=0 \quad - \quad - \quad - \quad 3.$$

By means of these three equations of the first degree the two indeterminate quantities A, A′ can be eliminated, and a single equation obtained in terms of the quantities P, Q, P′, Q′.

For, if from equation 1, multiplied by P, or $AP-PA'=(P-P')P$,

is subtracted equation 2, or $AP'-PA'=Q-Q'$,

the remainder is $AP-AP'=(P-P')P-(Q-Q')$

$$\text{Whence } A=\frac{(P-P')P-(Q-Q')}{P-P'},$$

$$\text{Similarly, } A'=\frac{(P-P')P'-(Q-Q')}{P-P'}.$$

If these values of A, A′ are substituted in equation 3,

$$\frac{(P-P')P-(Q-Q')}{P-P'}\times Q'-\frac{(P-P')P'-(Q-Q')}{P-P'}\times Q=0.$$

$$\text{or } (P-P')PQ'-(Q-Q')Q'-(P-P')QP'+(Q-Q')Q=0,$$

$$\text{or } (P-P')(PQ'-QP')+(Q-Q')(Q-Q')=0,$$

$$\text{or } (P-P')(PQ'-QP')+(Q-Q')^2=0.$$

The quantities P, P′, Q, Q′, containing only y and known quantities, this is the final equation in y.

It has been already noticed that, if this equation is identical, the proposed equations have at least one common factor of the form $x-a$, whatever be the value of y; and that, if it contains only known quantities, these equations are contradictory.

When the final equation has the proper form, the factor $x-a$ is obtained by dividing the first of the proposed equations by $x+A$, thus,

$$\begin{array}{l}
x+A)x^2+Px+Q(x+P-A \\
\quad\ \ \underline{x^2+Ax} \\
\quad\quad\quad (P-A)x+Q \\
\quad\quad\quad \underline{(P-A)x+(P-A)A} \\
\quad\quad\quad\quad\quad\quad\quad Q-(P-A)A.
\end{array}$$

The quotient is $x+P-A$, and the remainder is considered equal to zero, because it is reduced to zero by the substitution, for y, of a value deduced from the final equation.

Making the quotient $x+P-A$ equal to zero, the value of x is $x=A-P$, and by substituting the value of A,

$$x=\frac{(P-P')P-(Q-Q')}{P-P'}-P.$$

$$\text{or } x=-\frac{Q-Q'}{P-P'}.$$

This example is given as an illustration of the general method. From its particular form it admits of resolution by another and a much shorter process.

$$\begin{array}{l}
\text{For if from } x^2+Px+Q=0 \\
\quad\quad\quad\quad\quad x^2+P'x+Q'=0 \text{ is subtracted,} \\
\text{the remainder is } (P-P')x+Q-Q'=0\,; \\
\quad\quad\quad\quad \therefore x=-\dfrac{Q-Q'}{P-P'}.
\end{array}$$

OF THE DEGREE OF THE FINAL EQUATION.

247. The degree of the final equation cannot be depressed by the reduction of each of the coefficients P, Q, R . . . P′, Q′, R′ . . . in the equations,

$$x^m+Px^{m-1}+Qx^{m-2}\ \dots\ +Tx+V=0,$$
$$x^n+P'x^{n-1}+Q'x^{n-2}\ \dots\ +T'x+V'=0,$$

to the term of the highest exponent in y which it contains; for the degree of each of the equations is not changed by the reduction. Therefore the reasoning may be applied to the equations,

$$x^m+ayx^{m-1}+by^2x^{m-2}\ \dots\dots\ +ty^{m-1}x+vy^m=0 \quad - \quad - \quad 1,$$
$$x^n+a'yx^{n-1}+b'y^2x^{n-2}\ \dots\dots\ +t'y^{n-1}x+v'y^n=0 \quad - \quad - \quad 2,$$

which are of the same degree respectively as the preceding equations. The latter are reducible to

$$\left(\frac{x}{y}\right)^m+a\left(\frac{x}{y}\right)^{m-1}+b\left(\frac{x}{y}\right)^{m-2}\ \dots\ +t\frac{x}{y}+v=0 \quad - \quad - \quad 3,$$

$$\left(\frac{x}{y}\right)^n+a'\left(\frac{x}{y}\right)^{n-1}+b'\left(\frac{x}{y}\right)^{n-2}\ \dots\ +t'\frac{x}{y}+v'=0 \quad - \quad - \quad 4,$$

in which the unknown quantity is $\frac{x}{y}$, and a, b, . . . t, v; a', b', . . . t', v', are numbers.

Denoting by α, β, γ . . . the numerical roots of equation 3,
and by α', β', γ' . . . the numerical roots of equation 4,
these equations may be decomposed into,

$$\left(\frac{x}{y}-\alpha\right)\left(\frac{x}{y}-\beta\right)\left(\frac{x}{y}-\gamma\right), \text{ \&c.}=0,$$

$$\left(\frac{x}{y}-\alpha'\right)\left(\frac{x}{y}-\beta'\right)\left(\frac{x}{y}-\gamma'\right), \text{ \&c.}=0,$$

Whence $(x-\alpha y)(x-\beta y)(x-\gamma y)$, &c.$=0$ - - - 5,
$(x-\alpha' y)(x-\beta' y)(x-\gamma' y)$, &c.$=0$ - - - 6.

Substituting in equation 5 the roots of x from equation 6, viz. $\alpha'y$, $\beta'y$, &c.,

$$(\alpha'y-\alpha y)(\alpha'y-\beta y)(\alpha'y-\gamma y), \text{ \&c.}=0,$$
$$(\beta'y-\alpha y)(\beta'y-\beta y)(\beta'y-\gamma y), \text{ \&c.}=0,$$
$$(\gamma'y-\alpha y)(\gamma'y-\beta y)(\gamma'y-\gamma y), \text{ \&c.}=0.$$

. . . . . .

Or, since the number of factors in equation 5 is m, and the number of roots in equation 6 is n,

$$y^m(\alpha'-\alpha)(\alpha'-\beta)(\alpha'-\gamma), \text{ \&c.}=0,$$
$$y^m(\beta'-\alpha)(\beta'-\beta)(\beta'-\gamma), \text{ \&c.}=0,$$
$$y^m(\gamma'-\alpha)(\gamma'-\beta)(\gamma'-\gamma), \text{ \&c.}=0.$$

. . . . . .

Consequently there are n equations, each of the degree m; these give all the solutions in y. The product of these roots (or solutions) of y is the final equation, since it becomes zero for all the values of y which render its factors zero, and only for these values. Now this product is evidently of the degree mn. Consequently the degree of the final equation (unless roots not belonging to the proposed equations are introduced by the process of elimination) cannot exceed the product of the degrees of the proposed equations.

It ought to be observed that the numerical values of the roots of y are changed by this process, but that their number remains undisturbed by it.

OF THE EQUATIONS OF THE DIFFERENCES AND THE SQUARES OF THE DIFFERENCES OF THE ROOTS OF AN EQUATION.

248. Any equation containing one unknown quantity being proposed, it may be required to find another equation whose roots shall be the differences of the roots of the proposed equation combined two by two.

Let the proposed equation be

$$x^m+Px^{m-1}+Qx^{m-2} \,.\,.\,.\,.\,.\, =0 \quad - \quad - \quad - \quad \text{A},$$

Also, let x', x be any two roots of equation A, and y the difference of these roots, so that $x'=x+y$.

Equation A must be satisfied by the substitution of $x+y$ for x'.

Whence $(x+y)^m+P(x+y)^{m-1}+Q(x+y)^{m-2}+$, &c.$=0$.

Expanding the powers of $x+y$, and expressing by X, X′, X″, &c. the polynomial $x^m+Px^{m-1}+Qx^{m-2}+$, &c., and its successive derivatives (Art. 236).

$$X+X'y+\frac{1}{2}X''y^2+\frac{1}{2.3}X'''y^3+ \,.\,.\,.\,.\, y^m=0,$$

x being a root of equation A, or $X=0$, the first term of the last equation may be suppressed; the other terms being divided by y,

$$X'+\frac{1}{2}X''y+\frac{1}{2.3}X'''y^2 \,.\,.\,.\,.\, +y^{m-1}=0 \quad - \quad - \quad \text{B}.$$

Previous to the suppression of the common factor y, x, x' were any two roots whatever of equation A. Suppose $x'=x$; the difference $x'-x=y=0$. Therefore the equation in y, before the suppression of the factor y, must have contained the root $y=0$; the equation being divisible by y, this is manifest.

The division by y being made, the root y is suppressed, and the remaining $m-1$ values of y in equation B are the differences which would be obtained by subtracting the root denoted by x from all the other roots of equation A.

For the sake of precision, let the m roots of equation A be denoted by a, b, c, d, ... and let it be supposed that in equation B each of these roots in succession is substituted for x, a number, m, of equations is the result of these substitutions. Of these m equations the first has for roots the differences between a and the other roots b, c, d, ...; the second, the differences between b and the other roots a, c, d, ...; the third, the differences between c and the other roots a, b, d, ... &c. Consequently if x is eliminated between equations A, B, the final equation in y will have for roots the differences between each of the roots of equation A and all the other roots of the same equation. This final equation is therefore the equation required. It is named the Equation of the Differences.

Since equation A is of the degree m and equation B of the degree $m-1$, the equation of the differences is of the degree $m(m-1)$ (Art. 247).

The roots of the equation of the differences are

$$\begin{array}{l} a-b,\ a-c,\ a-d,\ \dots\dots \\ b-a,\ b-c,\ b-d,\ \dots\dots \\ c-a,\ c-b,\ c-d,\ \dots\dots \\ d-a,\ d-b,\ d-c,\ \dots\dots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

The number of these roots is evidently equal to the number of arrangements of the m roots of equation A, taken two by two; that is, to $m(m-1)$. It is also evident that if one root, a, is equal to $a-b$, another root, $-a$, is equal to $b-a$; therefore the first member of the equation of the differences is composed of factors which may be arranged two by two in products such as $(y-a)(y+a)$ or y^2-a^2; consequently the equation can contain only even powers of y. Making $m(m-1)=2n$, it must therefore be of the form

$$y^{2n}+py^{2n-2}+qy^{2n-4}\dots\dots+ty^2+u=0. \quad - \quad - \quad \text{C.}$$

Or making $y^2=z$,

$$z^n+pz^{n-1}+qz^{n-2}\dots\dots+tz+u=0. \quad - \quad - \quad - \quad \text{D.}$$

This is named the Equation of the Squares of the Differences; and for this reason, that since $z=y^2$, the equation in z has for roots the squares of the differences of the roots of the proposed equation A.

249. As an application, let it be required to find the equation of the squares of the differences in an equation of the third degree.

It is evident that if all the roots of the equation A are augmented or diminished by the same number, the differences of the roots must remain constant.

Whence, if a given equation has its second term, and this is taken away by the transformation of Article 237, if the equation of the differences of the roots of the transformed equation is afterwards deduced, this equation must be the same as it would have been although the second term had not been taken away, and the calculation will be simplified.

Let it be supposed then that the second term of the general equation of the third degree is taken away, and that the equation is

$$x^3+Qx+R=0.$$

Denoting by X this equation, and by X′, X″, X‴ the polynomials derived from X,

$$X=x^3+Qx+R,\ X'=3x^2+Q,\ X''=6x,\ X'''=6.$$

Therefore the equation $X'+\frac{1}{2}X''y+\frac{1}{2.3}X'''y^2\dots\dots+y^{m-1}=0$ becomes

$$3x^2+Q+3xy+y^2=0.$$

And the elimination is between $x^3+Qx+R=0$,
and $3x^2+3xy+y^2+Q=0$.

First Division.

$$\begin{array}{rl} & x^3+Qx+R \\ & \phantom{x^3+Qx+{}}3 \\ \cline{2-2} 3x^2+3xy+y^2+Q\ ) & 3x^3+3Qx+3R\ (\ x-y \\ & \underline{3x^3+3x^2y+xy^2+Qx} \\ & \quad -3x^2y-(y^2-2Q)x+3R \\ & \quad \underline{-3x^2y-3y^2x-y^3-Qy} \\ & \qquad\quad \underline{2(y^2+Q)x+y^3+Qy+3R} \end{array}$$

Second Division.

$$\begin{array}{rl}
 & 3x^2+3xy+y^2+Q \\
 & \quad 2(y^2+Q) \\ \cline{2-2}
2(y^2+Q)x+y^3+Qy+3R) & 6(y^2+Q)x^2+6(y^2+Q)yx+2(y^2+Q)^2 \;(\; 3x \\
 & 6(y^2+Q)x^2+3(y^3+Qy+3R)x \\ \cline{2-2}
 & \quad 3(y^3+Qy-3R)x+2(y^2+Q)^2 \\
 & \qquad\qquad 2(y^2+Q) \\ \cline{2-2}
 & 6(y^2+Q)(y^3+Qy-3R)x+4(y^2+Q)^3 \;(\; 3(y^3+Qy-3R) \\
 & 6(y^2+Q)(y^3+Qy-3R)x+3(y^3+Qy+3R)(y^3+Qy-3R) \\ \cline{2-2}
 & \qquad 4(y^2+Q)^3-3(y^2+Qy+3R)(y^3+Qy-3R)
\end{array}$$

In the last division the dividend has been twice multiplied by $2(y^2+Q)$ to render the division possible; but if y^2+Q is made equal to zero, the divisor (which may be put under the form

$$2(y^2+Q)x+(y^2+Q)y+3R=0, \text{ or } (y^2+Q)(2x+y)+3R=0)$$

is reduced to 3R, a quantity which in general is different from zero. Making therefore the last remainder equal to zero, and performing the multiplications indicated, the equation of the differences is

$$y^6+6Qy^4+9Q^2y^2+4Q^3+27R^2=0.$$

Then making $y^2=z$, the equation of the squares of the differences is

$$z^3+6Qz^2+9Q^2z+4Q^3+27R^2=0.$$

For the numerical equation $x^3-7x+7=0$, $Q=-7$, $6Q=-42$, $9\ Q^2=9\times7\times7=441$, $4Q^3=-4\times7\times7\times7=-1372$, $27\ R^2=27\times7\times7=1323$.

Whence the equation in z becomes $z^3-42z^2+441z-49=0$.

Of Equal Roots.

250. If in the equation $X=0$, x is replaced successively by a, b, c, d, ...

$a^m+Pa^{m-1}+Qa^{m-2}+$, &c.$=0$. Let this equation be denoted by $A=0$.

$b^m+Pb^{m-1}+Qb^{m-2}+$, &c.$=0$. - - - - $B=0$.

. . . .

For these substitutions equation B (Art. 248) becomes

$A'+\frac{1}{2}A''y+\frac{1}{2.3}A'''y^2 \ldots +y^{m-1}=0$. Let this equation be denoted by $A_{,}=0$.

$B'+\frac{1}{2}B''y+\frac{1}{2.3}B'''y^2 \ldots +y^{m-1}=0$. - - - - $B_{,}=0$.

. . . .

And the roots of equation $A_{,}$ are $a-b$, $a-c$, $a-d$, ...

- - - $B_{,}$ are $b-a$, $b-c$, $b-d$, ...

. . . .

If the equation $X=0$ has two equal roots, a, b, one of the values of y in the equation $A_{,}=0$ is equal to zero. This value is $y=a-b=0$. Therefore this equation $A_{,}=0$ must be satisfied by making $y=0$. But if $y=0$, all the terms of $A_{,}=0$ disappear, with the exception of A', the term which is wholly known; therefore A' must be equal to zero. The value a ought therefore at once to satisfy the two equations $A=0$, $A_{,}=0$.

Supposing A' in the polynomial $A_{,}$ and dividing the remaining terms by y,

$\frac{1}{2}A''+\frac{1}{2.3}A'''y+ \ldots\ldots +y^{m-2}=0$. Let this equation be denoted by $A_{,,}=0$.

The roots of this equation are $a-c$, $a-d$, ...

If the equation $X=0$ has three equal roots, a, b, c, one of the values of y in the equation $A_{,,}=0$ is equal to zero; namely $a-c=0$; and therefore $A''=0$.

Consequently when the equation $X=0$ has three equal roots the value a must at once satisfy the three equations $A=0$, $A'=0$, $A''=0$.

In the same manner it can be shown that if the equation $X=0$ has four equal roots, a, b, c, d, the value a must satisfy the equations $A=0$, $A'=0$, $A''=0$, $A'''=0$.

The equations $B=0$, $B_{,}=0$, $B_{,,}=0$, . . . $C=0$, $C_{,}=0$, $C_{,,}=0$, ... serve to detect the equality of the roots b, c, &c. when a is not equal to b; c, d, &c. when a, b are not equal to c, and so on.

A similar process may be employed to discover whether the proposed equation $X=0$ has any roots repeated, whose values are unknown.

For in the equation $A'=0$ or $ma^{m-1}+m(m-1)Pa^{m-2} \ldots\ldots +T=0$, a may be regarded as a root of the equation $X'=0$,

$$\text{or } mx^{m-1}+m(m-1)Px^{m-2} \ldots\ldots +T=0,$$

x denoting any quantity whatever. Then, since a is also a root of the proposed equation $X=0$, or

$$x^m+Px^{m-1}+Qx^{m-2} \ldots\ldots =0,$$

it follows (Art. 239) that $x-a$ is a common factor of the equations $X=0$, $X'=0$.

Changing, in like manner, a into x in the derived polynomials A'', A'''. . ., $x-a$ will become a factor of the equations $X''=0$, $X'''=0$, if the root a renders the polynomials X'', X''' equal to zero.

What has been proved respecting the root a can be proved in the same manner of any other root which is repeated oftener than once. Seeking, therefore, by the method of the greatest common measure, the common factors of the equations $X=0$, $X'=0$, $X''=0$, &c., these factors will give the equal roots of the equation $X=0$.

The factors common to the equations $X=0$, $X'=0$, are double factors of the equation $X=0$; so that if the common measure of X, X' is of the form $(x-\alpha)(x-\beta)$ x will have two values equal to α and two equal to β, or the equation $X=0$ will have the four factors $(x-\alpha)$, $(x-\alpha)$, $(x-\beta)$, $(x-\beta)$.

The factors common to the equations $X=0$, $X'=0$, $X''=0$, indicate triple factors in the equation $X=0$; that is to say, if the factors common to the three equations are $(x-\alpha)(x-\beta)$, the factors of the equation $X=0$ are $(x-\alpha)^3$, $(x-\beta)^3$.

These considerations may be extended at pleasure.

As an application, let it be required to find the equal factors of the equation $x^5-13x^4+67x^3-171x^2+216x-108=0$, or $X=0$.

In this particular case X' is $5x^4-52x^3+201x^2-342x+216=0$.

The greatest common measure of the polynomials X, X' is

$$8x^3-8x^2+21x-18.$$

This divisor, being of the third degree, must involve several factors; it becomes therefore necessary to try whether it has any in common with the polynomial X'' or $20x^3-156x^2+402x-342$.

The greatest common measure of X'' and $8x^3-8x^2+21x-18$ is $x-3$; therefore the proposed equation has three roots equal to 3, or admits $(x-3)^3$ as one of its factors.

Dividing the first common measure as often as possible by $x-3$, after two divisions a quotient, $x-2$, is obtained. The factor $(x-2)$ being common to equations X, X' only, it is twice a factor of the proposed equation $X=0$. Whence this equation is equivalent to $(x-3)^3(x-2)^2=0$.

251. Let X be a polynomial and X', X'', X''' . . . the polynomials derived from X by the process of Article 236; if, in the polynomial X, x is changed into $x+y$, the result of the substitution is a polynomial Y, which is expressed by $Y=X+\frac{X'}{1}y+\frac{X''}{1.2}y^2+\frac{X'''}{1.2.3}y^3 \ldots\ldots +y^m$.

Supposing that $X=(x-a)(x-b)(x-c) \ldots\ldots$, and changing in this expression x into $x+y$ or $y+x$,

$$Y=[(y+x)-a]\,[(y+x)-b]\,[(y+x)-c] \ldots\ldots$$

Y is therefore the product of m factors, which may be regarded as binomial, having y for their common first term and $x-a$, $x-b$, $x-c$, ... for their respective second terms.

Consequently the part independent of y is equal to the product of the m factors $(x-a)(x-b)(x-c)$ . . ., this is X.

The quantity by which the first power of y is multiplied is equal to the sum of the products of these factors multiplied together, $m-1$ by $m-1$; this is X′ . . . (Art. 139).

Hence if a polynomial X of the degree m is given, the derived polynomial X′ is equal to the sum of the products of the simple factors of the polynomial X combined, $m-1$ by $m-1$. These products can be obtained by dividing X by each of its factors, $x-a$, $x-b$, $x-c$, . . . in succession; therefore $X'=\frac{X}{x-a}+\frac{X}{x-b}+\frac{X}{x-c}+$, &c.

Let it now be supposed that the equation $X=0$ contains equal roots, or in other words, involves factors raised to certain powers; and let

$$X=(x-a)^{n}(x-b)^{n'}(x-c)(x-d)\ldots$$

Then, since X′ is equal to the sum of the quotients obtained by dividing X by each of its m simple factors, and since n of the factors of X are equal to $x-a$, and n' equal to $x-b$, it follows that

$$X'=\begin{cases} n(x-a)^{n-1}(x-b)^{n'}(x-c)(x-d)\ldots \\ +n'(x-a)^{n}(x-b)^{n'-1}(x-c)(x-d)\ldots \\ +(x-a)^{n}(x-b)^{n'}(x-d)\ldots \\ +(x-a)^{n}(x-b)^{n'}(x-c)\ldots \\ + \quad \text{\&c.} \quad \text{\&c.} \end{cases}$$

To abridge this expression of the value of X′, let

$$H=\begin{cases} n(x-b)(x-c)(x-d)\ldots\ldots+n'(x-a)(x-c)(x-d)\ldots \\ +(x-a)(x-b)(x-d)\ldots+(x-a)(x-b)(x-c)\ldots \\ + \quad \text{\&c.} \quad \text{\&c.} \end{cases}$$

Then $X'=(x-a)^{n-1}(x-b)^{n'-1}H$.

Hence it is evident that the product $(x-a)^{n-1}(x-b)^{n'-1}$ is a measure of the polynomials X, X′; and since $x-a$ is not contained in the first term of H but is found in all the other terms, $x-b$ is not contained in the second term but found in all the other terms, &c., it follows that none of the factors of X is common to all the terms of H, therefore the product $(x-a)^{n-1}$ $(x-b)^{n'-1}$, containing all the factors common to X, X′, is their greatest common measure.

If the exponent $n=n'=1$, the equation $X=0$ becomes $X=(x-a)(x-b)$ $(x-c)(x-d)\ldots$, which contains no equal roots. In this case $X'=H$, and it follows from the preceding demonstration that X, X′ have no common factor.

Consequently, that an equation $X=0$ may contain equal roots, it is necessary and sufficient that X and X′ have a common measure; and the greatest common measure of X, X′ (determined by the proper rule) expresses the product of the equal factors of X, raised each to a power less by 1 than in the polynomial X.

Example. Let $X=x^3-3a^2x-2a^3$.

$$X'=3x^2-3a^2.$$

The greatest common measure of X, X′ is $x+a$; whence X is divisible by $(x+a)^2$.

The quotient being $x-2a$, $x=(x+a)^2(x-2a)$.

The equation $x^3-3a^2x-2a^3$ has therefore two roots equal to $-a$ and one equal to $+2a$.

SECTION XIV.

OF THE RESOLUTION OF NUMERICAL EQUATIONS.

252. A numerical equation is an equation in which the coefficients and term wholly known are numbers.

The roots of a numerical equation may be real or imaginary; among the real roots some may be commensurable.

To find the integer commensurable roots of a numerical equation:

Let the equation be $Ax^4+Bx^3+Cx^2+Dx+E=0$ - - 1
in which A, B, C, D, E represent whole numbers.

Denoting by a an integer root of this equation, and substituting a for x,

$$Aa^4+Ba^3+Ca^2+Da+E=0 \quad - \quad - \quad 2$$

Transposing, and dividing all the terms by a,

$$\frac{E}{a}=-Aa^3-Ba^2-Ca-D \quad - \quad - \quad 3$$

The second member of this equation being integer, a must be a divisor of E.

Therefore the integer commensurable roots may be discovered by substituting successively in the equation all the divisors of the last term, both positive and negative; but when the last term has many divisors, this process requires many substitutions; in such cases the calculation is abridged by new conditions, which are deduced from equation 3.

Making $\frac{E}{a}=E'$, transposing D to the first member, and dividing by a,

$$\frac{E'+D}{a}=-Aa^2-Ba-C \quad - \quad - \quad 4$$

Therefore the sum $E'+D$ ought to be divisible by a.

Making $\frac{E'+D}{a}=D'$, transposing C, and dividing again by a,

$$\frac{D'+C}{a}=-Aa-B \quad - \quad - \quad - \quad 5$$

Therefore $D'+C$ ought to be divisible by a.

Making, again, $\frac{D'+C}{a}=C'$, transposing B, and dividing by a,

$$\frac{C'+B}{a}=-A \quad - \quad - \quad - \quad - \quad 6$$

Making, lastly, $\frac{C'+B}{a}=B'$, and transposing A,

$$B'+A=0 \quad - \quad - \quad - \quad - \quad 7$$

Therefore the sum $B'+A$ ought to be zero.

If one only of the preceding conditions is wanting, the number a is not a root; but it is a root if they are all fulfilled, for then, if the letters B′, C′ . . . are successively replaced by the quantities which they represent, a return is made from the equation $B'+A=0$ to the equation marked 2.

An equation being therefore prepared in such a manner that all its coefficients are whole numbers (Art. 48), in which case the coefficient of the first term may differ from 1, the conditions by which it is discoverod that a whole number is a root of the equation are these:

That the whole number must be a divisor of the last term:

That it must be a divisor of the sum obtained by adding the quotient to the coefficient of the term in x:

That it must be a divisor of the sum obtained by adding this second quotient to the coefficient of the term in x^2; and so on until the coefficient of the first term of the equation is added to the preceding quotient; then the sum of the last quotient and first coefficient must be equal to zero.

In the application of these principles to examples, the easiest mode, with the divisors +1, −1, is to substitute them immediately in the equation; with other divisors the process is conducted in the manner detailed below.

1st Ex. It is required to find the integer commensurable roots of the equation $2x^3-12x^2+13x-15=0$?

Substituting +1 for x, $2-12+13-15=-12$;
Substituting −1 for x, $-2+12-13+15=+12$;

Therefore +1, −1 are not roots of the proposed equation.

The other divisors, positive and negative, of 15 are,

$$+3,\ +5,\ +15,\ -3,\ -5,\ -15.$$

The quotients of -15 by these divisors are, respectively,

$$-5,\ -3,\ -1,\ +5,\ +3,\ +1.$$

Adding $+13$, the coefficient of the term in x, to each quotient, the sums are, $+8$, $+10$, $+12$, $+18$, $+16$, $+14$.

The integer quotients obtained from the division of these sums by the corresponding divisors are $*$, $+2$, $*$, -6, $*$, $*$.

Adding -12, the coefficient of the term in x^2, to each of these results, the sums are -10, -18.

The integer quotients obtained from the division of these sums by the corresponding divisors of -15 are, -2, $+6$.

Adding $+2$, the coefficient of the term in x^3, to each of these results, the sums are 0, $+8$.

The substance of the preceding detail is commonly exhibited in a tabular form, thus:

Divisors -	+3	+5	+15	−3	−5	−15
Quotients -	−5	−3	−1	+5	+3	+1
	+8	+10	+12	+18	+16	+14
Quotients -	*	+2	*	−6	*	*
		−10		−18		
Quotients -		−2		+6		
		0		+8		

Since the only column in which the result $B'+A=0$ is found, is that of the divisor 5, it is concluded that $+5$ is the only integer root of the proposed equation.

The root $+5$ being known, the equation is divisible by the factor $x-5$; effecting the division the quotient is $2x^2-2x+3$; therefore $2x^2-2x+3=0$.

Whence $x=\frac{1}{2}\pm\frac{1}{2}\sqrt{-5}$ (Art. 104 and 120).

The proposed equation is thus completely resolved, its three roots being

$$x=5,\ x=\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-5},\ x=\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-5}.$$

This method does not indicate how often the roots which are determined by it enter into the equation; to discover this the most obvious process is to divide the equation by the factors corresponding to these roots, each being taken once, and afterwards to examine, either by immediate substitution or by the preceding method, whether these roots still belong to the equation; if some of them verify the equation, it is certain that these enter at least twice into the original equation; by a new division it is determined whether they enter into it a third time, and so on.

Another process by which the number of repetitions of the same root is determined is supplied by the properties of derived polynomials; for if an equation, $X=0$, involves the factor $(x-a)^n$, the derived polynomial, X', ought to involve the factor $(x-a)^{n-1}$, the polynomial X'', derived from X', the factor $(x-a)^{n-2}$, &c.

So that the number of the equations, $X=0$, $X'=0$, $X''=0$, . . . which are verified by the value $x=a$, will indicate how many roots equal to a are contained in the equation $X=0$.

253. All the integer roots, equal and unequal, being found, they may be taken from the equation (Art. 231); then, if any commensurable roots remain, they must be fractional.

The determination of the fractional roots depends on this principle:—If the first term of an equation has 1 for coefficient, and the other terms have integer coefficients, this equation cannot have any commensurable roots which are not whole numbers.

To prove this, let $x^m+Px^{m-1}+Qx^{m-2} \dots\dots +Tx+U=0$, be an equation in which P, Q . . . T, U express whole numbers.

Substituting for x a fractional number, $\frac{a}{b}$, reduced to the lowest terms, the proposed equation becomes

$$\frac{a^m}{b^m}+P\frac{a^{m-1}}{b^{m-1}}+Q\frac{a^{m-2}}{b^{m-2}} \dots\dots +T\frac{a}{b}+U=0.$$

Multiplying all the terms by b^{m-1}, and transposing,

$$\frac{a^m}{b}=-Pa^{m-1}-Qa^{m-2}b \dots\dots -Tab^{m-2}-Ub^{m-1}.$$

Now a, b being prime to each other, the first member of this equality is an irreducible fraction, while the second is a whole number. The equality is therefore impossible, and by consequence no fractional number can satisfy the equation.

a. If, after removing denominators, the coefficient of the first term is not 1, the equation can be transformed into another fulfilling this condition; for which purpose it is sufficient to make the unknown quantity, x, equal to a new unknown quantity, y, divided by the coefficient of the first term of the equation in x. Then it is evident that all the commensurable roots of the equation in x must have a fixed relation to the commensurable roots of the equation in y; and as this can have integer roots only, these can be determined by the preceding method, and from them the corresponding roots of the equation in x.

The transformation of the equation in x into the equation in y is effected as follows:

Let the proposed equation be $Ax^m+Bx^{m-1}+Cx^{m-2}+$, &c. in which A, B, C . . . . . are whole numbers.

Making $x=\frac{y}{z}$, the preceding equation becomes

$$\frac{A}{z^m}y^m+\frac{B}{z^{m-1}}y^{m-1}+\frac{C}{z^{m-2}}y^{m-2}+ \dots\dots =0.$$

Multiplying all the terms by z^{m-1},

$$\frac{A}{z}y^m+By^{m-1}+Czy^{m-2}+ \dots\dots =0.$$

The coefficient of y^m in this equation becomes 1, and the other coefficients are whole numbers if $z=A$.

In some particular cases z may be chosen less than A, by which means a transformed equation with smaller coefficients is obtained.

As a second example, let it be required to apply these principles to the discovery of the commensurable roots of the equation $4x^4-11x^2+7x-6=0$?

The integer roots might be first sought; but as the coefficients are small numbers, it seems preferable at once to transform the proposed equation into another of which the first coefficient shall be 1, and the other coefficients whole numbers.

Following the general rule, it is necessary to make $x=\frac{y}{4}$; but in the present instance it is sufficient to make $x=\frac{y}{2}$; substituting $\frac{y}{2}$ for x and reducing, it is found that

$$4\left(\frac{y}{2}\right)^4-11\left(\frac{y}{2}\right)^2+7\left(\frac{y}{2}\right)-6=0,$$

$$\text{or } 4\times\frac{y^4}{16}-11\times\frac{y^2}{4}+7\times\frac{y}{2}-6=0,$$

$$\text{or } y^4-11y^2+14y-24=0,$$

the only commensurable roots of which equation are whole numbers.

Since the equation is not satisfied by the substitution for y of $+1$ and -1, $+1$ and -1 are not roots of the equation.

Process for finding the roots.

Divisors -	+2	+3	+4	+6	+8	+12	+24	−2	−3	−4	−6	−8	−12	−24
Quotients -	−12	−8	−6	−4	−3	−2	−1	+12	+8	+6	+4	+3	+2	+1
	+2	+6	+8	+10	+11	+12	+13	+26	+22	+20	+18	+17	+16	+15
Quotients -	+1	+2	+2	*	*	+1	*	−13	*	−5	−3	*	*	*
	−10	−9	−9			−10		−24		−16	−14			
Quotients -	−5	−3	*			*		+12		+4	*			
Quotients -	*	−1						−6		−1				
		0								0				

The conclusion to which this calculation leads is that $+3$ and -4 are roots of the equation in y.

Dividing the equation in y by the product $(y-3)(y+4)$ the quotient is

$$y^2-y+2=0;$$

$$\text{whence } y=\tfrac{1}{2}\pm\tfrac{1}{2}\sqrt{-7}.$$

The four values of y being determined, the relation $x=\frac{y}{2}$ gives the corresponding values of x, viz.

$$x=\tfrac{3}{2},\ x=-2,\ x=\tfrac{1}{4}+\tfrac{1}{4}\sqrt{-7},\ x=\tfrac{1}{4}-\tfrac{1}{4}\sqrt{-7}.$$

When the equation whose integer roots are sought wants any of its terms, it is necessary to consider the coefficients of these terms as equal to zero; consequently the quotients to which these coefficients ought to be added may be immediately subjected to new divisions; this has been done in the case of the equation $y^4-11y^2+14y-24=0$, which wants the term in y^3.

OF THE LIMITS OF THE ROOTS OF EQUATIONS.

254. The process which has been followed in determining the commensurable roots of an equation consists in the employment of a series of trials so ordered as certainly to discover all the roots of that kind which can exist in an equation.

It is also by trials that the approximate values of the incommensurable roots are obtained.

In investigating the rules which ought to direct these trials, the first problem is, to find the limits of the roots; in other words, to determine two numbers between which are comprised all the positive roots of the equation, and two numbers between which are comprised all the negative roots.

a. To determine a superior limit of the positive roots, suppose, first, that the proposed equation $X=0$ is composed of certain positive roots, and that these are followed by certain negative roots, all of a less degree in x than the terms which are positive.

Let Y denote the assemblage of positive terms,
Y' - the assemblage of negative terms,
and x^r the least power of x involved in Y;

$$\text{then } X=Y-Y'=x^r\left(\frac{Y}{x^r}-\frac{Y'}{x^r}\right).$$

In the quotient of Y by x^r no term ought to contain x in the denominator, but in the quotient of Y' by x^r all the terms must contain x in the denominator; making, therefore, x to begin with any positive value and to increase, the quotient $\frac{Y}{x^r}$ ought to increase, or, if Y is a monomial, to remain constant, while $\frac{Y'}{x^r}$ necessarily diminishes.

It being, therefore, found that a positive value, $x=l$, renders positive the polynomial X or $Y-Y'$, it is certain that by substituting in X greater values of x than l, the results must be always positive, and must go on increasing to infinity; above l there cannot, therefore, be any number which renders $Y-Y'=0$; and consequently l is a superior limit of the positive roots.

Whence it is to be concluded, that if in X are successively substituted for x increasing positive numbers until a positive result is obtained, the number which gives this result is the limit sought.

b. Suppose, next, that the equation contains positive mixed with negative terms, which is commonly the case; the first term of an equation always either is or can be rendered positive; combine with it the positive terms which occur before the first negative term, and (neglecting the other positive terms) compare this sum with that of the negative terms of the equation. Then the preceding reasoning proves that a superior limit of the positive roots will be obtained by determining a positive value of x, which renders the first sum greater than the second; and this is easily accomplished by substituting for x increasing numbers beginning from 0.

In the case of the positive and negative terms intermingled, it is often possible to separate the first member of the equation into several parts, in each of which care is to be taken to put one or more positive terms followed by negative terms of a lower degree; then determining as before, by successive trials, one positive value, departing from which all these parts are positive, this value may be taken for the limit required.

The terms of the equation may be separated into parts in different ways, and different limits may be obtained in consequence. Of these the lowest is always to be chosen.

c. In all cases the superior limit has this property, that if greater numbers are substituted in the equation the results are positive, and augment (if the numbers are increased) to infinity. The first condition is the only one necessary in order that l may be a superior limit of the positive roots; for it results from a theorem (for which see Article 254. a.) that every number greater than the greatest positive root gives a positive result.

As an example of the application of these principles, let it be required to find a superior limit of the roots of the equation

$$x^4-3x^3+2x^2-3x-8=0?$$

Comparing the first term of this equation with the negative terms, the polynomial x^4-3x^3-3x-8 is to be considered.

Making x successively 0, 1, 2, 3, 4, the value of the polynomial is -8, -10, -22, -17, $+44$. Whence, as $x=4$ gives a positive result, 4 is a superior limit of the roots of the proposed equation.

But since the first member of this equation is the sum of the two quantities x^4-x^3 and $2x^2-3x-8$, of which the first becomes 0, and the second $+1$, on the hypothesis of $x=3$, it appears that 3 is also a superior limit of the positive roots of the proposed equation.

2d Example. It is required to find a superior limit of the positive roots of the equation

$$x^7+8x^6+2x^5-10x^4-40x^3+10x^2-14x-100=0?$$

The first member of this equation can be separated into the parts,

$$(x^7-100),\ +(8x^6+2x^5-10x^4-40x^3),\ +(10x^2-14x),$$
$$\text{or } (x^7-100),\ +8x^3(x^3+\tfrac{1}{4}x^2-\tfrac{5}{4}x-5),\ +10x(x-\tfrac{7}{5}).$$

Making x successively 0, 1, 2, it is found that each part is positive, for $x=2$. Therefore 2 is a superior limit of the positive roots of this equation.

255. Newton employed a process by which a more approximate limit is frequently found than by other methods; it consists in finding a number such that if it is subtracted from the rootsof the proposed equation the result is a transformed equation whose terms are all positive. Then it is evident that no positive value can satisfy this transformed equation, and that by consequence the number which is to be found necessarily exceeds the greatest positive root of the proposed equation. Let this unknown number be l; to diminish all the roots of equation $X=0$ by l, it is necessary to make $x=l+y$. The result is a transformed equation in y, which may be put under the form

$$L+L'y+\tfrac{1}{2}L''y^2+\tfrac{1}{2.3}L'''y^3 \;.\;.\;.\;.\; +y^m=0.$$

L, L', L'', . . . representing the polynomial X and its derivatives X', X'' . . . . , in which x is replaced by l.

Now it is required that all the coefficients of the equation in y shall be positive; consequently the question is reduced to this, to find a value of x which shall render positive all the quantities X, X', X'', &c.?

The quantity X being of the degree m, X' is of the degree $m-1$, X'' of the degree $m-2$ . . . ., and so on to the last quantity which does not contain x, and which is essentially positive.

It is convenient, therefore, first to determine a value of x such as shall render the last term but one positive, which can be easily done, since this term is of the first degree in x.

Then this value is augmented, if necessary, until it renders the preceding term (or term in x^2) also positive; and this process is continued up to the term X.

For convenience it is usual to employ whole numbers only in these trials.

Let it be required to find by this method a superior limit of the positive roots of the equation

$$x^5+x^4-4x^3-6x^2-7000x+800=0\,?$$

$$X=x^5+x^4-4x^3-6x^2-7000x+800.$$

$$X'=5x^4+4x^3-12x^2-12x-7000.$$

$$\tfrac{1}{2}X''=10x^3+6x^2-12x-6.$$

$$\tfrac{1}{2.3}X'''=10x^2+4x-4.$$

$$\tfrac{1}{2.3.4}X''''=5x+1.$$

X'''' and X''' are positive, for $x=1$;
X'' is positive, for $x=2$;
X' is positive, for $x=7$.

As the value $x=7$ renders X also positive, it is concluded that 7 is a superior limit of the positive roots of the proposed equation.

In these trials it is never necessary to return to the preceding derived polynomials. Suppose, for example, that, commencing with the polynomial of the first degree in x, and ascending step by step to X'', a value $x=l$ has been found which renders positive all the polynomials X'', X''', X'''' . . . , then, if x is augmented by being changed into $x+h$, as the polynomials derived from X'' are X''', X'''' . . . . , it is evident that X'' will become $X''+X'''h+\frac{1}{2}X''''h^2+$ . . . . Consequently, when the value $x=l$ renders X'', X''' . . . . positive, it is certain that a value greater than l will not only render X'' positive, but also augment its value.

256. To determine an inferior limit of the positive roots of an equation, make $x=\frac{1}{y}$ in the proposed equation, and represent by l the superior limit of the positive roots of the transformed equation. Then it is evident that $\frac{1}{l}$ will be a quantity less than the least positive root of the proposed equation (Art. 233); consequently this quotient may be taken for the limit sought.

In making applications it is convenient that the inferior limit be the greatest possible; the nearest approximation possible should in consequence be made in determining l.

257. The limits of the negative roots are found by changing x into $-x$ in the proposed equation. By this transformation the signs of the roots are changed, so that by seeking the limits of the positive roots of the transformed equation, and affecting them with the sign $-$, the limits of the negative roots of the proposed equation are obtained.

When all the terms of an equation have the same sign, it is evident that no positive number can satisfy the equation. In such cases there is, therefore, no occasion to seek the limits of the positive roots.

In like manner there is no occasion to seek the limits of the negative roots of an equation, of which all the terms of an even degree have the same sign, those of an odd degree having the contrary sign; for it is evident that, any negative number whatever being substituted for x in an equation of this form, all the terms become of the same sign, and their sum cannot by consequence be equal to zero.

When the last term of an equation is wanting, one or more of its roots are equal to zero. Making abstraction of these roots, the preceding remarks are true alike for complete and incomplete equations.

258. Let X be a polynomial of the form

$$X=Mx^m+Nx^n+Px^p+, \&c.,$$

in which the exponents m, n, p . . . are integer or fractional, but positive and decreasing numbers, a positive value of x can be found such that, if any greater number is substituted for x in the proposed equation, the polynomial X will acquire values of the same sign as its first term, and which will continue to augment until they become infinite.

The proposed equation may be put under the form

$$X=M(x^m+\frac{N}{M}x^n+\frac{P}{M}x^p+, \&c.).$$

Representing by Y the sum of those terms between the parentheses which have positive coefficients, and by Y′ the sum of those terms which have negative coefficients,

$$X=M(x^m+Y-Y')=M[x^m(1-\frac{Y'}{x^m})+Y].$$

By hypothesis x^m is a positive power of x. Now, whatever the value of the expression $-Y'$, it is possible to find a positive value of x such that x^m shall be greater than Y'; therefore in this case $\frac{Y'}{x^m}<1$. Therefore $1-\frac{Y'}{x^m}$ is a positive quantity. But x^m is positive and Y is positive, therefore the whole quantity within the parentheses is positive; consequently, if M is positive the whole expression, or X, is positive, and if M is negative X is negative.

Therefore X is of the same sign as M or Mx^m.

Denoting by λ that value of x for which $\frac{Y'}{x^m}<1$, and supposing that x is made to increase from λ, the quotient of Y' by x^m is continually diminished while x^m and Y are indefinitely augmented; therefore the quantity X will go on increasing to infinity.

The value of λ may be obtained, as in the case of limits, by substituting successively increasing positive numbers for x until x^m exceeds Y'.

OF THE INDICATIONS GIVEN BY THE SUBSTITUTION OF ANY TWO NUMBERS FOR THE UNKNOWN QUANTITY.

259. Let X be a polynomial of the form $Ax^m+Bx^{m-1}+Cx^{m-2}+$, &c. (in which A, B, C, . . . are real quantities); if in this polynomial x is made to vary continuously the polynomial X will also vary continuously; in other words, if the difference of two values of x is less than any assigned quantity

the difference of the corresponding values of X shall also be less than any assigned quantity.

Let any value whatever be given to x; change x into $x+h$, denote by $X_{,}$ what X becomes when x becomes $x+h$; denote also by X', X''.... the polynomials derived from X.

$$\text{Then } X_{,}=X+X'h+\tfrac{1}{2}X''h^2 \dots\dots +Ah^m \text{ (Art. 236).}$$

The algebraic sum of the terms which follow X, and which contain the different powers of h, expresses the change produced in X by means of the change of x into $x+h$. Now, it is evident that by taking h sufficiently small each of the terms of this sum may be rendered indefinitely small, and consequently also the sum of the terms.

260. If two numbers substituted successively instead of x in an equation $X=0$, of the form

$$Ax^m+Bx^{m-1}+Cx^{m-2}+, \&c. =0,$$

give results of contrary signs, the equation has at least one real root comprehended between these two numbers. Let the two numbers which, being substituted for x in this equation, give results of contrary signs be represented by α and β, of which α is less than β, and let it be conceived that instead of x all the values from α to β are substituted successively in X, the polynomial X must vary in a continuous manner. But by hypothesis the results obtained by making $x=\alpha$, $x=\beta$ are of contrary signs; therefore, among the values which X obtains in consequence of the continuous increase of x from α to β, the value 0 must be found at least once. Now the value of x for which X becomes zero is a root of the equation; therefore the theorem is demonstrated.

It is necessary to observe, that there might be an error in affirming that there can be only one root of the equation $X=0$ between α and β; for the polynomial X, without ceasing to vary in a continuous manner, might have, for example, first, decreasing values which might terminate at a certain limit, after which it might increase to another limit, then decrease again, &c.; consequently there is no reason why x, in the interval between $x=\alpha$ and $x=\beta$, may not pass several times through zero.

261. If two quantities comprehend between them only one root of the equation $X=0$, and these quantities are successively substituted for x in the equation, they give two results of contrary signs; and in general, if two quantities comprehend between them an odd number of roots, the results obtained by substituting these quantities for x are of contrary signs. But if two quantities comprehend between them no root or an even number of roots, the results obtained by substituting these quantities for x are of the same sign.

Let it be supposed that a is a real root comprehended between α and β, and the only root between them. The polynomial X is divisible by $x-a$; denoting the quotient by Y, X may be put under the form $X=Y(x-a)$. Making successively $x=\alpha$, $x=\beta$, the values of Y must have the same sign, otherwise there must be between α and β a root of the equation $Y=0$, and which would be a new root of $X=0$, comprehended between α and β, which is contrary to the hypothesis. But if x is made successively equal to α and β, the factor $x-a$ must change its sign; for the differences $\alpha-a$, $\beta-a$ are of contrary signs, because a is between α and β; therefore the two results which are obtained by substituting α and β in X are of contrary signs.

Taking the more general case, that of an uneven number of roots, a, b, c, ... between α, β; since a, b, c ... are roots of the equation X, X is divisible by the product $(x-a)(x-b)(x-c)$ ... Let the quotient be Y; then $X=(x-a)(x-b)(x-c) \dots \times Y$.

In this case, as in the preceding, if x is made successively equal to α, β, the two corresponding values of Y will have the same sign, while the sign of each of the factors $x-a$, $x-b$, $x-c$ ... is changed, and the number of these factors is odd; whence the results of the substitutions of α and β in X ought to have contrary signs.

It is, in like manner, evident that if the number of factors $x-a$, $x-b$... is even, the sign of the product of these factors is $+$. Whence the preceding demonstration also proves that if there are no roots between α and β, or if there is an even number of roots, the results of the two substitutions will have the same sign.

When an equation has no real roots, if any real quantities whatever are substituted in it instead of x, the results will always be of the same sign. For if two results of contrary signs are obtained, it follows that the equation must have at least one real root comprehended between two of the quantities substituted for x, which is contrary to the hypothesis.

When an equation has been divided by all the factors corresponding to the real roots, the resulting equation ought to be in the case of this theorem; but an equation containing real roots may still give results of the same sign if each root enters an even number of times into the equation. For, if these roots are denoted by a, b, &c. the equation may be written thus,

$$(x-a)^{2n}(x-b)^{2n'}\dots\times Y=0.$$

Now, whatever real values are substituted for x, the factors $(x-a)^{2n}$, $(x-b)^{2n'}$ cannot become negative (since $(-a)^2$ as well as $(+a^2)=+a^2$); and Y cannot change its sign, because the equation $Y=0$ contains only imaginary roots.

262. Every equation of an odd degree has at least one real root of a sign contrary to its last term.

The first term being always supposed positive, let, in the first instance, the last term U be negative, and denote by l a superior limit of the positive roots. If, in the proposed equation, x is made equal to $+l$, the result is positive, and if x is made zero the result is $-U$. Therefore between zero and $+l$ the equation has at least one real root which must be positive.

Next, let the last term of the proposed equation be positive. If $-x$ is substituted for x in the equation, the first term (its exponent being odd) becomes negative. It is rendered positive by changing the signs of all the terms; by this change the last term is made negative; then, by the preceding case, the transformed equation has one positive root; therefore the proposed equation has one negative root.

263. Every equation of an even degree, whose last term is negative, has at least two real roots, the one positive and the other negative.

For, making successively $x=0$ and $x=+l$, two results of contrary signs are obtained; therefore the equation has at least one real root.

Then, changing x into $-x$ in the proposed equation, the first term of the transformed equation is still positive, but its last term is negative; it therefore has a positive, and by consequence the proposed equation a negative, root.

When the proposed equation is of an even degree and its last term positive the preceding method does not give the means of discovering whether the equation has a real root, because then the substitutions do not give results of contrary signs. It is thus that equations which have only imaginary roots are necessarily of this form, for if they were of an odd degree they would have at least one real root; and if, being of an even degree, the last term were negative, they would have at least two real roots.

264. In an equation of an odd degree the real roots of a sign contrary to the last term are odd in number, and the roots, if any, of the same sign are even in number. When the last term is negative, the substitutions $x=0$ and $x=+l$ giving results of different signs, it is inferred from Articles 261, 262, that the positive roots are odd in number.

Changing x into $-x$, the equation is transformed into another, of which the first and last terms are negative, but which are rendered positive by changing the signs of all the terms. Now, whether in this equation x is made equal to zero or equal to $+l$, the result is positive. Therefore the positive roots of the transformed equation, and by consequence the negative roots of the proposed equation, can only be of an even number.

When the last term of the proposed equation is positive the last term of the transformed equation must be negative. Whence, by the last case, the proposed equation has an odd number of negative roots and an even number of positive roots.

265. In an equation of an even degree, if the last term is negative, the positive roots are odd in number and the negative roots are also odd in number. But if the last term is positive the roots, if any, both positive and negative, are even in number.

For $x=0$ and $x=+l$, the results are of contrary signs; therefore the positive roots are odd; and making $x=-x'$, the first term of the transformed equation is positive and the last negative; therefore the positive roots of the transformed, and by consequence the negative roots of the proposed, equation are odd.

When the last term of the proposed equation is positive the last term of the transformed equation is also positive, and so is the first; therefore the real roots, if any, whether positive or negative, must be even in number.

In all cases, therefore, the total number of real roots is odd if the degree of the equation is odd, and even if the degree of the equation is even; hence, if imaginary roots are found in an equation the number of such roots must be even.

266. If an equation is composed of a series of positive terms after which there are only negative terms, it has one positive root, and only one.

Since the last term of the proposed equation is negative, the equation has certainly one positive root (Art. 262, 263).

To demonstrate that it cannot have more than one,

denote by Y the sum of the positive terms,
denote by Y′ the sum of the negative terms,
and by x^r the lowest power of x contained in Y;

The proposed equation may, consequently, be put under the form

$$x^r\left(\frac{Y}{x^r}-\frac{Y'}{x^r}\right)=0.$$

Let $x=a$ be the positive value which satisfies the equation: for this value $\frac{Y}{x^r}=\frac{Y'}{x^r}$.

If in this expression x is augmented, $\frac{Y}{x^r}$ is augmented, or (in the case of Y composed of one term) remains constant, while $\frac{Y'}{x^r}$ is diminished; but if x is diminished $\frac{Y'}{x^r}$ is augmented, and $\frac{Y}{x^r}$ is diminished. In either case the equation is not satisfied by values of x greater or less than a; therefore $x=a$ is the only positive root of the proposed equation.

SEPARATION OF THE ROOTS, BY THE METHOD OF LAGRANGE.

267. When an equation is proposed for resolution it is convenient, first, to find the commensurable roots, equal and unequal, then the other equal roots, if the equation contains any, and to suppress them by division. As this can be done by the methods of Article 252 and Article 251, it is assumed, in the following investigation, that the equation to be resolved has neither commensurable nor equal roots. Consequently, when the expression Real roots is employed, the meaning is real roots that are incommensurable and unequal.

The investigation of these roots is divided into two distinct parts. The first treats of the determination for each root of two numbers between which one root only of the equation is comprehended. This part is named the Separation of the roots. The second part treats of the evaluation of the roots with that degree of approximation which is required.

If in an equation the unknown quantity x is changed into $-x$, and the positive roots of the transformed equation are taken with the sign $-$, the negative roots of the proposed equation are obtained. It will therefore be sufficient to show in what manner the positive roots of equations are determined.

268. The first discovered rigorous method of effecting the separation of the roots of an equation is named the method of Lagrange. By it the separation of the roots is effected in the following manner:

Let $X=0$ be any equation which contains no equal roots; and let it be conceived that, instead of the unknown quantity x, there are substituted successively in X the positive numbers $p, q, r, s, \ldots$, forming an increasing series which commences at the inferior limit of the positive roots, ends at the superior limit of these roots, and is, besides, chosen in such a manner that between each number and the following there cannot fall more than a single root of the equation. From the theorem of Article 260 it is known with certainty, by the signs of the results of the substitutions, which of these numbers (p, q, r, &c.) comprehend between them a root, and which of them do not. Then the separation of the roots will be complete.

The essential condition to be fulfilled by the numbers $p, q, r, s, \ldots$ is, that no two of them, taken consecutively, shall include between them more than one root of the proposed equation. Now this condition would be fulfilled if for $p, q, r, s, \ldots$ were chosen the terms of an arithmetical progression, of which the ratio δ is less than the least difference which exists between the positive roots of the proposed equation.

Hence the difficulty is reduced to that of discovering the ratio δ. The method of Lagrange gives δ by means of the equation of the squares of the differences of the roots of the proposed equation. Thus,

Let $X=0$ be the proposed equation;

Let $Z=0$ be the equation of the squares of the differences of the roots of the proposed equation;

Let λ be the inferior limit of the roots of the equation $Z=0$.

Then it is evident that λ is less than the square of the least difference between any two roots of the equation $X=0$; and, consequently, that $\sqrt{\lambda}$ is less than the least difference between any two roots of the equation $X=0$, δ may therefore be assumed equal to $\sqrt{\lambda}$.

As $\sqrt{\lambda}$ is generally incommensurable, a rational number less than $\sqrt{\lambda}$ is usually taken in preference.

When for λ is found an incommensurable number greater than 1, the whole number immediately greater than $\sqrt{\lambda}$ is taken. Then, to effect the separation of the roots it is only necessary to substitute certain whole numbers in the equation $X=0$. When it has been discovered by this means that there is a root between two of these numbers, the intermediate whole numbers may then be substituted in the equation $X=0$, and it can be determined by the signs of the results between which of the consecutive whole numbers the root is comprehended.

Most frequently λ is a fraction less than 1; consequently δ is also a fraction less than 1; hence it becomes requisite to substitute fractional numbers in the equation $X=0$. But such substitutions can be avoided by a simple transformation, namely, by making $x=\delta x'$. Then, in order that two consecutive values of x may have between them a difference greater than δ, it is evident that the values of x' ought to differ from each other by more than 1. Consequently it is possible, by the substitution of whole numbers, to find the two numbers which comprehend between them each real root of the transformed equation in x'.

269. Considered under a point of view purely theoretical, the preceding method completely resolves the problem of the separation of the roots. But if, on the one hand, are considered the length of the calculations required to form the equation of the squares of the differences, when the degree of the proposed equation is high, and on the other the multitude of substitutions which must be successively made, it will appear that the application of this method is very laborious.

To diminish the labour of the substitutions, care ought to be taken, in each particular case, to determine for the superior and inferior limits of the roots of the proposed equation the nearest numbers which can be obtained, and for δ the greatest value possible.

After the limits of the roots, positive and negative, have been determined it is convenient to substitute, first, the consecutive whole numbers comprehended between these limits; and if it happens that the signs of the results indicate as many real roots as there are *ones* in the degree of the equation, since the equation cannot have a greater number of roots than this, it will not be necessary to calculate the equation of the squares of the differences of the roots.

The case in which it is known from the beginning that all the roots are real, deserves particular notice. In this case it is evident, that if numbers whose differences are less at each substitution, and which are comprehended between the extreme limits, are successively substituted in the proposed equation, the process must end in the determination of certain results exhibiting a number of changes of sign equal to the degree of the equation; consequently all the roots must be separated.

270. Examples of the separation of the roots of numerical equations:

1st. Let $x^3+x-3=0$.

By Article 266 this equation has only one positive root. To obtain the integer part of this root it is sufficient to substitute for x the numbers 0, 1, 2, 3 . . . until two results of contrary signs are obtained.

Now, if, in the expression x^3+x-3,

$x=0$, the result is $0+0-3$, or -3;
$x=1$, - - $1+1-3$, or -1;
$x=2$, - - $8+2-3$, or $+7$.

Consequently, since the results given by the substitutions $x=1$, $x=2$, are of contrary signs, the required root is comprehended between the numbers 1 and 2.

The substitutions and results are briefly exhibited thus:

Numbers substituted - -	0,	1,	2
Corresponding results -	-3,	-1,	$+7$.

To separate the negative roots, if x is changed into $-x$, the transformed equation is $x^3+x+3=0$.

Since all the terms of this transformed equation are positive, it has no positive root; therefore the proposed equation has no negative root. Consequently its two remaining roots are imaginary.

2d Example. Let $x^4-5x-10=0$.

This equation also has only one positive root.

Changing x into $-x$, the transformed equation is

$$x^4+5x-10=0,$$

which, like the proposed, has only one positive root; therefore the proposed equation has one negative root; it has, consequently, two real roots, the one positive, the other negative.

To separate these roots,

Substitute, successively, for x,	0,	1,	2,	3	0,	-1,	-2.
The corresponding results are,	-10,	-14,	-4,	$+56$	-10,	-4,	$+16$.

Hence the positive root is comprehended between 2 and 3, and the negative root between -1 and -2.

3d Example. Let $x^4+x^2-65x+5=0$.

When a superior limit of the positive roots is known, it is not necessary to carry the substitutions beyond this limit.

To find a superior limit in the present instance, if the equation is put under the form $x(x^3+x-65)+5=0$, and a value of x is sought, for which the expression x^3+x-65 is positive, it is found that 4 is such a value; therefore 4 is a superior limit of the positive roots of the proposed equation.

Substituting, next, $-x$ for x, all the terms of the transformed equation are positive, therefore the proposed equation has no negative root.

Yet, as this equation has no second term, the sum of the roots is zero; in consequence they cannot be all positive.

Therefore, since the equation contains no negative, it must contain imaginary roots. But if the equation contains imaginary roots, they are even in number; therefore, since the degree of the equation is the fourth, and its roots 4, the real roots (if any) must be positive and two in number.

To separate these roots,

Substituting, successively, for x, 0, 1, 2, 3, 4,
the corresponding results are +5, −58, −105, −100, +17.

The signs of these results show that the equation contains two positive roots, the one comprehended between 0 and 1, the other between 3 and 4.

4th Example. Let $x^3-7x+7=0$.

Since this equation is of an odd degree, it has at least one real root which is of a sign contrary to the last term, and therefore negative (Art. 262).

Changing x into $-x$, the transformed equation is,

$$x^3-7x-7=0.$$

This equation has only one positive root, therefore the proposed equation cannot have more than one negative root. Substituting whole numbers in the transformed equation, 0, 1, 2, 3, 4,
the corresponding results are, −7, −13, −13, −1, +29;
its positive root is comprehended between 3 and 4; therefore the negative root of the proposed equation is comprehended between −3 and −4.

If the other roots of the proposed equation are real they must be positive.

Trying the substitution of whole numbers in the equation,

Numbers substituted, 0, 1, 2, 3,
Corresponding results, 7, 1, 1, 13,

the results are all of the same sign; and since, for $x=3$, $x^3+7>7x$, it is evident that all values of x greater than 3 must give positive results; it cannot be discovered whether the proposed equation contains two positive roots except by substituting for x numbers which differ from each other by less than one.

If, however, the two roots are equal, results of contrary signs cannot be given by any substitution; it therefore becomes necessary to enquire whether this equation contains two equal roots.

Applying the method of Article 250 to the polynomials $X=x^3-7x+7$, $X'=3x^2-7$, a last remainder $=1$ is obtained; therefore the equations $X=0$, $X'=0$, have no common factor, and the proposed equation, $X=0$, no equal roots.

To effect with certainty the separation of the two doubtful roots, the equation of the squares of the differences may be employed. This equation, for the example under consideration, has been found (Art. 249) to be

$$z^3-42z^2+441z^2-46=0.$$

Making $z=\frac{1}{u}$, this equation is transformed into

$$u^3-9u^2+\frac{42}{49}u-\frac{1}{49}=0,$$

$$\text{or } u^2(u-9)+\frac{42}{49}\left(u-\frac{1}{42}\right)=0.$$

Now 9 is evidently a superior limit of the positive values of z in this transformed equation; therefore $\lambda=\frac{1}{9}$ is an inferior limit of the positive values of z; therefore, in conclusion, $\delta=\sqrt{\lambda}=\sqrt{\frac{1}{9}}=\frac{1}{3}$ is a quantity less than the least difference between the roots of the proposed equation.

To avoid the substitution of the fractions $\frac{1}{3}$, $\frac{2}{3}$, $\frac{3}{3}$, &c. in the proposed equation, let $x=\frac{x'}{3}$; the result is the transformed equation,

$$x'^3-63x'+189=0,$$

in which it is not necessary to substitute any but whole numbers.

Unnecessary substitutions are avoided by determining the superior and inferior limits of the positive values of x'; now a superior limit of the positive roots of the proposed equation is evidently $\sqrt{7}$; and if x is changed into $\frac{1}{x}$ the proposed equation is transformed into

$$x^3-x^2+\tfrac{1}{7}=0,$$

of which the superior limit is 1; therefore 1 is an inferior limit of the positive roots of the proposed equation; the positive roots of the equation in x' are consequently between 3×1, or 3, and $3\times\sqrt{7}$, or between 3 and $\sqrt{63}$, or between 3 and 8; hence it is not necessary to substitute for x' any number greater than 8;

Numbers substituted,	3,	4,	5,	6,
Corresponding results,	+27,	+1,	−1,	+27.

Without proceeding further, the signs of the results indicate two positive roots; the one between 4 and 5, the other between 5 and 6; consequently the proposed equation has also two roots, the one between $\frac{4}{3}$ and $\frac{5}{3}$, the other between $\frac{5}{3}$ and $\frac{6}{3}$.

It has been already found that a root is comprehended between −3 and −4; therefore the three roots of the proposed equation are real and completely separated.

5th Example. Let $x^3-5x-3=0$.

This equation has three real roots; one positive, between 2 and 3; two negative, the first between 0 and −1, the second between −1 and −2.

METHODS OF APPROXIMATION.

271. Two limits being found between which each of the roots of an equation is comprehended, it remains to determine each with a required degree of accuracy.

The most obvious method is to substitute in the proposed equation numbers intermediate between these limits, until two numbers are found giving results of contrary signs, and differing from each other by less than the fraction which expresses the required degree of approximation. Then either of the two numbers may be taken for the value of the root.

For example, suppose that A, B are two limits which comprehend between them only the single root a; and that A being less than B, $B-A=D$. Substituting $A+\frac{1}{2}D$ for x in the proposed equation $X=0$, the sign of the result will show whether a is between A and $A+\frac{1}{2}D$, or between $A+\frac{1}{2}D$ and B; and consequently the root will be included between two limits which differ from each other by $\frac{1}{2}D$. By a repetition of the same operation the root is next included between two limits which differ from each other by only $\frac{1}{4}D$; and it is evident that by repetitions of this process two limits must be at length found whose difference is less than a given quantity, δ. Then each of these limits will be a value of a, approximated to within less than the quantity δ.

It is not necessary to take for each new substitution a number which is exactly equidistant between the two preceding limits; it is even frequently more convenient to employ another.

By this process the approximation may be carried to any degree of nearness; but the calculation becomes extremely laborious through the great number of substitutions which it is necessary to make. It is therefore employed to obtain only a certain degree of approximation; to within $\frac{1}{10}$, for example; and then the calculation is made by a much more expeditious method, that of Newton.

When the root is less than 1, it is convenient to determine its value, not to within less than $\frac{1}{10}$ of 1, but to within less than $\frac{1}{10}$ of the root itself. Suppose, for example, that the approximate value ought to differ from the true value, a, by a quantity less than $\frac{1}{10}a$; and let A, B be the limits of a, A being less than B. Since a is greater than A, if $B-A$ is less than $\frac{1}{10}A$, then à fortiori is $B-A$ less than $\frac{1}{10}a$; hence it will be sufficient to continue the successive substitutions until two limits are obtained, whose difference is less than the tenth part of the smallest of these limits; however, it is always possible to change roots which are less than 1 into others greater than 1, either by changing x into $\frac{1}{x}$, or by multiplying all the roots by a suitable number.

272. To determine the value of a root by the method of Newton, it is necessary that the root be already known with a certain degree of approximation; for the convenience of calculation, this preliminary approximation should be to within $\frac{1}{10}$, and in the following investigation it is taken for granted that the values of the roots are so determined.

Let a root of the equation $X=0$, determined to within $\frac{1}{10}$, be denoted by a; let $x=a+y$; and let $a+y$ be substituted for x in the equation $X=0$; this is transformed into

$$A+A'y+\tfrac{1}{2}A''y^2+\frac{1}{2.3}A'''y^3+, \&c.=0,$$

in which A, A′, A″ ... are the values given to the polynomial X and its derivatives X′, X″ ... by the substitution of $a+y$ for x; from this equation is obtained

$$y=-\frac{A}{A'}-\frac{A''y^2}{2A'}-\frac{A'''y^3}{2.3.A'}-, \&c.$$

The value of y, which is to be found, is less than $\frac{1}{10}$; consequently y^2, y^3 ... are respectively less than $\frac{1}{100}$, $\frac{1}{1000}$ ... Admitting that the sum of the terms which contain y^2, y^3, &c. is less than $\frac{1}{100}$, it is evident that, neglecting them, the value of y is obtained to within less than $\frac{1}{100}$, viz.

$$y=-\frac{A}{A'}.$$

Dividing, therefore, A by A′, carrying the division to hundredths, and adding the quotient to the first value a, a new value of the root a is obtained, which may be denoted by β, and regarded as an approximation to within less than $\frac{1}{100}$.

The value β is corrected as the value a has been corrected, by making $x=\beta+y$, and seeking for y a value less than $\frac{1}{100}$.

Denoting by B, B′, B″, ... what the polynomials X, X′, X″ ....., become by the substitution of $\beta+y$ for x, the transformed equation is

$$B+B'y+\tfrac{1}{2}B''y^2+\frac{1}{2.3}B'''y^3+, \&c.=0.$$

$$\text{Whence } y=-\frac{B}{B'}-\frac{B''y^2}{2B'}-\frac{B'''y^3}{2.3.B'}-, \&c.$$

$$\text{Rejecting the terms in } y^2,\ y^3,\ \&c.,\ y=-\frac{B}{B'}.$$

Since by supposition y is less than $\frac{1}{100}$, the higher powers of y are less than $\left(\frac{1}{100}\right)^2$. Admitting that the sum of the terms in y^2, y^3, &c. is less than $\left(\frac{1}{100}\right)^2$, the value $y=-\frac{B}{B'}$ will be approximated to within less than $\left(\frac{1}{100}\right)^2$; for this reason the evaluation of the quotient of $-B$ by B' should be carried to the fourth decimal figure. Adding the quotient to β, the result is a new value of the root a, which may be denoted by the character γ, and which may be regarded as an approximation to within less than the $\frac{1}{10000}$ of 1.

By continuing this process, and always rejecting the terms which contain powers of the correction y superior to the first, (and admitting that the part rejected is less in the first transformation than $\left(\frac{1}{10}\right)^2$, in the second than $\left(\frac{1}{100}\right)^2$, in the third than $\left(\frac{1}{10000}\right)^2, \ldots,$) successive approximate values will be obtained in which the number of decimal figures will each time be doubled. The approximation by this method must be rapid, for the third correction gives eight decimal figures, the fourth sixteen, &c.

The calculation of these corrections is simple and uniform, all of them being evidently derived from the same formula,

$$y=-\frac{X}{X'}, \qquad - \quad - \quad - \quad 1$$

by replacing x first by α, then by β, and so on, in such a manner that after the formula has given the first correction it gives again the second, then the third, &c.

273. The approximations obtained by this method are, however, far from certain, for they depend on suppositions which are not always true; for example, it may happen that in the first correction the sum of the terms in y^2, y^3, &c. may be greater than $\frac{1}{100}$, or in the second correction greater than $\frac{1}{10000}$. The habit of calculating will suggest in particular cases the precautions necessary to the employment of this method with safety; indeed it can be shown that it is easy to discover after each correction whether the presumed degree of approximation has been attained or not.

Beginning with the value β, which is supposed an approximation to within $\frac{1}{100}$; in order that the value of β may have this degree of approximation, it is necessary that the true value of the root be between $\beta-\frac{1}{100}$ and β, or between β and $\beta+\frac{1}{100}$. Consequently substituting in X the three numbers $\beta-\frac{1}{100}$, β, $\beta+\frac{1}{100}$, if the result of the substitution of $\beta-\frac{1}{100}$ or $\beta+\frac{1}{100}$ is of a sign contrary to that which is given by the substitution of β, it is certain that the value β is truly approximated to within the $\frac{1}{100}$ of 1; but if the results are of the same sign, it is, on the other hand, certain that the degree of approximation is less considerable. In this case it will be necessary to re-commence the calculation with a more nearly approximate value than a; for example, instead of the approximation to $\frac{1}{10}$, an approximation may be sought to $\frac{1}{20}$, then an examination may be made in the manner already explained to discover whether the correction given by formula 1 is exact to hundredths; if it is not, the calculation must be again re-commenced with a still more approximate first value, and so on until the root is obtained exact to two decimal figures. The approximate value deduced from the following correction must be verified in the same manner; and if the fourth decimal figure is inaccurate, it must be suppressed. If three figures are accurate, and a new correction is deduced, the approximation given by this is presumed to extend to the sixth decimal figure. It will be necessary to verify this correction by the same process.

LAGRANGE'S METHOD OF APPROXIMATION.

274. It follows from the Articles on the separation of roots that an equation may be so transformed that its roots shall differ from each other by more than 1. The integer part of each root can also be determined. This premised, let $X=0$ be the equation under consideration, and let α be the integer part of a real and positive root.

If x is made equal to $\alpha+\frac{1}{y}$, the equation is transformed into another in Y, which may be represented by $Y=0$, and which is of the same degree as the proposed equation $X=0$. The value of y, which it is necessary to know in order to obtain the root sought, must be positive, and greater than 1; besides, it is evident that y can have only a single value of this kind, otherwise there must be two values of x between α and $\alpha+1$. Therefore, if in $Y=0$ is substituted the series 1, 2, 3, ... two results of contrary signs must be obtained, and two of the numbers substituted must comprehend between them the required value, so that the less of the two is the integer part of the value of y. Let this integer part be equal to β, and let $y=\beta+\frac{1}{z}$. Thus a transformed equation in z, namely, $Z=0$, is obtained, which has a single root greater than 1. Consequently by the substitution for z of the numbers 1, 2, 3, ... the integer part of z must be determined.

Let γ be this integer part, and let $z=\gamma+\frac{1}{u}$. This gives a new transformed equation, $U=0$, in which there is a single root greater than 1, the determination of the integer part of which is conducted as before. Also the series of transformed equations is continued as far as may be considered necessary.

The substitutions which have been made are

$$x=\alpha+\frac{1}{y},\ y=\beta+\frac{1}{z},\ z=\gamma+\frac{1}{u},\ \ldots.$$

whence it is concluded that the required root is expressible by the continued fraction

$$x=\alpha+\cfrac{1}{\beta+\cfrac{1}{\gamma+}}, \text{ \&c.}$$

In calculating the successive convergents any required degree of approximation can be obtained. But to avoid useless calculations it will be proper, successively as the numbers α, β, γ, ... are found, to form the corresponding convergents; and when two consecutive convergents have been obtained such that, if unity is divided by the product of their denominators, the result is a fraction less than the allowed error, it is certain that the first of the two convergents will have the degree of approximation required.

The calculation of the transformed equations in y, z, u ... is tedious; it is facilitated by attention to the manner in which the coefficients of each equation are deduced from the preceding equation. When x is replaced by $\alpha+\frac{1}{y}$ n the equation $X=0$, if the results, which are obtained by substituting a for x in X and its derivatives, are represented by A, A′, A″, ... the transformed equation may be at once written thus,

$$A+A'\frac{1}{y}+\frac{A''}{2}\,\frac{1}{y^2}+\frac{A'''}{2.3}\,\frac{1}{y^3}+, \&c. =0.$$

Consequently, multiplying by y^m (m being the degree of the equation) the transformed equation $Y=0$ becomes

$$Ay^m+A'y^{m-1}+\frac{1}{2}A''y^{m-2}+\frac{1}{2.3}A'''y^{m-3}+, \&c. =0.$$

And it is known how the coefficients A, A′, ... of this equation are derived from X.

The equation $Z=0$ is calculated in the same manner; it is of the form

$$Bz^m+B'z^{m-1}+\frac{1}{2}B''z^{m-2}+, \&c. =0;$$

the coefficients B, B′, B″, ... are obtained by substituting β for y in the polynomial Y and its derivatives; and in the same manner are obtained all the other transformed equations.

APPLICATIONS OF THE METHODS OF NEWTON AND LAGRANGE.

275. Let $x^3+7x+7=0$. It is required to find the values of the roots of this equation by the method of Newton?

It has been found (Art. 270) that the proposed equation has a positive root between $\frac{4}{3}$ and $\frac{5}{3}$, another between $\frac{5}{3}$ and $\frac{6}{3}$, and a negative root between -3 and -4.

It is necessary, first, to approximate to within one tenth of the true value, by bringing the limits $\frac{4}{3}$ and $\frac{5}{3}$ closer together. $\frac{4}{3}$ expressed in decimals is 1·33 (to the nearest hundredths), and $\frac{5}{3}$, 1·67.

The signs of the results which are obtained by substituting 1·33 and 1·67 for x in the proposed equation are + and −; and the sign of the result obtained by the substitution of 1·5 (the arithmetical mean between 1·33 and 1·67) is −; therefore the root is comprehended between the limits 1·33 and 1·5.

The intermediate number 1·4 differs from 1·33 by $\frac{7}{100}$, and from 1·5 by $\frac{1}{10}$; therefore the value $x=1{\cdot}4$ is an approximation to within $\frac{1}{10}$. To obtain a closer approximation it is necessary to employ the formula $y=-\frac{X}{X'}$.

$$\text{In this example } X=x^3-yx+y,\ X'=3x^2-y;$$

$$\text{therefore } -\frac{X}{X'}, \text{ or } y=\frac{x^3-yx+y}{3x^2-y}.$$

Substituting in this expression the approximate value $x=1{\cdot}4$,

$$y=-\frac{0{\cdot}056}{1{\cdot}12}=-0{\cdot}05;$$

Whence the corrected value of x is $x=1{\cdot}4-0{\cdot}05=1{\cdot}35$.

This approximation is supposed true to hundredths. The supposition is verified by substituting in equation $X=0$ the values 1·35 and 1·36.

Now 1·35 substituted for x gives +0·010375,
and 1·36 - - - - −0·004544.

These results being of contrary signs, it is certain that the hundredths in the value of x are correct.

Substituting, next, the value $x=1{\cdot}35$ in the formula $y=-\frac{X}{X'}$,

$$y=\frac{0{\cdot}010375}{1{\cdot}5325}=+0{\cdot}0068\,;$$

Consequently the value given by the second approximation is

$$x=1{\cdot}35+0{\cdot}0068=1{\cdot}3568.$$

To verify this value it is sufficient to substitute for x in the proposed equation the values 1·3568 and 1·3569.

$$\begin{array}{lll} 1{\cdot}3568 & \text{gives} & +0{\cdot}000141586432 \\ 1{\cdot}3569 & \text{-} & -0{\cdot}000006100991 \end{array}$$

The difference of sign in these results proves that the value of x is exact to four decimal figures.

By continuing this process of approximation the value of x may be determined with any required degree of accuracy.

The second positive root, which is between $\frac{5}{3}$ and $\frac{6}{3}$, is found in the same manner.

To obtain the negative root, which is comprehended between the limits -3 and -4, the most convenient process is to change x into $-x$; the transformed equation has a positive root between $+3$ and $+4$. The value of this being determined, as in the case of the preceding positive root, if the sign is changed the result is the negative root of the proposed equation.

276. If the method of Lagrange is applied to the calculation of the roots of the same equation, it is necessary first to transform the proposed equation into another whose roots differ from each other by more than 1.

Since $\delta=\frac{1}{3}$, make $x=\frac{1}{3}x'$; the transformed equation is

$$x'^3-63x'+189=0\,;$$

it is necessary, therefore, to determine, by means of continued fractions, the value of x' comprehended between 4 and 5; then dividing this value by 3, the result is the value of x comprehended between $\frac{4}{3}$ and $\frac{5}{3}$.

Since in the equation $x'^3-63x'+189=0$ the value of x' is between 4 and 5, let $x'=4+\frac{1}{y}$, and let this value be substituted for x'; the transformed equation is

$$Ay^3+A'y^2+\frac{1}{2}A''y+\frac{1}{2.3}A'''=0\,;$$

$$\begin{aligned} \text{or, since } A&=4^3-63\times4+189=+1, \\ A'&=3\times4^2-63=-15, \\ \tfrac{1}{2}A''&=3\times4=+12, \\ \tfrac{1}{2.3}A'''&=+1. \end{aligned}$$

The first transformed equation is

$$y^3-15y^2+12y+1=0.$$

Substituting 14 for y, the result is -27,
Substituting 15 for y, the result is $+181$.

Making $y=14+\frac{1}{z}$, the second transformed equation is

$$Bz^3+B'z^2+\frac{1}{2}B''z+\frac{1}{2.3}B'''=0\,;$$

$$\text{or, since } B=14^3-15\times 14^2+12\times 14+1=-27,$$
$$B'=3\times 14^2-2\times 15\times 14+12=+180,$$
$$\frac{1}{2}B''=3\times 14-15=+27,$$
$$\frac{1}{2.3}B'''=+1.$$

The second transformed equation is

$$-27z^3+180z^2+27z+1=0, \text{ or, changing the signs,}$$
$$27z^3-180z^2-27z-1=0.$$
$$\text{Making } z=6, \text{ the result is } -811,$$
$$\text{Making } z=7, \text{ the result is } +251.$$

The value of z being between 6 and 7, let $z=6+\frac{1}{u}$; the third transformed equation is

$$Cu^3+C'u^2+\frac{1}{2}C''u+\frac{1}{2.3}C'''=0;$$

$$\text{or, since } C=-811,\ C'=+729,\ \frac{1}{2}C''=+306,\ \frac{1}{2.3}C'''=+27,$$
$$811u^3-729u^2-306u-27=0,$$
$$u=1 \text{ gives } -251,$$
$$u=2 \text{ gives } +2933.$$

The value of u being between 1 and 2, let $u=1+\frac{1}{v}$.

Without continuing further this series of transformations, the result already obtained is

$$x'=4+\cfrac{1}{14+\cfrac{1}{6+\cfrac{1}{1+,\ \&c.}}}$$

and the convergents, calculated by the rule of Article 195, are

$$\frac{4}{1},\ \frac{57}{14},\ \frac{346}{85},\ \frac{403}{99},\ \&c.$$

In the first, $\frac{4}{1}$, the error is in defect, and less than $\frac{1}{14}$.

In the second, $\frac{57}{14}$, the error is in excess, and less than $\frac{1}{1190}$.

In the third, $\frac{346}{85}$, the error is in defect, and less than $\frac{1}{8415}$.

In the fourth, $\frac{403}{99}$, the error is in excess, and less than $\frac{1}{18216}$.

To estimate the degree of approximation of the fourth convergent without knowing the denominator of the fifth, it is evident that this denominator must be at least equal to 99+85, or 184; therefore the error of the fourth is less than $\frac{1}{99\times 184}$ or $\frac{1}{18216}$.

Dividing these convergents by 3, the values of x are

$$x=\frac{4}{3},\ \frac{19}{14},\ \frac{346}{255},\ \frac{403}{297},$$

which are approximations, alternately in defect and excess, to within less than the fractions $\frac{1}{42},\ \frac{1}{3570},\ \frac{1}{25245},\ \frac{1}{54648}$.

The approximate values of the other positive root can be found in the same manner. The negative root, which is between −3 and −4, can be obtained from the proposed equation without the intervention of the transformed equation in x'.

277. By means of proper substitutions it can always be determined how many real roots are contained in an equation. If this number is not equal to the degree of the equation it must contain imaginary roots. These are comprised in the formula $x=a+b\sqrt{-1}$, a and b being real quantities which it is necessary to determine.

If the expression $a+b\sqrt{-1}$ is substituted for x in the equation $X=0$, this is changed into another equation of the form $A+B\sqrt{-1}=0$, in which A, B ought also to be real quantities.

Now, in order that $A+B\sqrt{-1}$ may be equal to zero, it is necessary that $A=0$, $B=0$.

The investigation is therefore reduced to that of the real values of a, b, which satisfy the equations $A=0$, $B=0$. To obtain these values it is necessary to eliminate one of the quantities a, b, (a, for example), to seek the real roots of the final equation in b, and then the corresponding values of a, rejecting the solutions in which this unknown quantity is imaginary.

Whether the coefficients of the equation in x are real or imaginary, these remarks are alike applicable. But when the coefficients are real the imaginary roots are even in number, and arranged in pairs of the form $a\pm b\sqrt{-1}$.

SECTION XV.

STURM'S THEOREM. HORNER'S METHOD FOR THE SOLUTION OF NUMERICAL EQUATIONS. SOLUTIONS OF LITERAL EQUATIONS OF THE THIRD AND FOURTH DEGREE.

278. If two consecutive terms of an equation have like signs there is a Permanence, and if they have unlike signs, a Variation.

The equation $x^7-2x^5-5x^2+8x-10=0$ has three variations; the first from $+x^7$ to $-2x^5$, the second from $-5x^2$ to $+8x$, the third from $+8x$ to -10, and one permanence from $-2x^5$ to $-5x^2$.

The signs of this equation, disengaged from the expressions which are connected by them, are +, −, −, +, −.

279. Let $X=0$ be an equation of the form

$$Px^m+Qx^{m-1}+Rx^{m-2}\ldots+Tx+U=0,$$

of which the coefficients are real, and which contains no equal roots. Form $X_{\prime}$, the derived polynomial of X, divide X by $X_{\prime}$, and continue the division until the remainder is of a degree inferior to $X_{\prime}$. Change the signs of all the terms of the remainder, then denote it by $X_{\prime\prime}$. Divide $X_{\prime}$ by $X_{\prime\prime}$ in the same manner as X has been divided by $X_{\prime}$, change the signs of the terms of this remainder, denote it by $X_{\prime\prime\prime}$; divide $X_{\prime\prime}$ by $X_{\prime\prime\prime}$, and continue this series of operations in all respects (except that of changing the signs of the terms of each remainder) as if it were required to find the greatest common measure of $X, X_{\prime}$. A remainder must be found which does not contain x, and which cannot be equal to zero, since the equation $X=0$ contains no equal roots. Change the signs of this last remainder, and denote it by X_r.

A series of functions, X, $X_{\prime}$, $X_{\prime\prime}$, $X_{\prime\prime\prime}$ . . . . X_{r-1}, X_r, is thus obtained, of which all, except the last, contain x. This series is named the series (x), and when x is replaced by a particular value, a, it is called the series (a). The name Intermediate function is given to any of the functions between X and X_r.

Then, to determine how many real roots of the proposed equation $X=0$

are comprehended between any two numbers, α, β, $(\alpha<\beta)$, substitute α in the series (x); write the signs of all the results, and count the number of variations; next, substitute β in the series (x), write the signs of all the results, and count the number of variations. As many variations as the second set of signs has fewer than the first, so many real roots of the equation $X=0$ are comprehended between α and β.

In dividing X by $X_{\prime}$ let $Q_{\prime}$ be the quotient; then, since $X_{\prime\prime}$ is the remainder with all the signs changed,

$$X=X_{\prime}Q_{\prime}-X_{\prime\prime}.$$

Similarly,

$$\begin{aligned} X_{\prime}&=X_{\prime\prime}Q_{\prime\prime}-X_{\prime\prime\prime} \\ X_{\prime\prime}&=X_{\prime\prime\prime}Q_{\prime\prime\prime}-X_{\text{IV}} \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ X_{r-\prime\prime}&=X_{r-\prime}-X_{r}. \end{aligned}$$

Hence, 1st, two consecutive functions of the series (x) cannot become zero for the same value of x. For if, for the same value of x, $X_{n-1}=0$ and $X_{n}=0$, the equation $X_{n-1}=X_{n}Q_{n}-X_{n+1}$, which is comprehended in the series (x), becomes $0=0-X_{n+1}$.

Therefore $X_{n+1}=0$; and, proceeding from equation to equation, it would be proved that $X_{r}=0$, which is contrary to the hypothesis.

2d. If an intermediate function of the series (x) becomes zero for a certain value of x, the values of the preceding and following function are of contrary signs. For when X_{n}, for example, becomes zero, $X_{n-1}=X_{n}Q_{n}-X_{n+1}$ becomes $X_{n-1}=-X_{n+1}$.

a. Let it be supposed that the value of x is made to begin with α, and to increase in a continuous manner, and that these continuous values are introduced into the series (x); so long as x has not reached a value which destroys one or more functions of the series (x), it is evident that the signs of none of these functions are changed, and, consequently, that the series presents always the same succession of signs.

When x has attained the value a, for which one or more of the intermediate functions $X_{\prime}$, $X_{\prime\prime}$, . . . becomes zero; then if X_{n} is one of these functions, the functions X_{n-1}, X_{n+1}, which precede and follow X_{n}, ought for this value to be different from zero, and to have contrary signs; so that, giving to the vanishing function either the sign + or —, the series of the three functions X_{n-1}, X_{n}, X_{n+1} will present always one variation, and only one.

Since neither of the functions X_{n-1}, X_{n+1}, changes sign between $x=\alpha$ and $x-a$, they must, for $x=a$, be of contrary signs; therefore, whatever was the sign of X_{n} before the substitution of a for x, the series of these three functions must have presented one variation, and only one.

b. When x becomes $a+h$ (h being positive, and so small that no root of the equations $X_{n-1}=0$, $X_{n+1}=0$ falls betweens a and $a+h$,) in this interval neither of the functions X_{n-1}, X_{n+1} can change its sign. They must, therefore, have contrary signs. Consequently, whatever the sign of X_{n}, there is still only one variation in the series of the three functions.

If any other intermediate function become 0 for $x=a$, it must in like manner be situated between two functions which do not become zero, and the preceding considerations must apply to these three functions. Therefore, after the substitution of the value a for x, there are in the series (x) the same number of variations as before, although they may be differently distributed.

c. The succession of signs which is obtained by substituting for x values a little greater than a remains unchanged until x is increased so as to exceed the value which makes some function of the series (x) to disappear. But if this value does not reduce the first function, X, to zero, it follows from the preceding part of this demonstration that the variations of the series (x) are the same in number after the substitution of this value as before it, and that this number of variations cannot change unless x is increased so as to exceed a root of the equation $X=0$. It is necessary, therefore, to compare the signs which then belong to the series (x) with those which it has for values of x, a little less than a.

d. Let a be a root of the equation $X=0$; make $x=a+h$ (h being an indefinitely small positive quantity); denote by A, A', A'' . . . the values of the polynomial X and the derived polynomials for $x=a$, and by H the value of X corresponding to $a+h$.

$$\text{Then } H=A+A'h+\tfrac{1}{2}A''h^2+, \text{ \&c.}$$

But since a is a root of X, $A=0$; therefore $H=A'h+\frac{1}{2}A''h^2+$, &c., in which expression A' cannot be zero, because the equation $X=0$ has no equal roots.

Making h a common factor of the terms of the second member of the last equation,

$$H=h(A'+\tfrac{1}{2}A''h+, \text{ \&c.}),$$

the value of h being indefinitely small and positive, the sign of the whole quantity $A'+\frac{1}{2}A''h+$, &c. is the same as the sign of the term A'; therefore the sign of H must be the same as the sign of A'.

To discover what was the sign of X before x reached the value a it is necessary to make $x=a-h$. The result, H', of this substitution is obtained by changing h into $-h$ in the expression of H;

$$\text{wherefore } H'=-h(A'-\tfrac{1}{2}A''h+, \text{ \&c.})$$

From which it is evident that for very small values of h the quantity H' has a sign contrary to that of A'. Therefore, when the value of x is less than a the series (x) has a variation from X to $X_{\prime}$; and when the value of x exceeds a this variation is replaced by a permanence. But for values of x a little less than a or a little greater than a, the remaining part of the series (x) ought always to have the same number of variations, even although the value $x=a$ should reduce some intermediate function to zero. Therefore when x, by increasing, comes to exceed a root of the equation $X=0$, the series (x) loses one variation.

Continuing to augment x, the remaining number of variations of the series (x) is not changed until x comes again to exceed a root; then another variation is lost.

e. The same thing happens for every instance in which x exceeds a new root; so that the number of variations lost by the series (x) is always equal to the number of roots of the equation $X=0$ comprehended between the value $x=\alpha$, with which the substitutions are commenced, and the value $x=\beta$, with which they are concluded.

280. In the successive divisions made to find $X_{\prime\prime}$, $X_{\prime\prime\prime}$, &c. the dividends and divisors may be multiplied by any positive numbers; for by such multiplication none of the signs are changed.

a. If in the series (x) a function X_k is found which does not become zero for any value between $x=\alpha$ and $x=\beta$, and if it is required to discover the number of roots comprehended between α and β, the series may be made to end with the function X_k.

Then making x to vary from α to β, the principles which are applicable to complete series may be applied to the partial series $X, X_{\prime}, X_{\prime\prime}, X_{\prime\prime\prime}, \ldots X_k$.

b. If either of the limits α, β reduces an intermediate function to zero, this function is always situated between two others which do not disappear, and which are of contrary signs; so that the function which becomes zero may be taken, indifferently, with the sign + or the sign −, or omitted altogether.

But if for either of the limits, β for example, X vanishes, it is to be concluded, first, that β is a root. Next it follows, from Article 279 d, that if values a little greater than β are given to x, there is a permanence from X to $X_{\prime}$, whilst in the remaining part of the series there is the same number of variations as for $x=\beta$. Consequently the general rule 279 e. gives the number of real roots comprehended between α and a quantity a little greater than β

281. If $X_{\prime}$ is not the derived polynomial of X, but a polynomial which has no factor in common with X, and which takes a sign contrary to X for values of x which differ very little in defect from any root whatever of $X=0$, and if this polynomial $X_{\prime}$ is employed to find $X_{\prime\prime}$, $X_{\prime\prime\prime}$, &c. in the same manner

as the derived function of X is employed in Article 279, the theorem of Sturm subsists equally with the new series X, $X_{,}$, $X_{,,}$, $X_{,,,}$, &c. For if the details of the demonstration are resumed, it will be found that the new intermediate functions have the same properties; likewise in passing from a value of x a little less than a root of $X=0$ to a value of x a little greater than a root of $X=0$, the function X must change its sign, whilst the function $X_{,}$, which has no factors in common with X, must preserve the same sign.

282. If the equation $X=0$ contains equal roots; if, for example, $X=(x-a)^n(x-b)^{n'}(x-c)(x-d)$ . . . . the divisions made to deduce from the polynomial X and its derived polynomial $X_{,}$ the functions $X_{,,}$, $X_{,,,}$, &c. must lead to an exact divisor X_r, a function of x which is the greatest common measure of X, $X_{,}$; and from the composition of this common measure (Art. 251) if X, $X_{,}$ are divided by X_r the quotients are

$$\frac{X}{X_r}=V=(x-a)(x-b)(x-c)(x-d)\ \ldots\ldots$$

$$\frac{X_{,}}{X_r}=V_{,}=\begin{cases} n(x-b)(x-c)(x-d)\ \ldots +n'(x-a)(x-c)(x-d)\ \ldots \\ +(x-a)(x-b)(x-d)\ \ldots +(x-a)(x-b)(x-c)\ \ldots \\ +\ \&c. \end{cases}$$

of which V contains no equal factors, and $V_{,}$ has no factor in common with V.

Let $x-a$ be a real factor of X, and make

$$H=(x-b)(x-c)(x-d)\ \ldots$$

$$H_{,}=n'(x-c)(x-d)\ \ldots\ldots +(x-b)(x-d)\ \ldots +,\ \&c.,$$

$$\text{Therefore } V=(x-a)H;\ V_{,}=nH+(x-a)H_{,}.$$

Now, from the nature of the operations which serve to discover the greatest common measure of X, $X_{,}$, and to find the quotients V, $V_{,}$, these quotients, and by consequence also H, $H_{,}$, ought not to have imaginary coefficients, since there are none in either X or $X_{,}$; whence, for values of x very little less than a, the factor $x-a$ being very small, $V_{,}$ will have the sign of H, which sign is evidently contrary to that of V. Therefore, in applying the theorem of Sturm to the equation $V=0$, the quotient $V_{,}$ may be employed instead of the derived function of V.

Likewise the functions $V_{,,}$, $V_{,,,}$, which it is necessary to deduce from V, $V_{,}$ by the rule (Art. 236), are the quotients of $X_{,,}$, $X_{,,,}$ . . . . by X_r.

Therefore the two series X, $X_{,}$, $X_{,,}$ . . . . X_r,
V, $V_{,}$, $V_{,,}$ . . . . $V_{,}$,

differ from each other only by a common factor; and whatever the sign of this factor, it is evident that the two ought to present the same variations when a value of x which does not destroy X is substituted in them. Therefore, by the application of the theorem to the first series, it can be determined how many roots of the equation $V=0$ there are between two numbers, α and β, neither of which renders $X=0$; or (which is the same thing) how many real roots there are in the equation $X=0$, abstraction being made of the degree of multiplicity. Hence, in the equation $X=0$, the number of real roots comprehended between α and β is equal to the excess of the number of variations in the signs of the functions X, $X_{,}$, $X_{,,}$ . . . . X_r for $x=\alpha$ above the number of variations for $x=\beta$, abstraction being made of the multiplicity of the roots.

283. This theorem affords a simple means of establishing the conditions of the reality of the roots of an equation. Since the real roots are all comprehended between two limits $-L'$ and $+L$, the one negative and the other positive (which may be chosen $=\infty$), the question is reduced to the investigation of the conditions necessary in order that from $x=-L'$ to $x=+L$ the series X, $X_{,}$, $X_{,,}$ . . . . X_r may lose a number of variations equal to the degree of the equation.

Supposing m the degree of the equation, the series (x) must lose m variations. To lose m variations it must contain at least $m+1$ terms; and since it cannot have more, the functions X, $X_{,}$, $X_{,,}$, ... are $m+1$ in number.

They are consequently of the respective degrees m, $m-1$, $m-2$, &c., the last which does not contain x being X_m.

When in the polynomials in x very large numbers, positive or negative, are substituted for x, the results are of the same sign as if each polynomial were reduced to its first term (Art. 258); whence, in this investigation, attention need be given to the first term only of each of the functions X, $X_{,}$, $X_{,,}$, &c.

Taking the equation $X=0$ in the ordinary form,

$$x^m+Px^{m-1}+Qx^{m-2}+\&c.=0,$$

the first term of X is x^m, that of $X_{,}$ is mx^{m-1}, and those of $X_{,,}$, $X_{,,,}$ . . . are certain functions composed of the coefficients P, Q, &c., which are determined by successive divisions. Representing these functions by $G_{,,}$, $G_{,,,}$, . . . and arranging the $m+1$ quantities in order, thus

$$x^m,\ mx^{m-1},\ G_{,,}x^{m-2},\ G_{,,,}x^{m-3}\ .\ .\ .\ .\ G_m,$$

the question is reduced to the investigation of the conditions which cause this series to lose m variations in passing from $x=-L'$ to $x=+L$. Now, that this may be possible, it is necessary that there be m variations after the substitution of $-L'$, and m permanences after the substitution of $+L$. On the other hand, in this series the powers of x diminish by 1 from term to term; consequently, if it has permanences only when $x=+L$ it must have variations only when $x=-L'$. Whence the conditions sought are reduced to those which are necessary in order that the series may have positive coefficients only; that is to say, to the conditions that

$$G_{,,}>0,\ G_{,,,}>0\ .\ .\ .\ .,\ G_m>0.$$

It is evident that these conditions cannot be more in number than $m-1$; but they may be fewer, for some of the inequalities $G_{,,}>0$, $G_{,,,}>0$ . . . may be reducible to each other.

As an application, let it be required to find the conditions necessary for the reality of the roots of the equation,

$$x^3+Qx+R=0.$$

In this example $m=3$ and $m-1=2$. The number of conditions is therefore 2; these are, that $G_{,,}>0$ and $G_{,,,}>0$. To obtain $G_{,,}$ and $G_{,,,}$ it is necessary to find $X_{,,}$ and $X_{,,,}$, which is accomplished as follows:

$$X=x^3+Qx+R\ \therefore\ X_{,}=3x^2+Q$$

$$\begin{array}{r} x^3+Qx+R \\ 3 \\ \hline 3x^2+Q)\,3x^3+3Qx+3R\,(x \\ 3x^3+Qx \\ \hline 2Qx+3R \end{array}$$

$$\therefore\ -2Qx-3R=X_{,,}.$$

$$\begin{array}{r} 3x^2+Q \\ 4Q^2 \\ \hline -2Qx-3R)\,12Q^2x^2+4Q^3\,(-6Qx+9R \\ 12Q^2x^2+18QRx \\ \hline -18QRx+4Q^3 \\ -18QRx-27R^2 \\ \hline 4Q^3+27R^2 \end{array}$$

$$\therefore\ -4Q^3-27R^2=X_{,,,}.$$

To avoid fractional expressions the dividend in the first division is multiplied by 3, and in the second by $4Q^2$; these factors being positive the signs of $X_{,,}$, $X_{,,,}$ are not affected by the multiplication.

It is thus found that $X_{,,}=-2Qx-3R$, and that $X_{,,,}=-4Q^3-27R^2$. Whence the inequalities $G_{,,}>0$, $G_{,,,}>0$, become

$$-2Q>0,\ -4Q^3-27R^2>0,$$

and changing the signs, $2Q<0$, $4Q^3+27R^2<0$.

The first of these conditions is comprehended in the second; for the term $27R^2$ being necessarily positive, the second condition requires that Q be negative.

The only condition absolutely necessary is therefore

$$4Q^3+27R^2<0.$$

When this is fulfilled the roots of the cubic equation are all real; when it is not, one root is real and two are imaginary.

284. In order that the roots of the equation $X=0$ may be all real, it is necessary that the signs of the series (x) present only variations for $x=-\infty$ and permanences for $x=+\infty$.

Let it be supposed (as in Art. 283) that the series (x) contains $m+1$ terms, which diminish in degree regularly by 1; then, since any two consecutive terms of the series are, the one of an even, the other of an odd degree, if these two functions have the same sign for $x=+\infty$, they must have contrary signs for $x=-\infty$; and if they have the same sign for $x=-\infty$, they must have contrary signs for $x=+\infty$.

If therefore the signs of the functions

$$X, X_{\prime}, X_{\prime\prime} \ldots\ldots X_m,$$

or of the first terms of these functions, viz.

$$x^m, mx^{m-1}, G_{\prime\prime}x^{m-2}, \ldots\ldots G_m,$$

for $x=-\infty$, and for $x=+\infty$, are written the one series under the other, each variation in either series will correspond to a permanence in the other series. Therefore the number of permanences for $x=-\infty$ is equal to the number of variations for $x=+\infty$.

Let n=the number of permanences for $x=-\infty$.

$\therefore n$=the number of variations for $x=+\infty$.

Whence, the whole number of signs being $m+1$, the number of permanences for $x=+\infty$ is $m-n$; and the number of variations lost between $x=-\infty$ and $x=+\infty$ is $m-n-n$, or $m-2n$. Consequently, if the number of permanences for $x=-\infty$ is n, the number of imaginary roots of the equation is expressed by $2n$, and the number of real roots by $m-2n$.

USE OF STURM'S THEOREM.

285. Let the equation which is to be resolved contain no equal roots, and let the series (x) be formed; it is evident that by the substitution of 0 for x each of the functions $X, X_{\prime}, X_{\prime\prime}$, &c. is reduced to its last term, and that by the substitution of values above a certain positive limit $(+L)$ each function is made to take the same sign as its first term. Therefore by inspection of the series (x) the number of variations of the series (0) and also the number of variations of the series $(+L)$ are known. The difference between these two numbers indicates how many positive roots are contained in the equation.

The number of negative roots is obtained with equal facility, by observing that beyond a certain limit, $-L'$, all negative numbers impart to each function of the series (x) the same sign as to its first term.

For convenience $+\infty$ and $-\infty$ are taken instead of $+L$ and $-L'$; this being an abridged manner of denoting two numbers which may be as great as we please.

As the negative roots are rendered positive by changing x into $-x$, it is necessary to treat of the positive roots only. The separation of these is effected by substituting in the series (x) increasing positive numbers, $0, \alpha, \beta, \gamma$, &c.; then by counting the number of variations lost at each substitution, the number of roots comprehended between 0 and α, α and β, &c. will be obtained. These substitutions ought to be continued until a series is found whose variations are the same in number as those of the series $(+\infty)$. Farther it is needless to proceed, for there can be no root above the last number substituted.

Admitting that there are several roots between α and β, one or more intermediate substitutions must be made, then the number of variations lost by the series (x) after each substitution indicates the number of roots comprehended between the numbers substituted.

A few trials may suffice to effect the complete separation of the roots; that is, to assign to each root two limits between which it alone is comprehended. But when two or more of the roots are very nearly equal, it becomes necessary to multiply the number of substitutions in order to separate the roots; and as the equation has no equal roots, it is certain that the separation can always be effected. The tediousness of the calculations is compensated by the advantage of determining these roots with a close degree of approximation.

The rate of increase of the numbers $0, \alpha, \beta, \gamma, \ldots$ is arbitrary.

But it is convenient to commence with the numbers 0, 1, 10, 100, &c., for thus to the advantage of having only calculations easily performed is joined that of determining how many roots there are between 0 and 1, 1 and 10, 10 and 100, &c.

If there are roots between 1 and 10, the integer part of each is determined by substituting the numbers 1, 2, 3, ... 9; if there are roots between 10 and 100, the figure of the tens of each is obtained by the substitution of the numbers 10, 20, 30, ... 90, &c.

In like manner, for roots less than 1, the highest decimal figure is obtained by substituting the numbers from ·1 to ·9, or from ·01 to ·09, &c.

By proceeding in this manner the highest figure contained in each root is obtained. To determine the next figure, suppose, for the sake of perspicuity, that it is required to find certain roots between 100 and 1000, and that it has been found that one of these roots is between 300 and 400; it is evident that the substitution of the numbers 300, 310, 320, ... in the series (x) will lead to the discovery of the figure of the tens. But, to facilitate the calculation, let x be changed into $300+x'$; the result is a new series (x') in which it is only necessary to substitute the numbers 0, 10, 20, ...

Admitting, next, that after the determination of the tens it has been found that there are roots between 350 and 360, the substitution of the numbers 350, 351, 352, ... in the series (x) will lead to the discovery of the figure of the units. But in this instance also the calculation may be simplified. By the change of x into $300+x'$ the series (x) has been already changed into another series (x'). Now this may be in like manner transformed into a series (x'') if x' is replaced by $50+x''$; and it is evident, that if in this last series the numbers 0, 1, 2, 3, ... are substituted, the same results must be obtained as by the substitution of the numbers 50, 51, 52, ... in the series (x'), or the numbers 350, 351, 352 . . . . in the series (x).

These calculations may be continued so as to discover the tenths, the hundredths, &c.; and thus the roots are not only separated, but also calculated with any required accuracy of approximation.

286. As an application of Sturm's Theorem, let it be required to determine the roots of the equation $x^4+4x^3-4x^2-11x+4=0$?

1st. To determine the functions $X_{\prime}$, $X_{\prime\prime}$, $X_{\prime\prime\prime}$, $X_{\prime\prime\prime\prime}$.

Since $X=x^4+4x^3-4x^2-11x+4$, $X_{\prime}=4x^3+12x^2-8x-11$.

Dividing X by $X_{\prime}$ as follows,

$$\begin{array}{rl}
 & x^4+4x^3-4x^2-11x+4 \\
 & \underline{\qquad\qquad\qquad\qquad\quad 4} \\
4x^3+12x^2-8x-11) & 4x^4+16x^3-16x^2-44x+16(x+1 \\
 & \underline{4x^4+12x^3-8x^2-11x} \\
 & \qquad 4x^3-8x^2-33x+16 \\
 & \qquad \underline{4x^3+12x^2-8x-11} \\
\text{The remainder is} \quad - \quad - & \qquad \underline{-20x^2-25x+27}
\end{array}$$

Changing the signs of this remainder, $X_{\prime\prime}=20x^2+25x-27$.

Dividing, next, $X_{,}$ by $X_{,,}$:

$$\begin{array}{rl}
 & 4x^3+12x^2-8x-11 \\
 & \qquad\qquad\qquad\quad 5 \\ \hline
20x^2+25x-27) & 20x^3+60x^2-40x-55(x+7 \\
 & 20x^3+25x^2-27x \\ \hline
 & \qquad 35x^2-13x-55 \\
 & \qquad\qquad\qquad\quad 4 \\ \hline
 & \qquad 140x^2-52x-220 \\
 & \qquad 140x^2+175x-189 \\ \hline
\text{The remainder is} \quad - \; - \; - & \qquad\quad -227x-31
\end{array}$$

Changing the signs of this remainder, $X_{,,,}=227x+31$:

Dividing, lastly, $X_{,,}$ by $X_{,,,}$:

$$\begin{array}{rl}
 & 20x^2+25x-27 \\
 & \qquad\qquad\quad 227 \\ \hline
227x+31) & 4540x^2+5675x-6129(20x+5055 \\
 & 4540x^2+620x \\ \hline
 & \qquad 5055x-6129 \\
 & \qquad\qquad\quad 227 \\ \hline
 & \qquad 1147485x-1391283 \\
 & \qquad 1147485x+156705 \\ \hline
\text{The last remainder is} \quad - \; - & \qquad\qquad -1547988
\end{array}$$

Changing the sign of this remainder, $X_{,,,,}=+1547988$:

Whence the different functions of the series (x) are,

$$\begin{aligned}
X&=x^4+4x^3-4x^2-11x+4, \\
X_{,}&=4x^3+12x^2-8x-11, \\
X_{,,}&=20x^2+25x-27, \\
X_{,,,}&=227x+31, \\
X_{,,,,}&=+1547988.
\end{aligned}$$

2d. To determine how many real roots are contained in the proposed equation, let, first, $-\infty$, and, second, $+\infty$, be substituted for x in the first term of each function of the series (x), viz. in x^4, $4x^3$, $20x^2$, &c.; then, since for $x=-\infty$ the series of signs is $+, -, +, -, +$, four variations, and for $x=+\infty$ the series of signs is $+, +, +, +, +$, no variation, the four roots of the proposed equation are real.

To ascertain how many of the four roots are positive, and how many negative, make $x=0$. (By this substitution each of the functions of the series (x) is reduced to its last term.)

Now for $x=0$ the series of signs is $+, -, -, +, +$, two variations; therefore 2 roots are comprehended between $-\infty$ and 0, and 2 between 0 and $+\infty$; that is, 2 are negative and 2 positive.

Before attempting the complete separation of the roots, it is convenient to determine closer limits than $-\infty$ and $+\infty$.

Making $x=-10$, the series of signs is $+, -, +, -, +$, four variations.

The number of variations for $x=-10$ being the same as for $x=-\infty$, it follows that -10 is a limit of the negative roots.

Next, making $x=+10$, the series of signs is $+, +, +, +, +$, no variation; whence $+10$ is a superior limit of the positive roots.

The separation of the positive roots is effected by substituting 1, 2, 3 ... successively for x in the series (x).

For $x=+1$ the series of signs is $-, -, +, +, +$, one variation; and for $x=+2$, $+, +, +, +, +$, no variation.

One variation being lost between 0 and $+1$, and a second between $+1$ and $+2$, one of the positive roots is comprehended between 0 and 1, and the other between 1 and 2.

The separation of the negative roots is, in like manner, made by successively substituting −1, −2, −3, &c. in the series (x), until the numbers between which the two negative roots are comprehended have been determined.

For $x=-1$ the series of signs is +, +, −, −, +, two variations.
- -2 - - −, +, +, −, +, three variations.
- -3 - - −, +, +, −, +, three variations.
- -4 - - −, −, +, −, +, three variations.
- -5 - - +, −, +, −, +, four variations.

One variation being lost between −2 and −1, and another between −5 and −4, one of the negative roots is comprehended between −1 and −2, and the other between −4 and −5. The complete separation of the roots is consequently accomplished.

When different numbers are successively substituted for x, the signs of the functions X, $X_{\prime}$, $X_{\prime\prime}$, &c. are found as in the following instances for $x=+2$ and for $x=-2$:

For $x=+2$,

$$X=16+32-16-22+4=+14,$$
$$X_{\prime}=32+48-16-11 \quad =+53,$$
$$X_{\prime\prime}=80+50-27 \quad - \quad =+103,$$
$$X_{\prime\prime\prime}=454+31 \quad - \quad - \quad =+485,$$
$$X_{\prime\prime\prime\prime}=\text{a positive number} =+1547, \text{ \&c.};$$

Whence (as above) for $x=2$ the signs are +, +, +, +, +,

For $x=-2$,

$$X=16-32-16+22+4=-6,$$
$$X_{\prime}=-32+48+16-11=+21,$$
$$X_{\prime\prime}=80-50-27 \quad - \quad =+3,$$
$$X_{\prime\prime\prime}=-454+31 \quad - \quad =-423,$$
$$X_{\prime\prime\prime\prime}=\text{a positive number} =+1547, \text{ \&c.},$$

and the series of signs for $x=-2$ is −, +, +, −, +.

Let it, next, be required to find an approximate value of the root which is comprehended between 1 and 2.

Since 1 is the integer part of the root, it is necessary to transform the equation X=0 into another equation, whose roots shall be less by 1 than the roots of the equation X=0. Making $x=x'+1$, and employing the method of Article 236, the transformation is made as follows:

$$\begin{array}{lllll} 1, & +4, & -4, & -11, & +4 \\ & 5, & +1, & -10, & -6 \\ & 6, & +7, & -3 & \\ & 7, & +14 & & \\ & 8 & & & \end{array}$$

and the transformed equation in x' is $X'=0$, or

$$x'^4+8x'^3+14x'^2-3x'-6=0.$$

The tenths of the root are to be determined from this equation; to obtain the limits between which the figure expressing these tenths is comprehended, it is necessary to form the series (x').

The functions $X'_{\prime}$, $X'_{\prime\prime}$, &c. may be formed from X', as $X_{\prime}$, $X_{\prime\prime}$ .... from X; or x, in each of the functions $X_{\prime}$, $X_{\prime\prime}$ ...., may be replaced by $x'+1$, and the function transformed into another in x', in the same manner as the function X has been transformed into X'.

The functions of the series (x') are,

$$X'=x'^4+8x'^3+14x'^2-3x'-6,$$
$$X'_{\prime}=4x'^3+24x'^2+28x'-3,$$
$$X'_{\prime\prime}=20x'^2+65x'+18,$$
$$X'_{\prime\prime\prime}=227x+258,$$
$$X'_{\prime\prime\prime\prime}=+1552074.$$

If in these functions ·1 is substituted for x, the signs are −, +, +, +, +, one variation.

And if 1 is substituted for x, the signs are +, +, +, +, +, no variation.

Therefore one root of the equation in x' is comprehended between ·1 and 1.

Making $x=\cdot 2$, the signs of the series (x') are −, +, +, +, +, one variation.
0·3 - - - −, +, +, +, +, one variation.
0·4 - - - −, +, +, +, +, one variation.
0·5 - - - −, +, +, +, +, one variation.
0·6 - - - −, +, +, +, +, one variation.
0·7 - - - +, +, +, +, +, no variation.

Consequently a root of the equation in x' is comprehended between 0·6 and 0·7, and therefore a root of the equation in x between 1·6 and 1·7.

To obtain the figure expressing the hundredths of the root, it is necessary to diminish the roots of the equation in x' by ·6; making $x'=x''+\cdot 6$, and transforming the equation in x' into an equation in x'', as follows;

$$\begin{array}{llllll}
1, & +8, & +14, & -3, & -6 \\
 & 8\cdot 6, & +19\cdot 16, & +8\cdot 496, & -0\cdot 9024 \\
 & 9\cdot 2, & +24\cdot 68, & +23\cdot 304 & \\
 & 9\cdot 8, & +30\cdot 56 & & \\
 & 10\cdot 4 & & &
\end{array}$$

the transformed equation in x'' is, $x''^4+10\cdot 4x''^3+30\cdot 56x''^2+23\cdot 304x''-0\cdot 9024=0$;

And the functions of the series (x'') are,

$$X''=x''^4+10\cdot 4x''^3+30\cdot 56x''^2+23\cdot 304x''-0\cdot 9024,$$
$$X''_{,}=4x''^3+31\cdot 2x''^2+61\cdot 12x''+23\cdot 304,$$
$$X''_{,,}=20x''^2+89x''+64\cdot 2,$$
$$X''_{,,,}=227x''+394\cdot 2,$$
$$X_{,,,,}=\text{a positive number.}$$

By substituting ·01 and ·1 successively for x'' in the series (x''), it is found that one root of the equation $X''=0$ is comprehended between these limits; and by similar substitutions of ·02, ·03, ·04, it is ascertained that this root falls between ·03 and ·04; therefore the required root of the equation in x falls between 1·63 and 1·64.

Making $x''=x'''+\cdot 03$, effecting the transformation, and forming the series (x'''), it is found that the figure which expresses the thousandths of the root is 6; and, similarly, by making $x'''=x''''+0\cdot 006$, effecting the transformation, and forming the series (x''''), that the figure which expresses the ten thousandths of the root is 9, &c. &c.

Consequently the required root of the equation in x is 1·6369+.

2d Example. It is required to separate the positive roots of the equation, $x^3+11x^2-102x+181=0$?

$$X=x^3+11x^2-102x+181,$$
$$X_{,}=3x^2+22x-102,$$
$$X_{,,}=122x-393,$$
$$X_{,,,}=+1566114.$$

Making $x=-\infty$, the series of signs is −, +, −, +, three variations.
$x=0$ - - - +, −, −, +, two variations.
$x=+\infty$ - - - +, +, +, +, no variation.

Whence the three roots are real; and of the three, two are positive and one negative.

To effect the separation of the positive roots let $x=+10$, the series of signs is +, +, +, +, no variation.

Therefore 10 is a superior limit of the positive roots.

Making $x=1$, the series of signs is $+, -, -, +$, two variations.
$x=2$ - - - $+, -, -, +$, two variations.
$x=3$ - - - $+, -, -, +$, two variations.
$x=4$ - - - $+, +, +, +$, no variation.

The two positive roots are, consequently, both comprehended between 3 and 4.

Making $x=x'+3$, and transforming the functions of the series (x),

$$X'=x'^3+20x'^2-9x'+1,$$
$$X'_{,}=3x'^2+40x'-9,$$
$$X'_{,,}=122x-27,$$
$$X'_{,,,}=+\text{ an absolute number.}$$

Making $x'=0$, the series (x') gives the signs $+, -, -, +$, two variations.
$x={\cdot}1$ - - - $+, -, -, +$, two variations.
$x={\cdot}2$ - - - $+, -, -, +$, two variations.
$x={\cdot}3$ - - - $+, +, +, +$, no variation.

Therefore the figure which expresses the tenths of both the positive roots is 2 (these roots being comprehended between ·2 and ·3).

Making $x'=x''+{\cdot}2$, and transforming the functions of the series (x') into functions of the series (x''),

$$X''=x''^3+20{\cdot}6x''^2-{\cdot}88x''+{\cdot}008,$$
$$X''_{,}=3x''^2+41{\cdot}2x''-{\cdot}88,$$
$$X''_{,,}=122x''-2{\cdot}6,$$
$$X''_{,,,}=+\text{ an absolute number.}$$

Making $x''=0$ the series of signs is $+, -, -, +$, two variations.
$x''={\cdot}01$ - - - $+, -, -, +$, two variations.
$x''={\cdot}02$ - - - $+, +, -, +$, one variation.
$x''={\cdot}03$ - - - $+, +, +, +$, no variation.

Consequently the hundredths of one root are comprehended between ·01 and ·02, and the hundredths of the other root between ·02 and ·03.

Therefore the figure which expresses the hundredths of the first root is 1; the figure which expresses the hundredths of the other root is 2, and the roots 3·21 and 3·22.

To obtain the figures which express the thousandths of the two roots, it is necessary to proceed as if for two distinct roots; the figure of the root 3·21 is obtained by making $x''=x'''+{\cdot}01$, and forming the series (x'''); and the figure of the root 3·22 by $x''=x'''+{\cdot}02$, and forming another series (x''').

Calculating by the known method, it is found that the figure which expresses the thousandths of the root 3·21 is 3, and the figure which expresses the thousandths of the root 3·22 is 9; so that the roots approximated to thousandths are 3·213 and 3·229.

HORNER'S METHOD FOR THE RESOLUTION OF NUMERICAL EQUATIONS.

287. If the absolute term of an equation is transposed to the right of the sign =, and x is replaced by an exact numerical value, the equation is reduced to an identity; but if the number substituted for x is less than a root of the equation, the numerical value of the left member is less than the absolute term (the signs, if negative, being left out of consideration); and if the number substituted for x is greater than a root, the numerical value of the left member is greater than the absolute term.

In determining the approximate value of a positive root by finding the successive figures of that root, it is evident that both members of the equation are successively diminished by a quantity less than that which corresponds to an exact root; and that the absolute number after each operation expresses the numerical value of the transformed equation which has the undetermined figures of the root x for one of its roots.

Let $x^m+A_m x^{m-1}\ .\ .\ .\ .\ .\ +A_{\prime\prime}x^2+A_{\prime}x=N$, be a numerical equation which contains a real and incommensurable root, and let the first figure of this root (found by Sturm's Theorem, or any of the other known methods) be r, making $x=x'+r$, and transforming the equation in x into an equation in x',

$$x'^m+A'_m x'^{m-1}\ .\ .\ .\ .\ +A'_{\prime\prime}x'^2+A'_{\prime}x'+A=N;$$

or, transposing A and making $N-A=N'$,

$$x'^m+A'_m x'^{m-1}\ .\ .\ .\ .\ +A'_{\prime\prime}x'^2+A'_{\prime}x'=N'.$$

One root of this equation is composed of the remaining figures of that root whose first figure is r.

Let r' be the first figure of the root x' of this equation; and let r' be substituted for x' in the equation $X'=N'$,

$$\text{then } r'^m+A'_m r'^{m-1}\ .\ .\ .\ .\ +A'_{\prime\prime}r'^2+A'_{\prime}x'=N'.$$

Dividing all the terms by r',

$$r'^{m-1}+A'_m r'^{m-2}\ .\ .\ .\ .\ +A'_{\prime\prime}r'+A'_{\prime}=\frac{N'}{r'}.$$

Whence, if N' is divided by r', the quotient is

$$r'^{m-1}+A'_m r'^{m-2}\ .\ .\ .\ .\ +A'_{\prime\prime}r'+A'_{\prime};$$

and, conversely, if the polynomial in r' is made the divisor, the quotient is r'; r' being unknown, all the terms of this polynomial except the last, $A'_{\prime}$, are also unknown. But since $r=10r'$, and

$$A'_{\prime}=mr^{m-1}+(m-1)A_m r^{m-2}+, \text{ \&c.},$$

it is evident that the last term, $A'_{\prime}$, must form a great part of the numerical value of the whole polynomial divisor in r'. The coefficient $A'_{\prime}$ of the first power of x' may, therefore, be employed as a trial divisor to find the value of r', the second figure of the root. By substituting this value for the symbol r' in the polynomial

$$r'^m+A'_m r'^{m-1}\ .\ .\ .\ .\ +A'_{\prime\prime}r'^2+A'_{\prime}r',$$

it can be ascertained whether this is or is not an accurate determination of the second figure of the root x.

Making, in like manner, $x'=x''+r'$, transforming the equation $X'=N'$ into an equation in x'', viz.,

$$x''^m+A''_m x''^{m-1}\ .\ .\ .\ .\ +A''_{\prime\prime}x''^2+A'_{\prime\prime}x''+A_{\prime\prime},$$

transposing $A_{\prime\prime}$, denoting $N'-A_{\prime\prime}$ by N'', and dividing N'' by $A'_{\prime\prime}$, the coefficient of the first power of x'', the first figure of the root x'', and by consequence the third figure of the root x, is determined, &c.

This method resembles the processes for the extraction of the square and cube root; and the coefficient of the first power of x', x'', &c. is employed for the same purpose as $2a$ in the extraction of the square, and $3a^2$ in the extraction of the cube, root.

288. Let it be required to find, by this method, a positive root of the equation,

$$x^4+3x^3+2x^2+6x=148{\cdot}6.$$

The first figure of the root is found to be 2; making $x=x'+2$, and effecting the transformation,

$$x'^4+11x'^3+44x'^2+82x'=88{\cdot}6.$$

The first figure of the root x' cannot exceed ·9, otherwise the first figure of x must be greater than 2; therefore the quotient of $88{\cdot}6\div82=1$ is too great. Substituting successively ·9, ·8, ·7 for x' in the equation $X'=88{\cdot}6$, it is found that ·7 is the first figure of x', and consequently that the first and second figures of x are 2·7.

Making $x'=x''+{\cdot}7$ and effecting the transformation,

$$x''^4+13{\cdot}8x''^3+70{\cdot}04x''^2+161{\cdot}142x''=5{\cdot}6269.$$

Dividing 5·6269 by 161·142 the quotient is ·03, which is the correct figure. (The accuracy of this figure may be tested as that of the second figure of the root has been; it will, however, be found that the results given by the trial divisor after the second figure of the root are in general accurate.)

Making $x''=x'''+{\cdot}03$, and effecting the transformation,

$$x'''^4+13{\cdot}92x'''^3+71{\cdot}2874x'''^2+165{\cdot}381768x={\cdot}72923059.$$

Dividing ·72923059 by 165·381768, the quotient is ·004, which is the first figure of the root x''' and the fourth of x.

Making $x'''=x''''+{\cdot}004$, and effecting the transformation,

$$x''''^4+13{\cdot}936x''''^3+71{\cdot}454536x''''^2+165{\cdot}952745616x''''={\cdot}066562038464.$$

The first figure of x'''' found from N'''' and the coefficient of x'''' is ·0004; and as the fifth and sixth decimal figures of the root are zeros, the value $x=2{\cdot}7344$ is a very near approximation.

The calculation of the transformations is annexed.

In practice it will not be found necessary to preserve all the decimal figures in the coefficients of the transformed equations.

It appears, in the annexed calculation, that after the third transformation the trial divisors or coefficients of the first powers of x'', x''', &c. are nearly constant, and that consequently the last figures of the root may be found by division.

In this example, if ·72923059 is divided by 165·381768, the quotient is ·004409, a result which differs by less than $\frac{1}{100000}$ from that already obtained.

1	+3 +5 +7 +9 +11	+2 +12 +26 +44	+6 +30 +82	=148·6(2 88·6
1	+11 +11·7 +12·4 +13·1 +13·8	+44 +52·19 +60·87 +70·04	+82 +118·533 +161·142	=88·6(·7 5·6269
1	+13·8 +13·83 +13·86 +13·89 +13·92	+70·04 +70·4549 +70·8707 +71·2874	+161·142 +163·255647 +165·381768	=5·6269(·03 ·72923059
1	+13·92 +13·924 +13·928 +13·932 +13·936	+71·2874 +71·343096 +71·398808 +71·454536	+165·381768 +165·667140384 +165·952745616	=·72923059(·004 ·066562038464
1 ·	+13·936 ·	+71·454536 ·	+165·952745616 ·	=·066562038464(·0004 ·

Required one value of x in each of the following equations?

1st. $x^3-12x^2+45x-53=0$?..............................*Ans.* $x=5{\cdot}879385+$.

2d. $x^3-12x-28=0$?..*Ans.* $x=4{\cdot}30213+$.

3d. $x^4-12x^2+12x=3$?...................................*Ans.* $x=2{\cdot}858083+$.

4th. $x^4+4x^3-4x^2-11x+4=0$?.........................*Ans.* $x=1{\cdot}6369135+$.

5th. $x^5+6x^4-10x^3-112x^2-207x-110=0$?........*Ans.* $x=4{\cdot}4641016+$.

6th. $x^6+2x^5+3x^4+4x^3+5x^2+6x-654321=0$?...*Ans.* $x=8{\cdot}9569795+$.

CARDAN'S RULE FOR THE RESOLUTION OF A CUBIC EQUATION.

289. Let the proposed equation be

$$Ax^3+Bx^2+Cx+D=0.$$

And let it be reduced (Art. 237) to the form

$$x^3+qx+r=0.$$

If in this reduced equation $y+z$ is substituted for x,

$$(y+z)^3+q(y+z)+r=0;$$

$$\text{or } y^3+3yz(y+z)+z^3+q(y+z)+r=0;$$

$$\text{or } y^3+(3yz+q)(y+z)+z^3+r=0.$$

In order that the second term of this equation may be reduced to zero, assume $3yz+q=0$,

$$\text{or } z=-\frac{q}{3y};$$

$$\therefore\ y^3+z^3+r=0,$$

$$\text{or } y^3+\left(-\frac{q}{3y}\right)^3+r=0,$$

$$\text{or } y^3-\frac{q^3}{27y^3}+r=0.$$

$$\text{Whence } y^6+ry^3=\frac{q^3}{27},$$

$$y^6+ry^3+\frac{r^2}{4}=\frac{r^2}{4}+\frac{q^3}{27},$$

$$y^3+\frac{r}{2}=\pm\sqrt{\frac{r^2}{4}+\frac{q^3}{27}},$$

$$\text{and } y^3=-\frac{r}{2}\pm\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}.$$

$$\text{Now } z^3=-\frac{q^3}{27y^3};$$

$$\therefore\ z^3=-\frac{q^3}{27}\times\frac{1}{-\frac{r}{2}\pm\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}}.$$

Multiplying both terms of the second member of this equation by

$$-\frac{r}{2}\mp\sqrt{\frac{r^2}{4}+\frac{q^3}{27}},$$

$$z^3=-\frac{q^3}{27}\times\frac{-\frac{r}{2}\mp\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}}{-\frac{q^3}{27}};$$

$$\text{or } z^3=-\frac{r}{2}\mp\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}.$$

y and z differ only in respect of the sign prefixed to the radical.

Consequently, if $y=\sqrt[3]{-\frac{r}{2}+\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}}$,

$$z=\sqrt[3]{-\frac{r}{2}-\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}};$$

$$\text{and } x=y+z=\sqrt[3]{-\frac{r}{2}+\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}}+\sqrt[3]{-\frac{r}{2}-\sqrt{\frac{r^2}{4}+\frac{q^3}{27}}}.$$

If x^3-qx+r be the reduced equation, the value of x is

$$x=\sqrt[3]{-\frac{r}{2}+\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}}+\sqrt[3]{-\frac{r}{2}-\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}}.$$

In this case if $\frac{q^3}{27}$ is greater than $\frac{r^2}{4}$, the expression $\sqrt{\frac{r^2}{4}-\frac{q^3}{27}}$ is negative, and therefore impossible.

This solution consequently appears under the form of the sum of two impossible quantities, expressible by

$$\sqrt[3]{a+\beta\sqrt{-1}}+\sqrt[3]{a-\beta\sqrt{-1}};$$

$$\text{or by } a^{\frac{1}{3}}\left(1+\frac{\beta}{a}\sqrt{-1}\right)^{\frac{1}{3}}+a^{\frac{1}{3}}\left(1-\frac{\beta}{a}\sqrt{-1}\right)^{\frac{1}{3}}.$$

$$\text{Now } a^{\frac{1}{3}}\left(1+\frac{\beta}{a}\sqrt{-1}\right)^{\frac{1}{3}}=a^{\frac{1}{3}}\left(1+\tfrac{1}{3}\frac{\beta}{a}\sqrt{-1}+\tfrac{1}{3}\left(\tfrac{1}{3}-1\right).\tfrac{1}{2}.\frac{\beta^2}{a^2}\sqrt{-1}^2+, \&c.\right)$$

$$a^{\frac{1}{3}}\left(1-\frac{\beta}{a}\sqrt{-1}\right)^{\frac{1}{3}}=a^{\frac{1}{3}}\left(1-\tfrac{1}{3}\frac{\beta}{a}\sqrt{-1}+\tfrac{1}{3}\left(\tfrac{1}{3}-1\right)\tfrac{1}{2}\frac{\beta^2}{a^2}\sqrt{-1}^2-, \&c.\right)$$

(Art. 175 and 176).

Whence the sum of these quantities is possible.

DES CARTES'S RULE FOR THE RESOLUTION OF A BIQUADRATIC EQUATION.

290. By rendering the coefficient of the first term 1, and the coefficient of the second term zero, the equation of the fourth degree is reduced to the form
$$x^4+qx^2+rx+s=0.$$

This equation may be supposed the product of the two quadratic factors x^2+ux+v, x^2-ux+z, the coefficients of x being $+u$ and $-u$, since the coefficient of x^3 in the product of these factors is zero.

Forming the product of these factors, and making it equal to the proposed equation,

$$x^4+qx^2+rx+s=x^4+(v+z-u^2)x^2+(uz-vz)x+vz.$$

Therefore (Art. 174) $q=v+z-u^2$ - - - 1,

$r=uz-vz$ - - - 2,

$s=vz$ - - - - 3.

From equation 1, $q+u^2=v+z$,

From equation 2, $\frac{r}{u}=z-v$;

$\therefore$ by addition - $z=\frac{1}{2}\left(q+u^2+\frac{r}{u}\right)$ - - 4,

and by subtraction $v=\frac{1}{2}\left(q+u^2-\frac{r}{u}\right)$ - - 5.

Multiplying equations 4 and 5, $vz=\frac{1}{4}\left((q+u^2)^2-\frac{r^2}{u^2}\right)$,

but, equation 3, $vz=s$;

$$\therefore\ s=\frac{1}{4}\left((q+u^2)^2-\frac{r^2}{u^2}\right),$$

$$4s=q^2+2qu^2+u^4-\frac{r^2}{u^2},$$

$$\text{and } 4u^2s=q^2u^2+2qu^4+u^6-r^2,$$

$$\text{or } u^6+2qu^4+(q^2-4s)u^2-r^2=0.$$

This, which is a cubic equation in u^2, can be resolved by Article 289.

The value of u being known, the value of z is given by equation 4, and the value of v by equation 5.

Lastly, u, z, v being known, two values of x are given by the equation

$$x^2+ux+v=0,$$

and the two others by the equation

$$x^2-ux+z=0.$$

No method of resolving literal equations of a higher degree than the fourth has been discovered.

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